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STEIN-TYPE ESTIMATORS FOR PARAMETERS RESTRICTED BY LINEAR INEQUALITIES

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ABSTRACT

In simultaneous estimation of p normal means, James and Stein (1961) introduced an estimator, called Stein-type estimator, and showed that it has uniformly smaller risk than the maximum likelihood estimator (MLE) if $p \geqq 3$ under the squared error loss. Since then many authors have discussed the Stein-type estimator under various situations. In this paper we discuss the cases where constraints of linear inequalities are given on p normal means and give Stein-type estimator for means. We also give Stein-type estimator for means of Hudson class of probability distributions when constraints of simpler linear inequalities are given on means.

1. Introduction

Suppose that a p-variate random variable $\boldsymbol{X}$ is normally distributed and has a mean vector $\boldsymbol{\mu}$ and the identity covariance matrix. This assumption is denoted by $\boldsymbol{X} \sim N(\boldsymbol{\mu}, I)$. Discussing simultaneous estimation of the mean vector $\boldsymbol{\mu}$ under the squared error loss, James and Stein (1961) introduced an estimator

$$\left(1-\frac{p-2}{\boldsymbol{X}'\boldsymbol{X}}\right)\boldsymbol{X} \tag{1.1}$$

which has uniformly smaller risk than $\boldsymbol{X}$, a very natural estimator and the maximum likelihood estimator (MLE) of $\boldsymbol{\mu}$, if $p \geqq 3$. In this paper, such a shrinked estimator as (1.1) is called Stein-type.

Since then, the Stein-type estimator has been extended by many authors in two directions. One direction is to extend the Stein-type estimator for nonnormal distributions. In simultaneous estimation k Poisson means, Clevenson and Zidek (1975) gave a Stein-type estimator which dominates the usual estimator if $k>1$. For a sub-class of the exponential family, which we call Hudson class. Hudson (1978) gave some improved estimators. Another direction is to extend the Stein-type estimator under more general loss functions. For a bibliography the reader is referred to Berger et al. (1977).

In this paper, we construct Stein-type estimators in cases where unknown population means are restricted by a set of linear inequalities, and prove that the MLE is not admissible in those cases also. There are many situations in which linear inequalities on parameters are essential in the statistical inference. The MLE in those situations were discussed, for example and among others, in Brunk et al. (1972).

In section 2, a Stein-type estimator which shrinks towards the origin will be discussed when $\boldsymbol{X}\sim N(\boldsymbol{\mu}, I)$ and $\boldsymbol{\mu}\equiv(\mu_1, \cdots, \mu_p)'$ satisfies linear inequalities $\sum_{j=1}^{p} t_{ij}\mu_j \geqq 0$, $i=1, \cdots, k$.

In section 3, a Stein-type estimator which shrinks towards a given point or a sub-space will be discussed under the same assumption on $\boldsymbol{X}$ and $\boldsymbol{\mu}$.

In section 4, a Stein-type estimator for Hudson class will be discussed when simpler linear inequalities

(i) $\mu_i \geqq c_i$, $i=1, \cdots, p$

or

(ii) $-\infty < \mu_1 \leqq \cdots \leqq \mu_p < \infty$

are given on means, $\boldsymbol{\mu}$.

2. Stein-type estimators for multivariate normal means restricted by linear inequalities

In this section we shall give a Stein-type estimator of multivariate normal mean vector $\boldsymbol{\mu}$ which satisfies linear inequalities $T\boldsymbol{\mu} \geqq \boldsymbol{0}$, where T is a known matrix.

Suppose that $\boldsymbol{Y}\sim N(\boldsymbol{\theta}, \Sigma)$, and partition $\boldsymbol{Y}$ and $\boldsymbol{\theta}$ in two parts,

$$\boldsymbol{Y}\equiv\begin{bmatrix}\boldsymbol{Y}^{(1)}\\ \cdots\cdots\\ \boldsymbol{Y}^{(2)}\end{bmatrix} \quad \text{and} \quad \boldsymbol{\theta}\equiv\begin{bmatrix}\boldsymbol{\theta}^{(1)}\\ \cdots\cdots\\ \boldsymbol{\theta}^{(2)}\end{bmatrix} \tag{2.1}$$

The sizes of $\boldsymbol{Y}^{(1)}$ and $\boldsymbol{\theta}^{(1)}$ are $k\times 1$ and those of $\boldsymbol{Y}^{(2)}$ and $\boldsymbol{\theta}^{(2)}$ are $l\times 1$. Assume further Σ to have the form

$$\Sigma\equiv\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix}, \tag{2.2}$$

namely $\boldsymbol{Y}^{(1)}\sim N(\boldsymbol{\theta}^{(1)}, \Sigma_1)$, $\boldsymbol{Y}^{(2)}\sim N(\boldsymbol{\theta}^{(2)}, \Sigma_2)$, and they are independent.

First we shall consider the simultaneous estimation of $\boldsymbol{\theta}$ under the loss function $(\hat{\boldsymbol{\delta}}-\boldsymbol{\theta})'\Sigma^{-1}(\hat{\boldsymbol{\delta}}-\boldsymbol{\theta})$ when $\boldsymbol{\theta}^{(1)} \geqq \boldsymbol{0}$, where $\hat{\boldsymbol{\delta}}$ is an estimator of $\boldsymbol{\theta}$. For this case a Stein-type estimator of $\boldsymbol{\theta}$ is given by the following lemma.

Lemma 1. Suppose that $\boldsymbol{Y}\sim N(\boldsymbol{\theta}, \Sigma)$, where $\boldsymbol{Y}, \boldsymbol{\theta}$ and Σ are defined as (2.1) and (2.2), respectively. Let

$$\boldsymbol{\delta}^1(\boldsymbol{Y})=\begin{cases}\left(1-\dfrac{a}{\boldsymbol{Y}'\Sigma^{-1}\boldsymbol{Y}}\right)\boldsymbol{Y}, & \text{if } \boldsymbol{Y}^{(1)}\geqq \boldsymbol{0}\\ \hat{\boldsymbol{\theta}}(\boldsymbol{Y}), \text{ the MLE of } \boldsymbol{\theta}, & \text{otherwise},\end{cases}$$

where a is a constant. Then, in simultaneous estimation of $\boldsymbol{\theta}$ under $\boldsymbol{\theta}^{(1)}\geqq\boldsymbol{0}$ and under the loss function $(\hat{\boldsymbol{\delta}}-\boldsymbol{\theta})'\Sigma^{-1}(\hat{\boldsymbol{\delta}}-\boldsymbol{\theta})$, $\boldsymbol{\delta}^1(\boldsymbol{Y})$ has uniformly smaller risk than the MLE, $\hat{\boldsymbol{\theta}}(\boldsymbol{Y})$, of $\boldsymbol{\theta}$, if $p=k+l\geqq 3$ and $0<a<2(p-2)$.

Proof: Define a convex cone S^+ in the sample space of $\boldsymbol{Y}$ by

$$S^+\equiv\{\boldsymbol{y}\,|\,\boldsymbol{y}^{(1)}\geqq\boldsymbol{0}\}.$$

Since $\hat{\boldsymbol{\theta}}(\boldsymbol{Y})=\boldsymbol{Y}$ for $\boldsymbol{Y}^{(1)}\geqq\boldsymbol{0}$, the difference of risks of $\hat{\boldsymbol{\theta}}(\boldsymbol{Y})$ and $\boldsymbol{\delta}^1(\boldsymbol{Y})$ is given by

$$\Delta R\equiv R(\hat{\boldsymbol{\theta}},\boldsymbol{\theta})-R(\boldsymbol{\delta}^1,\boldsymbol{\theta})=\int_{S^+}\left(\frac{2a\boldsymbol{y}'\Sigma^{-1}(\boldsymbol{y}-\boldsymbol{\theta})-a^2}{\boldsymbol{y}'\Sigma^{-1}\boldsymbol{y}}\right)p(\boldsymbol{y},\boldsymbol{\theta})d\boldsymbol{y}, \tag{2.3}$$

where $p(\boldsymbol{y},\boldsymbol{\theta})$ is the probability density function of $\boldsymbol{Y}$. To show (2.3) positive, note that

$$\boldsymbol{y}'\Sigma^{-1}(\boldsymbol{y}-\boldsymbol{\theta})=\boldsymbol{y}^{(1)'}\Sigma_1^{-1}(\boldsymbol{y}^{(1)}-\boldsymbol{\theta}^{(1)})+\boldsymbol{y}^{(2)'}\Sigma_2^{-1}(\boldsymbol{y}^{(2)}-\boldsymbol{\theta}^{(2)}),$$

and

$$\boldsymbol{y}'\Sigma^{-1}\boldsymbol{y}=\boldsymbol{y}^{(1)'}\Sigma_1^{-1}\boldsymbol{y}^{(1)}+\boldsymbol{y}^{(2)'}\Sigma_2^{-1}\boldsymbol{y}^{(1)}.$$

We have

$$\begin{aligned}&\int_{S^+}\frac{2a\boldsymbol{y}'\Sigma^{-1}(\boldsymbol{y}-\boldsymbol{\theta})}{\boldsymbol{y}'\Sigma^{-1}\boldsymbol{y}}p(\boldsymbol{y},\boldsymbol{\theta})d\boldsymbol{y}\\ &=\int_{S^+}\frac{2a\boldsymbol{y}^{(1)'}\Sigma_1^{-1}(\boldsymbol{y}^{(1)}-\boldsymbol{\theta}^{(1)})}{\boldsymbol{y}^{(1)'}\Sigma_1^{-1}\boldsymbol{y}^{(1)}+\boldsymbol{y}^{(2)'}\Sigma_2^{-1}\boldsymbol{y}^{(2)}}p(\boldsymbol{y},\boldsymbol{\theta})d\boldsymbol{y}\\ &+\int_{S^+}\frac{2a\boldsymbol{y}^{(2)'}\Sigma_2^{-1}(\boldsymbol{y}^{(2)}-\boldsymbol{\theta}^{(2)})}{\boldsymbol{y}^{(1)'}\Sigma_2^{-1}\boldsymbol{y}^{(1)}+\boldsymbol{y}^{(2)'}\Sigma_2^{-1}\boldsymbol{y}^{(2)}}p(\boldsymbol{y},\boldsymbol{\theta})d\boldsymbol{y}\end{aligned} \tag{2.4}$$

Integrating the first term of (2.4) by parts with respect to the ith component of $\boldsymbol{y}^{(1)}$, $i=1,\cdots,k$, we have

$$\int_{S^+}\left[\frac{2ak}{\boldsymbol{y}'\Sigma^{-1}\boldsymbol{y}}-\frac{4a\boldsymbol{y}^{(1)'}\Sigma_1^{-1}\boldsymbol{y}^{(1)}}{(\boldsymbol{y}'\Sigma^{-1}\boldsymbol{y})^2}\right]p(\boldsymbol{y},\boldsymbol{\theta})d\boldsymbol{y}.$$

Integrating the second term of (2.4) by parts with respect to the jth component of $\boldsymbol{y}^{(2)}, j=1,\cdots,l$, we have

$$\int_{S^+}\left[\frac{2al}{\boldsymbol{y}'\Sigma^{-1}\boldsymbol{y}}-\frac{4a\boldsymbol{y}^{(2)'}\Sigma_2^{-1}\boldsymbol{y}^{(2)}}{(\boldsymbol{y}'\Sigma^{-1}\boldsymbol{y})^2}\right]p(\boldsymbol{y},\boldsymbol{\theta})d\boldsymbol{y}.$$

Therefore, the difference ΔR turns out to be

$$\Delta R=\int_{S^+}\left[\frac{2a(k+l)-(4a+a^2)}{\boldsymbol{y}'\Sigma^{-1}\boldsymbol{y}}\right]p(\boldsymbol{y},\boldsymbol{\theta})d\boldsymbol{y},$$

which is positive if $p=k+l\geqq 3$ and $0<a<2(k+l-2)$.

Based on Lemma 1, we have the following theorem.

Theorem 1. Suppose that $\boldsymbol{X}\sim N(\boldsymbol{\mu}, I)$ and $T\boldsymbol{\mu}\geqq\boldsymbol{0}$, where T is a $k\times p$ known matrix of rank k. Let

$$\boldsymbol{\delta}^2(\boldsymbol{X})=\begin{cases}\left(1-\dfrac{a}{\boldsymbol{X}'\boldsymbol{X}}\right)\boldsymbol{X}, & \text{if} \quad T\boldsymbol{X}\geqq\boldsymbol{0}\\ \hat{\boldsymbol{\mu}}(\boldsymbol{X}), \text{ the MLE of } \boldsymbol{\mu}, & \text{otherwise},\end{cases}$$

where a is a constant. Then, in simultaneous estimation of $\boldsymbol{\mu}$ under the squared error loss, $\boldsymbol{\delta}^2(\boldsymbol{X})$ has uniformly smaller risk than $\hat{\boldsymbol{\mu}}(\boldsymbol{X})$, the MLE of $\boldsymbol{\mu}$, if $p\geqq 3$ and $0<a<2(p-2)$.

Proof: There exists a $(p-k)\times p$ matrix T_1 whose rows are orthogonal to those of T. Define a random variable $\boldsymbol{Z}$ by

$$\boldsymbol{Z}=C\boldsymbol{X}$$

where C is a nonsingular matrix defined by

$$C\equiv\begin{bmatrix} T \\ \cdots\cdots \\ T_1 \end{bmatrix}.$$

It is clear that $\boldsymbol{Z}\sim N(\boldsymbol{\xi}, \Delta)$, where $\boldsymbol{\xi}\equiv C\boldsymbol{\mu}$ and $\Delta\equiv CC'$. Now $\boldsymbol{Z}, \boldsymbol{\xi}$ and Δ are expressed by partitioned matrices as

$$\boldsymbol{Z}\equiv\begin{bmatrix} \boldsymbol{Z}^{(1)} \\ \cdots\cdots \\ \boldsymbol{Z}^{(2)} \end{bmatrix}, \quad \boldsymbol{Z}^{(1)}\equiv T\boldsymbol{X},\ \boldsymbol{Z}^{(2)}\equiv T_1\boldsymbol{X},$$

$$\boldsymbol{\xi}\equiv\begin{bmatrix} \boldsymbol{\xi}^{(1)} \\ \cdots\cdots \\ \boldsymbol{\xi}^{(2)} \end{bmatrix}, \quad \boldsymbol{\xi}^{(1)}\equiv TE(\boldsymbol{X})=T\boldsymbol{\mu},\ \boldsymbol{\xi}^{(2)}\equiv T_1E(\boldsymbol{X})=T_1\boldsymbol{\mu},$$

$$\Delta\equiv CC'=\begin{bmatrix} TT' & 0 \\ 0 & T_1T_1' \end{bmatrix}.$$

By applying Lemma 1 to the random variable $\boldsymbol{Z}, \boldsymbol{\delta}^1(\boldsymbol{Z})$ defined by Lemma 1 has uniformly smaller risk than $\hat{\boldsymbol{\xi}}(\boldsymbol{Z})$, the MLE of $\boldsymbol{\xi}$, when constraints $\boldsymbol{\xi}^{(1)}=T\boldsymbol{\mu}\geqq\boldsymbol{0}$ are given. The difference of risks between $\hat{\boldsymbol{\xi}}(\boldsymbol{Z})$ and $\boldsymbol{\delta}^1(\boldsymbol{Z})$ is given by

$$\begin{aligned}\Delta R&=\int_{\{\boldsymbol{z}|\boldsymbol{z}^{(1)}\geqq\boldsymbol{0}\}}\left(\frac{2a\boldsymbol{z}'\Delta^{-1}(\boldsymbol{z}-\boldsymbol{\xi})-a^2}{\boldsymbol{z}'\Delta^{-1}\boldsymbol{z}}\right)p(\boldsymbol{z}, \boldsymbol{\xi})\mathrm{d}\boldsymbol{z}\\ &=\int_{\{\boldsymbol{z}|\boldsymbol{z}^{(1)}\geqq\boldsymbol{0}\}}\left[\frac{2ap-(4a+a^2)}{\boldsymbol{z}'\Delta^{-1}\boldsymbol{z}}\right]p(\boldsymbol{z}, \boldsymbol{\xi})d\boldsymbol{z},\end{aligned}$$

where $p(\boldsymbol{z}, \boldsymbol{\xi})$ is the probability density function of $\boldsymbol{Z}$.

Changing the variable $\boldsymbol{z}$ to $\boldsymbol{x}$ by the relation $\boldsymbol{x}=C^{-1}\boldsymbol{z}$, we have

$$\Delta R=\int_{\{x|Tx\geqq 0\}}\left(\frac{2a\boldsymbol{x}'(\boldsymbol{x}-\boldsymbol{\mu})-a^2}{\boldsymbol{x}'\boldsymbol{x}}\right)p(\boldsymbol{x},\boldsymbol{\mu})d\boldsymbol{x}$$
$$=\int_{\{x|Tx\geqq 0\}}\left[\frac{2ap-(4a+a^2)}{\boldsymbol{x}'\boldsymbol{x}}\right]p(\boldsymbol{x},\boldsymbol{\mu})d\boldsymbol{x}\,,$$

which is positive if $p\geqq 3$ and $0<a<2(p-2)$.

Example 1. Suppose that $\boldsymbol{X}\sim N(\boldsymbol{\mu}, I)$ and that linear inequalities $-\infty<\mu_1\leqq\cdots\leqq\mu_p<\infty$ are given on $\boldsymbol{\mu}$. For this case, put

$$T=\underset{\updownarrow\, p-1}{\overset{\leftarrow\, p\,\rightarrow}{\begin{bmatrix}-1 & 1 & & & 0\\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & -1 & 1\end{bmatrix}}}\,,$$

then the constraints are given by $T\boldsymbol{\mu}\geqq \boldsymbol{0}$. Applying Theorem 1 to simultaneous estimation of $\boldsymbol{\mu}$ under the squared error loss, we see that an estimator of $\boldsymbol{\mu}$,

$$\begin{cases}\left(1-\dfrac{a}{\boldsymbol{X}'\boldsymbol{X}}\right)\boldsymbol{X}, & \text{if } -\infty<X_1\leqq\cdots\leqq X_p<\infty\\ \text{MLE of } \boldsymbol{\mu}, & \text{otherwise,}\end{cases}$$

has uniformly smaller risk than the MLE of $\boldsymbol{\mu}$ if $p\geqq 3$ and $0<a<2(p-2)$.

Lemma 1 can be extended as follows if we change the domain of the MLE of $\boldsymbol{\theta}$. Suppose again that $\boldsymbol{Y}\sim N(\boldsymbol{\theta},\Sigma)$. Let the elements of $\boldsymbol{Y}^{(1)}$ with negative value be $Y_{j_1}^{(1)},\cdots,Y_{j_r}^{(1)}$ and m be the maximum number of non-negative element of $Y^{(1)}$ whose covariance with $Y_{j_1}^{(1)},\cdots,Y_{j_r}^{(1)}$ are all zero, and let those non-negative elements be $Y_{i_1}^{(1)},\cdots,Y_{i_m}^{(1)}$. Define

$$\boldsymbol{Y}_{(m)}^{(1)+}\equiv\begin{bmatrix}Y_{i_1}^{(1)}\\ \vdots\\ Y_{i_m}^{(1)}\end{bmatrix},$$

if it exists, and

$$\boldsymbol{Y}_{(m)}^{*}\equiv\begin{bmatrix}\boldsymbol{Y}_{(m)}^{(1)+}\\ \cdots\cdots\cdots\\ \boldsymbol{Y}^{(2)}\end{bmatrix}.$$

Then a Stein-type estimator of $\boldsymbol{\theta}$ will be given by

$$\begin{cases}\left(1-\dfrac{a}{\boldsymbol{Y}_{(m)}^{*\prime}\Sigma_{(m)}^{*-1}\boldsymbol{Y}_{(m)}^{*}}\right)\boldsymbol{Y}_{(m)}^{*}\\ \text{MLE of } \boldsymbol{\theta}-\boldsymbol{\theta}_{(m)}^{*}\end{cases}\tag{2.5}$$

if $\boldsymbol{Y}\in S(m)$, where

$$S(m)\equiv\{\boldsymbol{Y}|\,\boldsymbol{Y}\equiv[Y_{i_1}^{(1)},\cdots,Y_{i_m}^{(1)}:(\boldsymbol{Y}^{(1)}-\boldsymbol{Y}_{(m)}^{(1)+})':\boldsymbol{Y}^{(2)\prime}]'\},$$

$\boldsymbol{\theta}_{(m)}^{*}$ and $\Sigma_{(m)}^{*}$ are the mean vector and covariance matrix of $\boldsymbol{Y}_{(m)}^{*}$, respectively.

Theorem 2. Suppose again $Y \sim N(\boldsymbol{\theta}, \Sigma)$, where $\boldsymbol{Y}, \boldsymbol{\theta}$ and $\boldsymbol{\Sigma}$ are defined as (2.1) and (2.2), respectively. Let

$$\boldsymbol{\delta}^3(\boldsymbol{Y}) = \begin{cases} \{\text{the estimator defined by (2.5)}\}, & \text{if } \boldsymbol{Y} \in S(m) \\ \text{MLE of } \boldsymbol{\theta}, & \text{otherwise.} \end{cases}$$

Then, in simultaneous estimation of $\boldsymbol{\theta}$ under $\boldsymbol{\theta}^{(1)} \geqq \boldsymbol{0}$ and under the loss function $(\hat{\boldsymbol{\delta}} - \boldsymbol{\theta})' \Sigma^{-1} (\hat{\boldsymbol{\delta}} - \boldsymbol{\theta})$, $\boldsymbol{\delta}^3(\boldsymbol{Y})$ has uniformly smaller risk than the MLE of $\boldsymbol{\theta}$ if $l + m \geqq 3$ and $0 < a < 2(l + m - 2)$.

Proof: Theorem 2 can be proved in a similiar way as Lemma 1 and the proof will be omitted here.

Example 2. Let $\boldsymbol{X} \equiv (X_1, \cdots, X_p)' \sim N(\boldsymbol{\mu}, I)$ and constrains $\boldsymbol{\mu} \geqq \boldsymbol{0}$ be given. Also let $m \geqq 3$ and

$$S(i_1, \cdots, i_m) \equiv \{\boldsymbol{X} | X_{i_1} \geqq 0, \cdots, X_{i_m} \geqq 0, X_{i_{m+1}} < 0, \cdots, X_{i_p} < 0\}.$$

If we define

$$\delta_{i_j}(\boldsymbol{X}) = \begin{cases} \left.\begin{cases} \left(1 - \dfrac{a}{\sum_{l=1}^{m} X_{i_l}^2}\right) X_{i_j} & j = 1, \cdots, m \\ \hat{\mu}_{i_j}, \text{ MLE of } \mu_{i_j} & j = m+1, \cdots, p \end{cases}\right\} & \text{if } \boldsymbol{X} \in S(i_1, \cdots, i_m) \\ \hat{\mu}_{i_j}, \text{ MLE of } \mu_{i_j}, \quad \text{otherwise,} \end{cases}$$

and $\boldsymbol{\delta}(\boldsymbol{X}) \equiv (\delta_1(\boldsymbol{X}), \cdots, \delta_p(\boldsymbol{X}))'$. Then $\boldsymbol{\delta}(\boldsymbol{X})$, in simultaneous estimation of $\boldsymbol{\mu} \equiv (\mu_1, \cdots, \mu_p)'$ under the squared error loss, has uniformly smaller risk than $\hat{\boldsymbol{\mu}}(\boldsymbol{X})$, the MLE of $\boldsymbol{\mu}$, if $p \geqq 3$ and $0 < a < 2(m-2)$. In fact, the difference of risks between $\hat{\boldsymbol{\mu}}(\boldsymbol{X})$ and $\boldsymbol{\delta}(\boldsymbol{X})$ is given by

$$\Delta R = \Sigma \int_{S(i_1, \cdots, i_m)} \frac{2ma - (4a + a^2)}{\sum_{l=1}^{m} x_{i_l}^2} p(\boldsymbol{x}, \boldsymbol{\mu}) \mathrm{d}\boldsymbol{x},$$

where the summation is taken over all possible sets of $(i_1, \cdots, i_m)$ $(m \geqq 3)$ which is a subset of $(1, \cdots, p)$.

Remark: In Lemma 1 if the constrains is $\boldsymbol{\theta}^{(1)} \geqq \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector, then a Stein-type estimator of $\boldsymbol{\theta}$ can be constructed by

$$\begin{cases} \left(1 - \dfrac{a}{\boldsymbol{Y}' \Sigma^{-1} \boldsymbol{Y}}\right) \boldsymbol{Y}, & \text{if } \boldsymbol{Y}^{(1)} \geqq \boldsymbol{c} \\ \text{MLE of } \boldsymbol{\theta}, & \text{otherwise.} \end{cases}$$

If we use the "positive part Stein-type estimator", then we get a better estimator.

3. Another class of estimators for multivariate normal means restricted by linear inequalities

Let $\boldsymbol{X} \sim N(\boldsymbol{\mu}, I)$, in this section we shall give a Stein-type estimators of $\boldsymbol{\mu}$ which shrinks towards a given point or towards a sub-space which is determined by data when some linear inequalities on $\boldsymbol{\mu}$ are given.

In order to construct a Stein-type estimator which shrinks towards the arithmetic mean, define a $p \times p$ matrix B by

$$B \equiv \begin{bmatrix} p-1 & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & p-1 \end{bmatrix},$$

which is a projection matrix with rank $p-1$. Decompose $\boldsymbol{X}$ and $\boldsymbol{\mu}$ as

$$\boldsymbol{X} = (I-B)\boldsymbol{X} + B\boldsymbol{X}$$

and

$$\boldsymbol{\mu} = (I-B)\boldsymbol{\mu} + B\boldsymbol{\mu},$$

and suppose that linear inequalities $AB\boldsymbol{\mu} \geqq \boldsymbol{0}$ are given. Consider an estimator

$$\boldsymbol{\delta}^4(\boldsymbol{X}) = \begin{cases} (I-B)\boldsymbol{X} + \left(1 - \dfrac{a}{\boldsymbol{X}'B\boldsymbol{X}}\right)B\boldsymbol{X}, & \text{if} \quad AB\boldsymbol{X} \geqq \boldsymbol{0} \\ \hat{\boldsymbol{\mu}}(\boldsymbol{X}), \text{ MLE of } \boldsymbol{\mu}, & \text{otherwise.} \end{cases}$$

Then we have the following theorem.

Theorem 3. Suppose again that $\boldsymbol{X} \sim N(\boldsymbol{\mu}, I)$, and linear inequalities $AB\boldsymbol{\mu} \geqq \boldsymbol{0}$ are given on $\boldsymbol{\mu}$. Then, in simultaneous estimation of $\boldsymbol{\mu}$ under the squared error loss, $\boldsymbol{\delta}^4(\boldsymbol{X})$ has smaller risk than the MLE of $\boldsymbol{\mu}$ if $p \geqq 4$, $0 < a < 2(p-3)$ and the rank of AB is equal to the number of rows of the matrix A.

Proof: The matrix B can be expressed as $B = QQ'$, where Q consists of $p-1$ column vectors $q_i, i = 1, \cdots, p-1$, which form an orthogonal basis of range space of B.

Let $\boldsymbol{\beta} \equiv Q'\boldsymbol{\mu}$ and $D \equiv AQ$. Then $Q'\boldsymbol{X} \sim N(\boldsymbol{\beta}, I)$ and linear inequalities $AB\boldsymbol{\mu} \geqq \boldsymbol{0}$ are equivalent to $D\boldsymbol{\beta} \geqq \boldsymbol{0}$. Applying Throrem 1, we see that if $p \geqq 4$ and $0 < a < 2(p-3)$, then

$$\boldsymbol{\delta}^*(Q'\boldsymbol{X}) = \begin{cases} \left(1 - \dfrac{a}{(Q'\boldsymbol{X})'(Q'\boldsymbol{X})}\right)Q'\boldsymbol{X}, & \text{if} \quad DQ'\boldsymbol{X} \geqq \boldsymbol{0} \\ \hat{\boldsymbol{\beta}}(Q'\boldsymbol{X}), \text{ MLE of } \boldsymbol{\beta}, & \text{otherwise,} \end{cases}$$

has uniformly smaller risk than the MLE of $\boldsymbol{\beta}$. Furthermore, we have

$$\begin{aligned}\Delta R &\equiv R(\hat{\boldsymbol{\mu}}, \boldsymbol{\mu}) - R(\boldsymbol{\delta}^4, \boldsymbol{\mu}) \\ &= R(\hat{\boldsymbol{\beta}}, \boldsymbol{\beta}) - R(\boldsymbol{\delta}^*, \boldsymbol{\beta}) \\ &> 0\,,\end{aligned}$$

which completes the proof.

Example 3. Suppose that $\boldsymbol{X} \sim N(\boldsymbol{\mu}, I)$ and define $\bar{\mu} \equiv \frac{1}{p} \sum_{i=1}^{p} \mu_i$, $\bar{X} \equiv \frac{1}{p} \sum_{i=1}^{p} X_i$, $\bar{\boldsymbol{\mu}}_{p\times 1} \equiv (\bar{\mu}, \cdots, \bar{\mu})'$ and $\bar{\boldsymbol{X}}_{p\times 1} \equiv (\bar{X}, \cdots, \bar{X})'$. Suppose further that linear inequalities $\mu_i \geqq \bar{\mu}$, $i = 1, \cdots, k (\leqq p-1)$, are given on $\boldsymbol{\mu}$. Then by applying Theorem 3, in simultaneous estimation of $\boldsymbol{\mu}$ under the squared error loss we can see that an estimator of $\boldsymbol{\mu}$

$$\begin{cases} \bar{\boldsymbol{X}} + \left(1 - \dfrac{a}{(\boldsymbol{X}-\bar{\boldsymbol{X}})'(\boldsymbol{X}-\bar{\boldsymbol{X}})}\right)(\boldsymbol{X}-\bar{\boldsymbol{X}})\,, & \text{if} \quad X_1 \geqq \bar{X}, \cdots, X_k \geqq \bar{X} \\ \text{MLE of } \boldsymbol{\mu}, \quad \text{otherwise,} & \end{cases}$$

has uniformly smaller risk than the MLE of $\boldsymbol{\mu}$ if $p \geqq 4$ and $0 < a < 2(p-3)$.

Example 4. Suppose that $\boldsymbol{X} \sim N(\boldsymbol{\mu}, I)$ and that linear inequalities $-\infty < \mu_1 \leqq \cdots \leqq \mu_p < \infty$ are given on $\boldsymbol{\mu}$. Then, in simultaneous estimation of $\boldsymbol{\mu}$ under the squared error loss, we can see that an estimator,

$$\begin{cases} \bar{\boldsymbol{X}} + \left(1 - \dfrac{a}{(\boldsymbol{X}-\bar{\boldsymbol{X}})'(\boldsymbol{X}-\bar{\boldsymbol{X}})}\right)(\boldsymbol{X}-\bar{\boldsymbol{X}})\,, & \text{if} \quad -\infty < X_1 \leqq \cdots \leqq X_p < \infty \\ \text{MLE of } \boldsymbol{\mu}, \quad \text{otherwise,} & \end{cases}$$

has uniformly smaller risk than the MLE of $\boldsymbol{\mu}$ if $p \geqq 4$ and $0 < a < 2(p-3)$.

Theorem 3 can be easily extended as follows.

Let P be a projector with rank $p-k$, and decompose $\boldsymbol{X}$ and $\boldsymbol{\mu}$ as

$$\boldsymbol{X} = (I-P)\boldsymbol{X} + P\boldsymbol{X}$$

and

$$\boldsymbol{\mu} = (I-P)\boldsymbol{\mu} + P\boldsymbol{\mu}\,.$$

Suppose that linear inequalities $KP\boldsymbol{\mu} \geqq \boldsymbol{0}$ on $\boldsymbol{\mu}$ are given. Then, in simultaneous estimation of $\boldsymbol{\mu}$ under the squared error loss, an estimator of $\boldsymbol{\mu}$,

$$\begin{cases} (I-P)\boldsymbol{X} + \left(1 - \dfrac{a}{(P\boldsymbol{X})'(P\boldsymbol{X})}\right)P\boldsymbol{X}\,, & \text{if} \quad KP\boldsymbol{X} \geqq \boldsymbol{0} \\ \text{MLE of } \boldsymbol{\mu}, \quad \text{otherwise,} & \end{cases}$$

has uniformly smaller risk than the MLE of $\boldsymbol{\mu}$ if $p-k \geqq 3$, $0 < a < 2(p-k-2)$ and the rank of KP is equal to the number of rows of the matrix K.

Remark 3. Theorem 3 can be extended further. We express the p-dimensional real space R^p as

$$R^p = R^{(0)} \oplus R^{(1)} \oplus \cdots \oplus R^{(l)}\,,$$

where $R^{(0)}, \cdots, R^{(l)}$ are orthogonal sub-spaces. Here we assume that the dimension of sub-space $R^{(i)}$ is p_i, $i=1, \cdots, l$ and each p_i is greater than or equal to three, $i=1, \cdots, l$. Then we decompose $\boldsymbol{X}$ and $\boldsymbol{\mu}$ as

$$\boldsymbol{X}=\boldsymbol{X}^{(0)}+\boldsymbol{X}^{(1)}+\cdots+\boldsymbol{X}^{(l)}, \quad \boldsymbol{X}^{(i)} \in R^{(i)}, \quad i=0,1, \cdots, l$$

and

$$\boldsymbol{\mu}=\boldsymbol{\mu}^{(0)}+\boldsymbol{\mu}^{(1)}+\cdots+\boldsymbol{\mu}^{(l)}, \quad \boldsymbol{\mu}^{(i)} \in R^{(i)}, \quad i=0,1, \cdots, l .$$

Suppose that linear inequalities $A_i \boldsymbol{\mu}^{(i)} \geqq \mathbf{0}$ are given on $\boldsymbol{\mu}^{(i)}, i=1, \cdots, l$. We assume that the rank A_i is less than or equal to p_i, $i=1, \cdots, l$. Then in simultaneous estimation of $\boldsymbol{\mu}$ under the squared error loss, an estimator of $\boldsymbol{\mu}$.

$$\begin{cases} \boldsymbol{X}^{(0)}+\sum_{i=1}^{l}\left(1-\dfrac{a_i}{\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)}}\right)X^{(i)}, & \text{if} \quad A_i \boldsymbol{X}^{(i)} \geqq \mathbf{0}, \quad i=1, \cdots, l \\ \text{MLE of } \mu, \quad \text{otherwise}, \end{cases}$$

has uniformly smaller risk than the MLE of $\boldsymbol{\mu}$ if $0<a_i<2(p_i-2)$, $i=1, \cdots, l$.

4. Stein-type estimator for means of Hudson class restricted by simple linear inequalities

In this section we shall give a Stein-type estimator for means, $\boldsymbol{\mu} \equiv (\mu_1, \cdots, \mu_p)'$, of continuous distributions in Hudson class when simpler linear inequalities (i) $\mu_i \geqq c_i, i=1, \cdots, p$ or (ii) $-\infty<\mu_1 \leqq \cdots \mu_p<\infty$ are given on $\boldsymbol{\mu}$.

Suppose that X is a continuous random variable with a density function $f(\cdot, \mu)$ which satisfies the equality

$$E((X-\mu)g(X))=E(t(X)g'(X)),$$

where $E(X)=\mu$, for some non-negative real function $t(\cdot)$ and for any absolutely continuous function $g(\cdot)$ such that $E|t(X)g'(X)|<\infty$.

Let $X_1, \cdots, X_p$ be independent random variables which have the density function $f(\cdot, \mu_i)$, where $E(X_i)=\mu_i$, $i=1, \cdots, p$. Then an estimator of $\boldsymbol{\mu}$,

$$X_i-\frac{p-2}{\sum_{i=1}^{p}(h(X_i))^2} h(X_i), \quad i=1, \cdots, p, \tag{4.1}$$

suggested by H. Hudson, has uniformly smaller risk than $X \equiv (X_1, \cdots, X_p)'$ if $p \geqq 3$. In (4.1)

$$h(x)=\int \frac{1}{t(x)} dx \tag{4.2}$$

is an indefinite integral.

If there exists linear inequalities on $\boldsymbol{\mu}$ of the form (i), then we have the following theorem.

Theorem 4. Suppose that $X_1, \cdots, X_p$ are independent random variables with the common density $f(\cdot, \mu_i)$ and $E(X_i)=\mu_i, i=1, \cdots, p$. Let $\boldsymbol{\delta}^5(\boldsymbol{X}) \equiv (\delta_1{}^5(\boldsymbol{X}), \cdots, \delta_p{}^5(\boldsymbol{X}))'$ be an estimator of $\boldsymbol{\mu}$ with the ith element

$$\delta_i{}^5(\boldsymbol{X}) = \begin{cases} X_i - \dfrac{a}{\sum\limits_{i=1}^{p} (h^*((X_i))^2} h^*(X_i), & \text{if } X_i \geqq c_i, \quad i=1, \cdots, p \\ \hat{\mu}_i(\boldsymbol{X}), \text{ MLE of } \mu_i, & \text{otherwise,} \end{cases}$$

where

$$h^*(x) \equiv \int_{c^*}^{x} \frac{1}{t(x)} dx ,$$

and

$$c^* \equiv \min \{c_1, \cdots, c_p\} .$$

Then, in simultaneous estimation of $\boldsymbol{\mu}$ under $\mu_i \geqq c_i, i=1, \cdots, p$ and the squared error loss, $\boldsymbol{\delta}^5(\boldsymbol{X})$ has uniformly smaller risk than $\hat{\boldsymbol{\mu}}(\boldsymbol{X}) \equiv (\hat{\mu}_1(\boldsymbol{X}), \cdots, \hat{\mu}_p(\boldsymbol{X}))'$, the MLE of $\boldsymbol{\mu}$, if $p \geqq 3$, $0<a<2(p-2)$.

Proof: The difference of risks between $\hat{\boldsymbol{\mu}}(\boldsymbol{X})$ and $\boldsymbol{\delta}^5(X)$ is given by

$$\begin{aligned} \Delta R &\equiv R(\hat{\boldsymbol{\mu}}, \boldsymbol{\mu}) - R(\boldsymbol{\delta}^5, \boldsymbol{\mu}) \\ &= \int_{c_p}^{\infty} \cdots \int_{c_1}^{\infty} \sum_{i=1}^{p} \left\{ \frac{2ah^*(x_i)(x_i-\mu_i)}{\sum\limits_{i=1}^{p} (h^*(x_i))^2} \right\} \left\{ \prod_{i=1}^{p} f(x_i, \mu_i) \right\} dx_1 \cdots dx_p \\ &\quad - \int_{c_p}^{\infty} \cdots \int_{c_1}^{\infty} \frac{a^2}{\sum\limits_{i=1}^{p} (h^*(x_i))^2} \left\{ \prod_{i=1}^{p} f(x_i, \mu_i) \right\} dx_1 \cdots dx_p , \end{aligned} \tag{4.3}$$

where $f(x_i, \mu_i)$ is the probability density function of X_i. For the first term of (4.3), by integrating by parts with respect to x_i, we have

$$\begin{aligned} &\int_{c_i}^{\infty} \frac{2ah^*(x_i)(x_i-\mu_i)}{\sum\limits_{i=1}^{p} (h^*(x_i))^2} f(x_i, \mu_i) dx_i \\ &= \frac{2at(c_i)h^*(c_i)f(c_i, \mu_i)}{\sum\limits_{\substack{j=1 \\ j \neq i}}^{p} (h^*(x_j))^2 + (h^*(c_i))^2} + \int_{c_i}^{\infty} \left\{ \frac{2a}{\sum\limits_{i=1}^{p} (h^*(x_i))^2} - \frac{4a(h^*(x_i))^2}{\left[\sum\limits_{i=1}^{p} (h^*(x_i))^2 \right]^2} \right\} f(x_i, \mu_i) dx_i . \end{aligned} \tag{4.4}$$

Since the first term of (4.4) is non-negative, we have

$$\begin{aligned} \Delta R &\geqq \int_{c_p}^{\infty} \cdots \int_{c_1}^{\infty} \left\{ \frac{2ap-(4a+a^2)}{\sum\limits_{i=1}^{p} (h^*(x_i))^2} \right\} \left\{ \prod_{i=1}^{p} f(x_i, \mu_i) \right\} dx_1 \cdots dx_p \\ &> 0 , \quad \text{if} \quad 0<a<2(p-2). \end{aligned}$$

If there exists linear inequalities on $\boldsymbol{\mu}$ of the form (ii), then we have the following theorem.

Theorem 5. Suppose that $X_1, \cdots, X_p$ are independent random variables with the common density $f(\cdot, \mu_i)$ and $E(X_i)=\mu_i, i=1, \cdots, p$. Let $\boldsymbol{\delta}^6(\boldsymbol{X})\equiv(\delta_1{}^6(\boldsymbol{X}), \cdots, \delta_p{}^6(\boldsymbol{X}))'$ be an estimator of $\boldsymbol{\mu}$ with the ith element

$$\delta_i{}^6(\boldsymbol{X})=\begin{cases} X_i-\dfrac{a}{\sum_{i=1}^{p}(h(X_i))^2}h(X_i), & \text{if } -\infty<X_1\leqq\cdots\leqq X_p<\infty \\ \hat{\mu}_i(\boldsymbol{X}), \text{ MLE of } \mu_i, & \text{otherwise,} \end{cases}$$

where $h(X_i)$ is defined by (4.2). Then, in simultaneous estimation of $\boldsymbol{\mu}$ under $-\infty<\mu_1\leqq\cdots\leqq\mu_p<\infty$ and under the squared error loss, $\boldsymbol{\delta}^6(\boldsymbol{X})$ has uniformly smaller risk than $\hat{\boldsymbol{\mu}}(\boldsymbol{X})$, the MLE of $\boldsymbol{\mu}$, if $p\geqq 3$, $0<a<2(p-2)$.

Proof: We prove only for the three dimensional case. The proof of general case is a complicated but straight extension of the three dimensional case. The difference of risks between $\hat{\boldsymbol{\mu}}(\boldsymbol{X})$ and $\boldsymbol{\delta}^6(\boldsymbol{X})$ is given by

$$\begin{aligned}
\Delta R &= R(\hat{\boldsymbol{\mu}}, \boldsymbol{\mu})-R(\boldsymbol{\delta}^6, \boldsymbol{\mu}) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{x_2}\int_{-\infty}^{x_2}\left\{\frac{2ah(x_1)(x_1-\mu_1)}{\sum_{i=1}^{3}(h(x_i))^2}\right\}\left\{\prod_{i=1}^{3}f(x_i, \mu_i)\right\}dx_1dx_2dx_3 \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{x_2}\int_{x_1}^{x_3}\left\{\frac{2ah(x_2)(x_2-\mu_2)}{\sum_{i=1}^{3}(h(x_i))^2}\right\}\left\{\prod_{i=1}^{3}f(x_i, \mu_i)\right\}dx_2dx_1dx_3 \\
&+ \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty}\left\{\frac{2ah(x_3)(x_3-\mu_3)}{\sum_{i=1}^{3}(h(x_i))^2}\right\}\left\{\prod_{i=1}^{3}f(x_i, \mu_i)\right\}dx_3dx_2dx_1 \\
&- \int_{-\infty}^{\infty}\int_{-\infty}^{x_3}\int_{-\infty}^{x_2}\left\{\frac{a^2}{\sum_{i=1}^{3}(h(x_i))^2}\right\}\left\{\prod_{i=1}^{3}f(x_i, \mu_i)\right\}dx_1dx_2dx_3 .
\end{aligned}$$

Integrating the first three terms, we have

$$\begin{aligned}
\Delta R &= -\int_{-\infty}^{\infty}\int_{-\infty}^{x_3}\frac{2at(x_2)h(x_2)}{2(h(x_2))^2+(h(x_3))^2}f(x_2, \mu_1)f(x_2, \mu_2)f(x_3, \mu_3)dx_2dx_3 \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{x_3}\int_{-\infty}^{x_2}\frac{2a}{\sum_{i=1}^{3}(h(x_i))^2}\left\{\prod_{i=1}^{3}f(x_i, \mu_i)\right\}dx_1dx_2dx_3 \\
&- \int_{-\infty}^{\infty}\int_{-\infty}^{x_3}\frac{2at(x_3)h(x_3)}{(h(x_1))^2+2(h(x_3))^2}f(x_3, \mu_2)f(x_1, \mu_1)f(x_3, \mu_3)dx_1dx_3 \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{x_3}\frac{2at(x_1)h(x_1)}{2(h(x_1))^2+(h(x_3))^2}f(x_1, \mu_2)f(x_1, \mu_1)f(x_3, \mu_3)dx_1dx_3 \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{x_3}\int_{x_1}^{x_3}\frac{2a}{\sum_{i=1}^{3}(h(x_i))^2}\left\{\prod_{i=1}^{3}f(x_i, \mu_i)\right\}dx_2dx_1dx_3
\end{aligned}$$

$$+\int_{-\infty}^{\infty}\int_{x_1}^{\infty}\frac{2at(x_2)h(x_2)}{(h(x_1))^2+2(h(x_2))^2}f(x_2,\mu_3)f(x_2,\mu_2)f(x_1,\mu_1)dx_2dx_1$$

$$+\int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty}\frac{2a}{\sum_{i=1}^{3}(h(x_i))^2}\left\{\prod_{i=1}^{3}f(x_i,\mu_i)\right\}dx_3dx_2dx_1$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{x_3}\int_{-\infty}^{x_1}\frac{4a+a^2}{\sum_{i=1}^{3}(h(x_i))^2}\left\{\prod_{i=1}^{3}f(x_i,\mu_i)\right\}dx_1dx_2dx_3\,.$$

Here if we exchange the variable x_2 and x_1 in the first term then it is equal to the fourth term but with opposite sign. In the same way, if we exchange the variable x_3 and x_2 in the third term then it is equal to the sixth term but with opposite sign. Therefore we have

$$\Delta R=\int_{-\infty}^{\infty}\int_{-\infty}^{x_3}\int_{-\infty}^{x_2}\left\{\frac{6a-(4a+a^2)}{\sum_{i=1}^{3}(h(x_i))^2}\right\}\left\{\prod_{i=1}^{3}f(x_i,\mu_i)\right\}dx_1dx_2dx_3$$

$$>0\,,\quad \text{if}\quad 0<a<2\,.$$

Finally we give an estimator which shrinks towards $\bar{h}$, the mean of $h(x_1),\cdots,h(x_p)$.

Theorem 6. Suppose again that $X_1,\cdots,X_p$ are independent random variables with the common density $f(\cdot,\mu_i)$ and $E(X_i)=\mu_i$, $i=1,\cdots,p$. Let $\boldsymbol{\delta}^7(\boldsymbol{X})=(\delta_1{}^7(\boldsymbol{X}),\cdots,\delta_p{}^7(\boldsymbol{X}))'$ be an estimator of $\boldsymbol{\mu}$ with the ith element

$$\delta_i{}^7(\boldsymbol{X})=\begin{cases}X_i-\dfrac{a}{\sum_{i=1}^{p}(h(X_i)-\bar{h}))^2}(h(X_i)-\bar{h})\,, & \text{if}\quad -\infty<X_1\leqq\cdots\leqq X_p<\infty\\ \text{MLE of }\mu_i, & \text{otherwise,}\end{cases}$$

where $h(X_i)$ is defined by (4.2) and $\bar{h}\equiv\frac{1}{p}\sum_{i=1}^{p}h(X_i)$. Then, in simultaneous estimation of $\boldsymbol{\mu}$ under $-\infty<\mu_1\leqq\cdots\leqq\mu_p<\infty$ and under the squared error loss, $\boldsymbol{\delta}^7(\boldsymbol{X})$ has uniformly smaller risk than the MLE of $\boldsymbol{\mu}$, if $p\geqq 4$, $0<a<2(p-3)$.

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