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OSCILLATORY FLOW OF A FLUID WITH COUPLE STRESS

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ABSTRACT

Applications of couple stress and micropolar theories to the problems of oscillatory flow in a circular tube are discussed. Couple stress and spin angular momentum are considered in this approach. The exact solutions for velocity, micro-rotation, vorticity, shearing stress, flow rate and energy dissipation are obtained mathematically. And the energy dissipation over the cross section are calculated numerically. These solutions are characterized by three parameters. One is dimensionless frequency W which is called Womersley number. The others are the ratios of viscosities ε and the size effect parameter λ which do not appear in Newtonian fluid. ε is the ratio of vortex viscosity to shearing viscosity. λ gives the relation of the size between the corpuscle and the tube radius. The solutions are compared with those for Newtonian fluid and investigated with variations of ε and λ.

Nomenclature

A: amplitude of sinusoidal pressure gradient
$\boldsymbol{b}$: body force
$\boldsymbol{D}$: deformation rate tensor
$\boldsymbol{e}$: antisymmetric tensor of the third order
$\boldsymbol{I}$: idemfactor
i: imaginary unit
$J_n(x)$: Bessel function of the first kind, of order n
k: $=\{4\mu\mu_1/\gamma(\mu+\mu_1)\}^{1/2}$
$\boldsymbol{l}$: body moment
l: volume averaged radius of gyration
$\boldsymbol{M}$: couple stress tensor
$\boldsymbol{m}^{(n)}$: couple stress vector

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$\boldsymbol{n}$: exterior unit normal
p: pressure
Q: volume flow rate
R: radius of tude
$\boldsymbol{s}$: spin angular momentum
$\boldsymbol{T}$: stress tensor
$\boldsymbol{t}^{(n)}$: stress vector
$\boldsymbol{v}$: velocity vector
$\boldsymbol{W}$: vorticity tensor
W: dimensionless frequency
δ: $=\{\phi^2/(1+\varepsilon)-\lambda^2\}^{1/2}$
ε: $=\mu_1/\mu$
λ: $=kR$
μ: shear viscosity
μ_1: vortex viscosity
ξ: $=r/R$
ρ: mass density
τ: shearing stress
Φ: dissipation function
ϕ: $=iW^{3/2}$
$\boldsymbol{\Omega}$: micro-rotation vector
$\boldsymbol{\omega}$: vorticity vector
ω_0: angular frequency of sinusoidal pressure gradient

Subscripts

a: refers to antisymmetric part
s: refers to symmetric part
t: refers to transpose
$-$: reduced to dimensionless form

I. Introduction

In general fluid dynamics the stress tensor is symmetric. The argument is based on conservation of angular momentum and makes no allowance for possible internal angular momentum. When neutrally buoyant corpuscles are contained in fluid, corpuscles rotate if velocity gradients exist, due to the shearing stresses. Corpuscles have spin angular momentum in addition to orbital angular momentum. The symmetry of stress tensor is not held in the fluid which has spin angular momentum. The fluid containing neutrally buoyant corpuscles, if observed macroscopically, is considered to be a non-Newtonian fluid which has a constitutive equation expressed by the stress tensor different from that of Newtonian fluid. Since the corpuscle is much larger than a fluid particle, i.e., the radius of gyration of the corpuscle is different from that of the fluid particle, the angular velocity of the corpuscle is not equal to that of the fluid particle. This difference produces a couple stress in the fluid. The fluid which has couple stress and spin angular momentum is called polar fluid.

The theory of polar fluid was developed by Eringen (1964)[6] and Allen and DeSilva (1966).[1] The behaviors of such a fluid have been studied from various viewpoints. Stokes (1966, 1971)[11),12] discussed a number of fundamental steady flows in order to determine material constants of the fluid which has couple stresses and investigated effects of couple stresses in fluids on the creeping flow past a sphere. However he did not consider spin angular momentum. Ramkissoon and Majumdar (1976)[9] proposed to consider the Stokes' law problem for axially symmetric bodies in micropolar fluids. They derived a simple formula for the drag in terms of the stream function. Ariman et al. (1974)[3] reported that the theoretical results of polar fluid agrees with the experimental steady and pulsatile blood flow data by Bugliarello and Sevilla (1970)[4]. Sawada and Tanahashi (1981)[10] analyzed a few flow patterns and discussed with apparent viscosity. Condiff and Dahler (1964)[5] argued with several alternative boundary conditions and examined the special case of a rapidly rotating electric field.

The present paper is devoted to analysing the oscillatory flow of polar fluid in a circular tube. We would concern ourselves with solving this problem using the theory developed by Eringen. Expression for velocity, micro-rotation, vorticity, shearing stress, flow rate and energy dissipation have been derived and they are compared with those of Newtonian fluid which are obtained by Womersley (1955)[16] and Uchida (1956)[15] et al.

II. Fundamental Equations

1. Cauchy's Equations of Motion

Let $\boldsymbol{v}$ be the velocity vector. The balance principle for mass, expressing the conservation of mass, is

$$\frac{d}{dt}\int_V \rho dV=0 \tag{1}$$

where ρ is the density of the material. The equations governing the balance of momentum and angular momentum are

$$\frac{d}{dt}\int_V \rho\boldsymbol{v}dV=\int_V \rho\boldsymbol{b}dV+\int_S \boldsymbol{t}^{(n)}dS \tag{2}$$

$$\frac{d}{dt}\int_V (\boldsymbol{r}\times\rho\boldsymbol{v}+\rho\boldsymbol{s})dV=\int_S \{\boldsymbol{r}\times\boldsymbol{t}^{(n)}+\boldsymbol{m}^{(n)}\}dS+\int_V (\boldsymbol{r}\times\rho\boldsymbol{b}+\rho\boldsymbol{l})dV \tag{3}$$

where $\boldsymbol{b}$ and $\boldsymbol{l}$ represent the body force and the body momentum, respectively. Stress vector $\boldsymbol{t}^{(n)}$ and couple stress vector $\boldsymbol{m}^{(n)}$ are given by

$$\boldsymbol{t}^{(n)}=\boldsymbol{n}\cdot\boldsymbol{T}, \quad \boldsymbol{m}^{(n)}=\boldsymbol{n}\cdot\boldsymbol{M} \tag{4}$$

where $\boldsymbol{n}$ is the extrior unit normal to the surface on which $\boldsymbol{t}^{(n)}$ and $\boldsymbol{m}^{(n)}$ act. $\boldsymbol{M}$ is the couple stress. It arises in such the same way that the stress tensor $\boldsymbol{T}$ does. Substituting Eq. (4) into Eqs. (2) and (3) leads the first and second Cauchy's equations of motion

$$\rho\frac{d\boldsymbol{v}}{dt}=\nabla\cdot\boldsymbol{T}+\rho\boldsymbol{b} \tag{5}$$

$$\rho\frac{d\boldsymbol{s}}{dt}=\nabla\cdot\boldsymbol{M}+\rho\boldsymbol{l}+\boldsymbol{e}:\boldsymbol{T}^{a} \tag{6}$$

2. Constitutive Equations

The rate of deformation tensor $\boldsymbol{D}$ and the vorticity tensor $\boldsymbol{W}$ are given by

$$\left.\begin{aligned}\boldsymbol{D}&=\frac{1}{2}(\boldsymbol{v}\overleftarrow{\nabla}+\vec{\nabla}\boldsymbol{v})\\ \boldsymbol{W}&=\frac{1}{2}(\boldsymbol{v}\overleftarrow{\nabla}-\vec{\nabla}\boldsymbol{v})\end{aligned}\right\} \tag{7}$$

They are the symmetric and antisymmetric parts of the velocity gradient, respectively. The vorticity vector $\boldsymbol{\omega}$ is defined as the curl of the velocity vector, so that

$$\boldsymbol{\omega}=\mathrm{curl}\,\boldsymbol{v} \tag{8}$$

The vorticity vector and the vorticity tensor are related by following equations,

$$\boldsymbol{W}=-\frac{1}{2}\boldsymbol{e}\cdot\boldsymbol{\omega}\,,\quad \boldsymbol{\omega}=-\boldsymbol{e}:\boldsymbol{W} \tag{9}$$

where $\boldsymbol{e}$ is the antisymmetric tensor of the third order.

The linear constitutive equations of stress tensor $\boldsymbol{T}$ and couple stress tensor $\boldsymbol{M}$ are necessary to solve Cauchy's equations of motion. The stress tensor is divided into two components,

$$\boldsymbol{T}=\boldsymbol{T}^{s}+\boldsymbol{T}^{a} \tag{10}$$

Eringen-Allen suggested,

$$\boldsymbol{T}^{s}=-p\boldsymbol{I}+\lambda(\mathrm{tr}\,\boldsymbol{D})\boldsymbol{I}+2\mu\boldsymbol{D} \tag{11}$$

$$\boldsymbol{T}^{a}=-2\mu_{1}(\boldsymbol{W}+\boldsymbol{e}\cdot\boldsymbol{\Omega})=-\mu_{1}\boldsymbol{I}\times(\boldsymbol{\omega}-2\boldsymbol{\Omega}) \tag{12}$$

It is explicit that the symmetric part of the stress tensor depends only upon the symmetric velocity gradient tensor. In general the micro-rotation is not equal to one-half of the vorticity which the velocity gradient produces. This difference makes the antisymmetric part of $\boldsymbol{T}$. The dimensions of material constants λ, μ and μ_1 are those of viscosity (namely, $[M/LT]$). μ_1 is called vortex viscosity by Kline et al. (1972)[8]. If vortex viscosity μ_1 is set equal to zero, the constitutive equation of the stress tensor reduces to the usual viscosity law.

The constitutive equation of couple stress tensor $\boldsymbol{M}$ is as follows:

$$\boldsymbol{M}=\alpha\,\mathrm{tr}\,(\boldsymbol{\Omega}\overleftarrow{\nabla})\boldsymbol{I}+\beta(\boldsymbol{\Omega}\overleftarrow{\nabla})+\gamma(\boldsymbol{\Omega}\overleftarrow{\nabla})^{t} \tag{13}$$

These coefficients α, β and γ are material constants. The dimensions of α, β and

γ are those of momentum (namely, $[ML/T]$). They are called spin viscosity by Kline et al. The ratio γ/μ has a dimension of length square. Allen and Kline (1970)[2] denoted this material constant by the volume averaged radius of gyration l, i.e.,

$$l^2=\gamma/\mu \tag{14}$$

It was shown by Eringen (1966)[7] that μ_1 and γ must be rearl and positive from the thermodynamical considerations, but α and β may be negative if they satisfy some inequalities.

The relation between the spin angular momentum $\boldsymbol{s}$ and the micro-rotation $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{s}=l^2\boldsymbol{\Omega} \tag{15}$$

3. Fundamental Equations

Eq. (1), which means the conservation of mass, is rewritten as the continuity equation:

$$\frac{d\rho}{dt}+\rho\,\mathrm{div}\,\boldsymbol{v}=0 \tag{16}$$

When material constauts $\mu, \mu_1, \lambda, \alpha, \beta$ and γ are assumed to be spatially constant and from Eqs. (11), (12), (13), (14) and (15), the following equations are obtained;

$$\rho\frac{d\boldsymbol{v}}{dt}=-\nabla p+(\lambda+2\mu)\nabla\nabla\cdot\boldsymbol{v}-(\mu+\mu_1)\nabla\times\nabla\times\boldsymbol{v}+2\mu_1\nabla\times\boldsymbol{\Omega}+\rho\boldsymbol{b} \tag{17}$$

$$\frac{\rho\gamma}{\mu}\,\frac{d\boldsymbol{\Omega}}{dt}=(\alpha+\beta+\gamma)\nabla\nabla\cdot\boldsymbol{\Omega}-\gamma\nabla\times\nabla\times\boldsymbol{\Omega}+2\mu_1\nabla\times\boldsymbol{v}-4\mu_1\boldsymbol{\Omega}+\rho\boldsymbol{l} \tag{18}$$

III. Oscillatory Flow in Circular Tube

Oscillatory flow of incompressible fluid axial symmetry through a rectilinear tube of circular section is considered. It will be convenient to take the cylindrical coordinates whose x axis is identified with the center line of the tube. We assume that neither body force nor body moment is present. We restrict ourselves to the case

$$v_r=0\,,\quad v_\theta=0\,,\quad v_x=v(r,t) \tag{19}$$

$$\Omega_r=0\,,\quad \Omega_\theta=\Omega(r,t)\,,\quad \Omega_x=0 \tag{20}$$

Here the continuity equation (16) is automatically satisfied. From Eqs. (19) and (20), Eqs. (17) and (18) in cylindrical coordinates are

$$(\mu+\mu_1)\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+2\mu_1\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\Omega)-\frac{\partial p}{\partial x}=\rho\frac{\partial v}{\partial t} \tag{21}$$

$$r\frac{\partial}{\partial r}\left\{\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\Omega)\right\}-2\mu_1\frac{\partial v}{\partial r}-4\mu_1\Omega=\frac{\rho\gamma}{\mu}\cdot\frac{\partial\Omega}{\partial t} \tag{22}$$

In the general oscillating flow, an arbitrary time-dependent pressure gradient will be expressed by Fourier series as follows;

$$-\frac{\partial p}{\partial x}=A_0+\sum_{n=1}^{\infty}\{A_n\cos(\omega_n t)+B_n\sin(\omega_n t)\} \tag{23}$$

Since the total pressure gradient (23) is a superposition of sinusoidal components of pressure gradient, it is sufficient to investigate the fundamental wave, which is given by the real part of the following equation;

$$-\frac{\partial p}{\partial x}=A\exp(i\omega_0 t) \tag{24}$$

where A is the amplitude of sinusoidal pressure gradient. If $\omega_0\to 0$, the flow becomes Poiseuille flow. The solutions of Eqs. (21) and (22) should be of the form

$$v(r,t)=\hat{v}(r)\exp(i\omega_0 t) \tag{25}$$

$$\Omega(r,t)=\hat{\Omega}(r)\exp(i\omega_0 t) \tag{26}$$

Only the real parts of these equations have the physical meaning. Substituting Eqs. (25) and (26) into Eqs. (21) and (22) leads simultaneous ordinary differential equations for $\hat{V}(r)$ and $\hat{\Omega}(r)$ are:

$$(\mu+\mu_1)\frac{1}{r}\cdot\frac{d}{dr}\left(r\frac{d\hat{v}}{dr}\right)+2\mu_1\frac{1}{r}\cdot\frac{d}{dr}(r\hat{\Omega})-i\rho\omega_0\hat{v}=-A \tag{27}$$

$$r\frac{d}{dr}\left\{\frac{1}{r}\cdot\frac{d}{dr}(r\hat{\Omega})\right\}-2\mu_1\frac{d\hat{v}}{dr}-4\mu_1\hat{\Omega}=i\frac{\rho\omega_0\gamma}{\mu}\hat{\Omega} \tag{28}$$

Thə general solutions for this system, which are finite at the tube axis $r=0$, are expressed by

$$\hat{v}(r)=-\frac{A}{m^2(\mu+\mu_1)}+\frac{2aC_1\mu_1}{(a^2+m^2)(\mu+\mu_1)}J_0(ar)+\frac{2bC_2\mu_1}{(a^2-b^2)(b^2+m^2)(\mu+\mu_1)}J_0(br) \tag{29}$$

$$\hat{\Omega}(r)=C_1J_1(ar)-\frac{C_2}{a^2-b^2}J_1(br) \tag{30}$$

where C_1 and C_2 are arbitrary constants. a, b and m are given by

$$a^2=-(4\mu\mu_1+i\rho\omega_0\gamma)/\gamma(\mu+\mu_1)$$

$$b^2=-i\rho\omega_0/\mu$$

$$m^2=i\rho\omega_0/(\mu+\mu_1)$$

To determine the arbitrary constants C_1 and C_2, some kinds of boundary conditions are necessary. Though many arguments have been made for spin boundary conditions, a clear conclusion has not been obtained yet. The typical conditions are no-spin condition and constant-spin condition at the wall. The former, which is suggested by Eringen, corresponds to the no-slip condition of velocity. The latter is used by Ariman et al. and is based on the experimental results that corpuscles rotate at the wall. Here we use no-spin condition, which is

$$v=0, \quad \Omega=0 \quad \text{at} \quad r=R, \tag{31}$$

Applying these conditions to Eqs. (29), (30), (25) and (26) yields the desired solutions for v and Ω as follows;

$$\begin{aligned}\bar{v}(\xi)&=\frac{V(\xi)}{AR^2/4\mu}\\ &=-\frac{4}{\phi^2}\left[1-\frac{J_1(\delta)J_0(\phi\xi)+\dfrac{\delta\phi\varepsilon}{\lambda^2(1+\varepsilon)}J_0(\phi)J_1(\delta\xi)}{J_1(\delta)J_0(\phi)+\dfrac{\delta\phi\varepsilon}{\lambda^2(1+\varepsilon)}J_0(\delta)J_1(\phi)}\right]\exp(i\omega_0 t)\end{aligned} \tag{32}$$

$$\begin{aligned}\bar{\Omega}(\xi)&=\frac{\Omega(\xi)}{AR/4\mu}\\ &=-\frac{2}{\phi}\cdot\frac{J_1(\phi)J_1(\delta\xi)-J_1(\delta)J_1(\phi\xi)}{J_1(\delta)J_0(\phi)+\dfrac{\delta\phi\varepsilon}{\lambda^2(1+\varepsilon)}J_0(\delta)J_1(\phi)}\exp(i\omega_0 t)\end{aligned} \tag{33}$$

where

$$\begin{aligned}&\varepsilon=\mu_1/\mu\\ &\phi=Wi^{3/2}\\ &k^2=4\mu\mu_1/\gamma(\mu+\mu_1)\\ &\delta=\sqrt{\phi^2/(1+\varepsilon)-\lambda^2}\\ &\xi=r/R\\ &W=R\sqrt{\rho\omega_0/\mu}\\ &\lambda=kR\end{aligned}$$

These solutions are characterized by three dimensionless parameters W, λ and ε, which are called Womersley number, the size effect parameter and the viscosity ratio, respectively.

1. Velocity Profiles

Velocity profiles at $\pi/8$ intervals during one half cycle are shown in Fig. 1. and Fig. 2. When $W\gg 1$ and $\phi\xi\gg 1$, Eq. (32) becomes

$$\bar{v}(\xi)\simeq-\frac{4i}{W^2}\left[1-\frac{1}{\sqrt{\xi}}\exp\{-(1-\xi)Wi^{1/2}\}\right]\exp(i\omega_0 t) \tag{34}$$

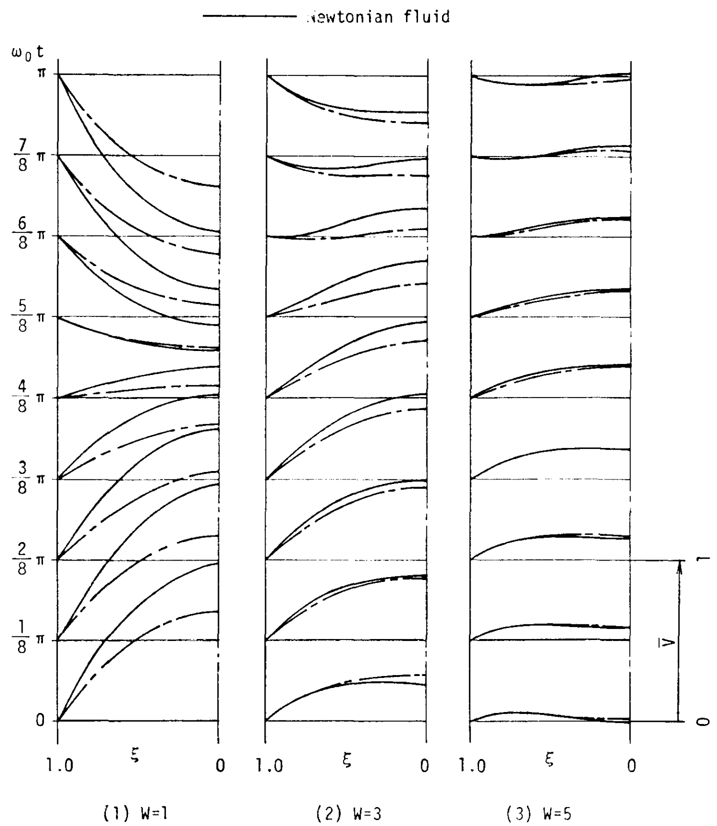


Fig. 1. Velocity profiles ($\lambda=1.0$, $\varepsilon=1.0$)

When W is large, the amplitude of the velocity decreases in proportion to $1/W^2$ and the fluid flows with the phase lag of $\pi/2$ from the phase of pressure gradient because of $-i$ of the right-hand side of Eq. (34). It will be found that in the rapid oscillation the velocity profile becomes flat near the center of the tube and the maximum of the velocity distributions exists in neighborhood of the wall. Whereas the velocity approaches $\exp(i\omega_0 t)$ multiplied by the velocity of the steady flow when W is small. There is no phase difference between the velocity and the pressure gradient.

As ε becomes small, the velocity profile goes to that of Newtonian fluid. It corresponds to the decrease of μ_1 that ε goes to zero with constant μ. Then the antisymmetric part of stress tensor is vanished. The polar effect does not appear in such case. The limit of $\varepsilon \to 0$ in Eq. (32) is the velocity of Newtonian fluid.

In the theory of polar fluid, it must be considered that corpuscles rotate because of shearing stresses. These rotations produce the rotation of the cell, which is relative to the volume averaged radius of gyration of the cell. If the radius of the circular tube changes, the velocity profile in this case varies. But in Newtonian fluid the velocity profile does not depend on the tube radius. Such size effect

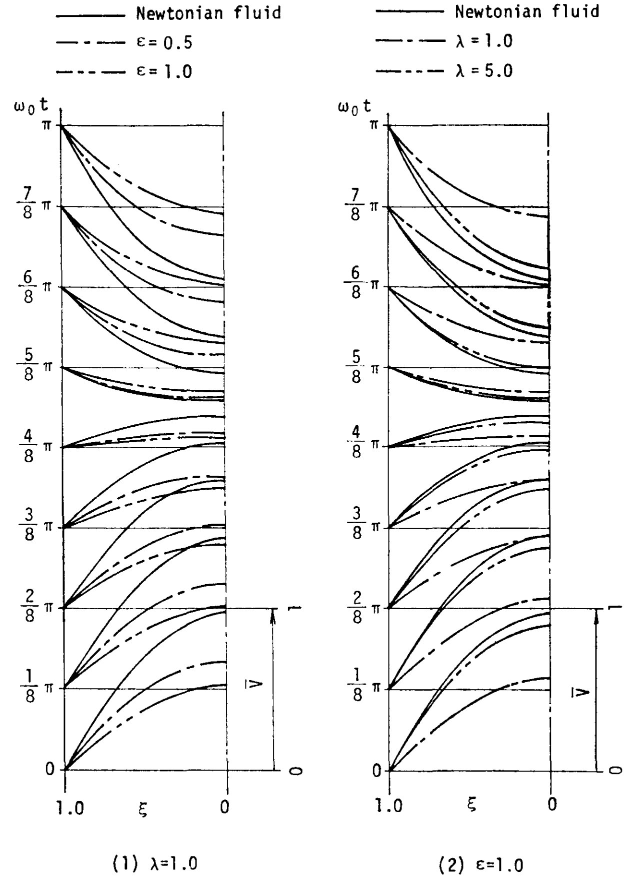


Fig. 2. Velocity profiles (W=1.0)

corresponds to λ. Now $\lambda \to \infty$ corresponds to the situation where the ratio of tube radius to that of a corpuscle is very large. Thus the velocity profile approaches that of Newtonian fluid.

2. Micro-rotation and Vorticity

From Eqs. (8) and (19) the nonzero component of vorticity vector $\boldsymbol{\omega}$ is

$$\omega = \omega_\theta = -\frac{\partial v}{\partial r} \tag{35}$$

Substituting Eq. (32) into Eq. (35) yields

$$\bar{\omega}(\xi) = \frac{\omega(\xi)}{AR/4\mu} = \frac{4}{\phi} \cdot \frac{J_1(\delta)J_1(\phi\xi) + \dfrac{\delta^2\varepsilon}{\lambda^2(1+\varepsilon)}J_1(\phi)J_1(\delta\xi)}{J_1(\delta)J_0(\phi) + \dfrac{\delta\phi\varepsilon}{\lambda^2(1+\varepsilon)}J_0(\delta)J_1(\phi)} \exp(i\omega_0 t) \tag{36}$$

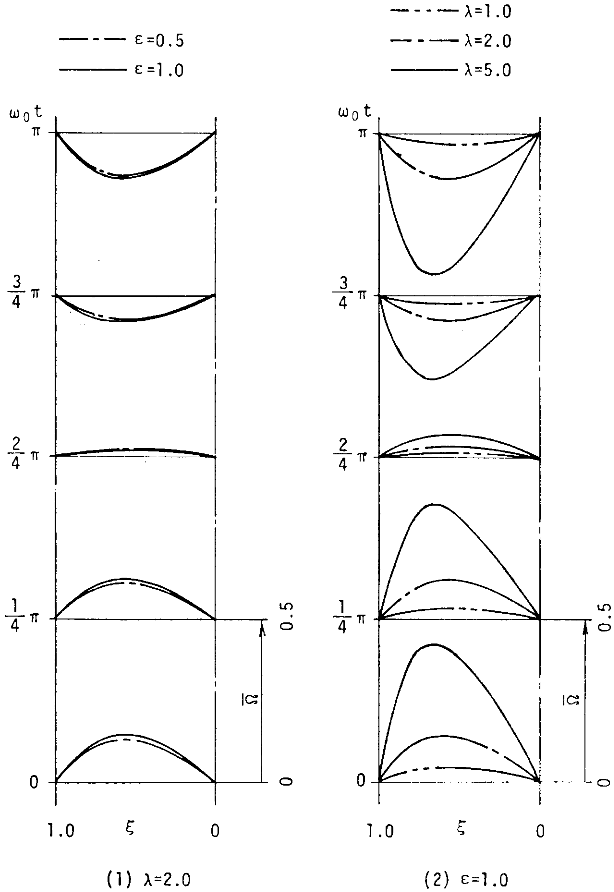


Fig. 3. Distributions of micro-rotation ($W=1.0$)

Distributions of micro-rotation and vorticity at $\pi/4$ intervals over a half cycle are shown in Fig. 3. Comparisons of micro-rotation with one-half of vorticity at $\pi/4$ intervals over a half cycle are shown in Fig. 4. The antisymmetric part of stress tensor mainly consists of the difference of micro-rotation and one-half of vorticity. When the oscillation is extremely slow, this difference becomes

$$\frac{\bar{\omega}(\xi)}{2}-\bar{\Omega}(\xi)=\frac{2}{\phi}\cdot\frac{\left\{\dfrac{1}{1+\varepsilon}+\dfrac{\phi^2\varepsilon}{\lambda^2(1+\varepsilon)^2}\right\}J_1(\phi)J_1(\delta\xi)}{J_0(\phi)J_1(\delta)+\dfrac{\phi\delta\varepsilon}{\lambda^2(1+\varepsilon)}J_0(\delta)J_1(\phi)}\exp(i\omega_0 t)$$
$$\rightarrow\frac{1}{1+\varepsilon}\cdot\frac{I_1(\lambda\xi)}{I_1(\phi)} \tag{37}$$

Then, micro-rofation is smaller than one-half of vorticity in the case of $W\rightarrow 0$. With regard to the phase, there is no difference between the pressure gradient and micro-rotation. For the opposite extreme the amplitude of the micro-rotation diminishes

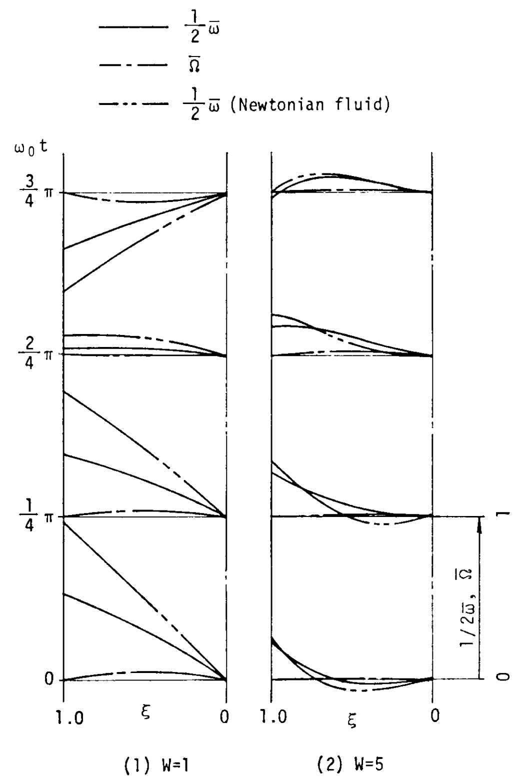


Fig. 4. Comparisons between distributions of micro-rotation and one-half of vorticity ($\lambda=1.0$, $\varepsilon=1.0$)

in proportion to $1/W$ and $\Omega\to 0$ in the limit of $W\to\infty$. There is little significant variation of micro-rotation with ε. For small values of ε, the vorticity becomes that of Newtonian fluid.

3. Shearing Stress

The component of shearing stress T_{rx} will be obtained as follows:

$$\tau = T_{rx} = (\mu+\mu_1)\frac{\partial v}{\partial r} + 2\mu_1\Omega \tag{38}$$

Substituting Eqs. (32) and (33) into Eq. (38) leads

$$\begin{aligned}\bar{\tau}(\xi) &= \frac{\tau(\xi)}{AR/2} \\ &= -\frac{2}{\phi}\cdot\frac{J_1(\delta)J_1(\phi\xi)+\varepsilon\left(1+\frac{\delta^2}{\lambda^2}\right)J_1(\phi)J_1(\delta\xi)}{J_1(\delta)J_0(\phi)+\frac{\delta\phi\varepsilon}{\lambda^2(1+\varepsilon)}J_0(\delta)J_1(\phi)}\exp(i\omega_0 t)\end{aligned} \tag{39}$$

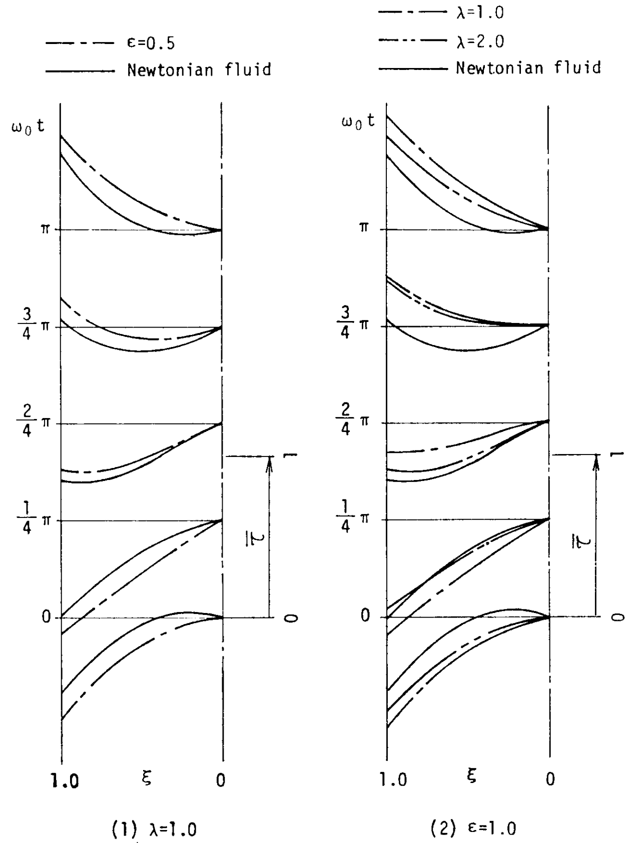


Fig. 5. Distributions of shearing stress ($W=3.0$)

Fig. 5 shows the distributions of shearing stress. The dimensionless wall shearing stress is given by

$$\bar{\tau}_w=\left[-\frac{1+\varepsilon}{2}\bar{\omega}+\varepsilon\bar{\Omega}\right]_{\xi=1} \tag{40}$$

Since we use the no-spin condition, the wall shearing stress is expressed only by the vorticity. In Fig. 6, distributions of the wall shearing stress shown W as a parameter over one cycle.

When the shearing stress is asymptotically expanded for large values of W, Eq. (39) is reduced to

$$\bar{\tau}(\xi)\simeq\frac{2(1+\varepsilon)i^{3/2}}{W}\cdot\frac{\exp\{-(1-\xi)i^{1/2}\}}{\sqrt{\xi}}\exp(i\omega_0 t) \tag{41}$$

The amplitude of the shearing stress is in inverse proportion to W. In the rapid oscillation ($W\rightarrow\infty$), the phase of the shearing stress delays by $\pi/4$ from the phase of the oscillatory pressure gradient because of the first term $i^{3/2}$ in Eq. (41). When

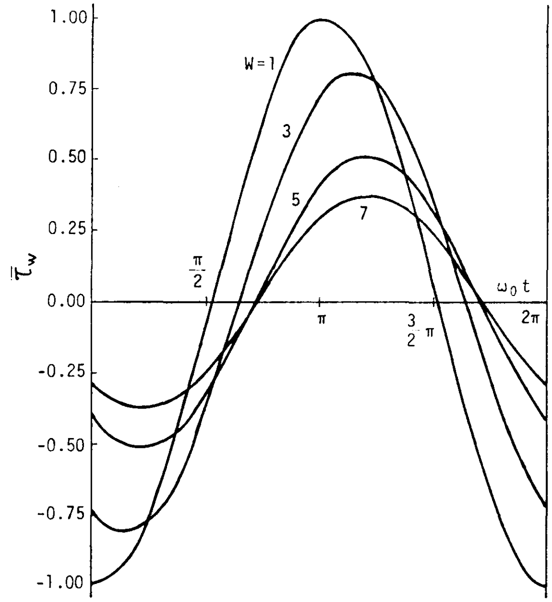


Fig. 6. Variations of distributions of wall shearing stress with W ($\lambda=1.0$, $\varepsilon=1.0$)

$W\rightarrow 0$, the shearing stress is equal to that of Newtonian fluid. This fact means that in the steady flow there is no difference of the shearing stress between polar fluid and Newtonian fluid. If the boundary conditions are different at the wall, it may be seen that some change occurs for the wall shearing stress from Eq. (40). For example, if we use the constant-spin condition which Ω is not zero at the wall, the second term of the right-hand side of Eq. (40) should influence on the wall shearing stress. But there is little variation with the wall shearing stress between no-spin and constant-spin conditions. Details about this phenomenon are discussed in Ref. 14).

4. Flow Rate

The volume flow rate is given by

$$Q(t)=\int_0^R 2\pi r v(r)dr \tag{42}$$

Hence the dimensionless flow rate is

$$\bar{Q}(t)=\frac{Q(t)}{\pi A R^4/8\mu}$$
$$=-\frac{8}{\phi^2}\left[1-\frac{\dfrac{2}{\phi}\left\{1+\dfrac{\phi^2\varepsilon}{\lambda^2(1+\varepsilon)}\right\}J_1(\delta)J_1(\phi)}{J_1(\delta)J_0(\phi)+\dfrac{\delta\phi\varepsilon}{\lambda^2(1+\varepsilon)}J_0(\delta)J_1(\phi)}\right]\exp\,(i\omega_0 t) \tag{43}$$

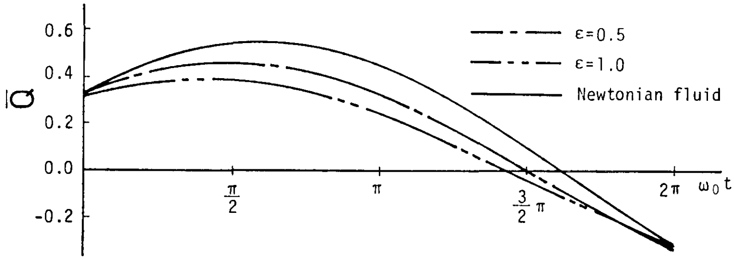


Fig. 7. Variations of flow rate with ε ($W=3.0$, $\lambda=1.0$)

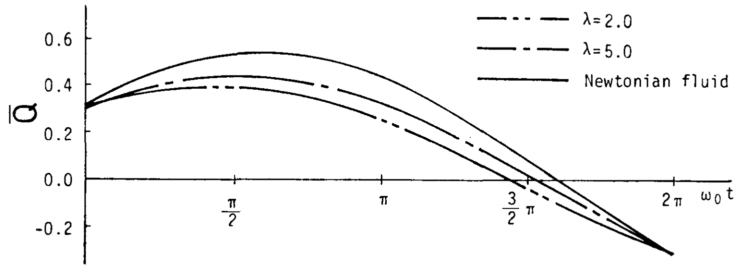


Fig. 8. Variations of flow rate with λ ($W=3.0$, $\varepsilon=1.0$)

Variations of the flow rate with λ and ε are shown in Fig. 7 and Fig. 8. The property of the flow rate is similar to that of the velocity. The influence of λ and ε on Q is almost analogous to that on the velocity profile.

5. Energy Dissipations

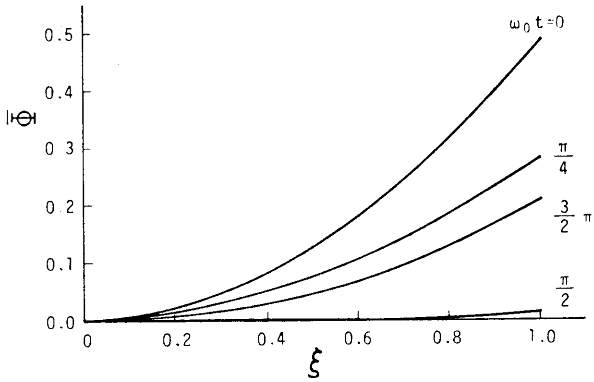
The dissipation of energy per unit volume due to internal friction is given by the dissipation function:

$$\Phi=\boldsymbol{T}^s:\boldsymbol{D}+\boldsymbol{M}:\nabla\boldsymbol{\Omega}+\left(\frac{1}{2}\boldsymbol{\omega}-\boldsymbol{\Omega}\right)\cdot\boldsymbol{e}:\boldsymbol{T}^a \tag{44}$$

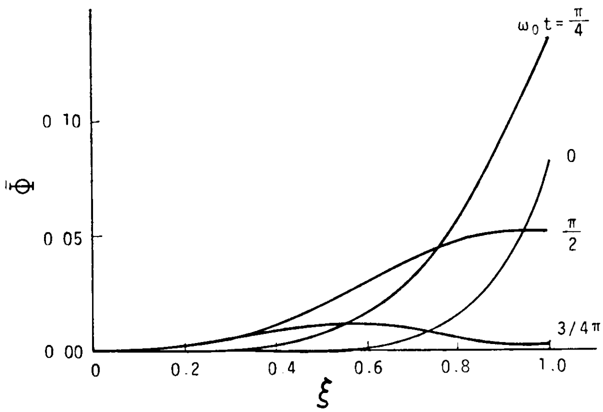
In cylindrical coordinates the dissipation function given by Eq. (44) is written by

$$\Phi=\mu\left(\frac{\partial v}{\partial r}\right)^2+4\mu_1\left(\frac{1}{2}\cdot\frac{\partial v}{\partial r}+\Omega\right)^2-2\beta\left(\frac{\Omega}{r}\cdot\frac{\partial v}{\partial r}\right)$$
$$+r\left\{\left(\frac{\Omega}{r}\right)^2+\left(\frac{\partial\Omega}{\partial r}\right)^2\right\} \tag{45}$$

Both the third and forth terms of the right-hand side of this equation can be neglected since their effects are small in comparison with the first and second terms. Substituting the solutions of the velocity and the micro-rotation into Eq. (45), we obtain



(1) W=1



(2) W=5

Fig. 9. Distributions of energy dissipation ($\lambda=1.0$, $\varepsilon=1.0$)

$$\bar{\Phi}(\xi)=\frac{\Phi(\xi)}{A^2R^2/4\mu}$$

$$=4\left[\mathrm{Re}\left\{-\frac{1}{\phi}\cdot\frac{J_1(\delta)J_1(\phi\xi)+\dfrac{\delta^2\varepsilon}{\lambda^2(1+\varepsilon)}J_1(\phi)J_1(\delta\xi)}{J_1(\delta)J_0(\phi)+\dfrac{\delta\phi\varepsilon}{\lambda^2(1+\varepsilon)}J_0(\delta)J_1(\phi)}\exp(i\omega_0 t)\right\}\right]^2$$

$$+4\varepsilon\left[\mathrm{Re}\left\{\frac{1}{\phi}\cdot\frac{\left\{1+\dfrac{\delta^2\varepsilon}{\lambda^2(1+\varepsilon)}\right\}J_1(\phi)J_1(\delta\xi)}{J_1(\delta)J_0(\phi)+\dfrac{\delta\phi\varepsilon}{\lambda^2(1+\varepsilon)}J_0(\delta)J_1(\phi)}\exp(i\omega_0 t)\right\}\right]^2 \qquad (46)$$

When Womersley number, W, is not large, the maximum of energy dissipations occurs on the wall of the tube. In the rapid oscillation, the maximum of energy dissipations takes place at a location slightly inside the wall, depending on the phase as shown in Fig. 9. The energy dissipations over the cross section are plotted in Fig. 10 and Fig. 11. The analytical expression for energy dissipation over the cross section can not be obtained because of the difficulty of the integration of

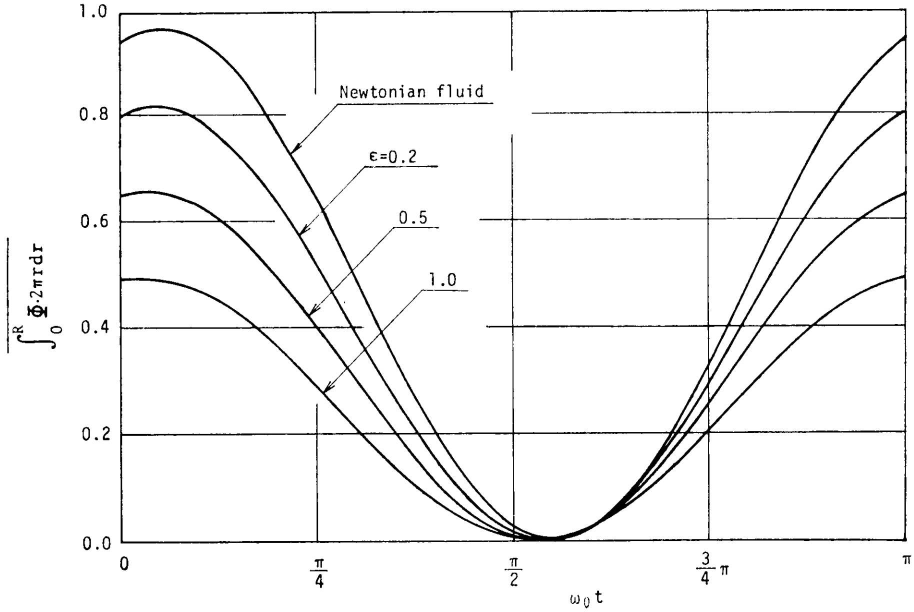


Fig. 10. Variations of energy dissipations over the cross section ($\lambda=1.0$)

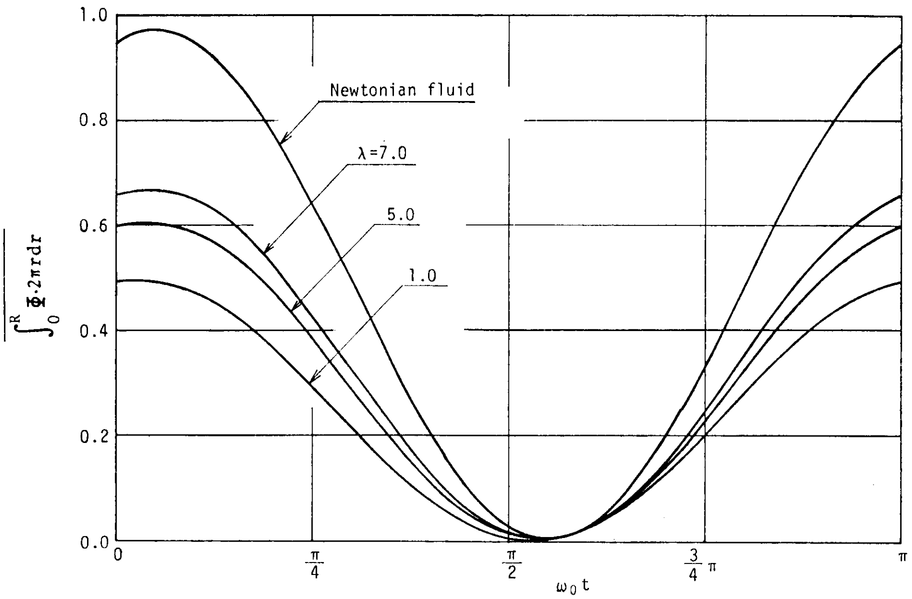


Fig. 11. Variations of energy dissipations over the cross section ($\varepsilon=1.0$)

Bessel functions. These curves are calculated numerically by means of Simpson's rule.

IV. Conclusions

In the present paper, the oscillatory flow of a fluid with couple stress which is called polar fluid have been studied mathematically on the basis of the theory advanced by Eringen. The conclusions are summarized as follows:

(1) Profiles of velocity and micro-rotation, distributions of vorticity and shearing stress, flow rates and energy dissipations are obtained as exact solutions of the first and second Cauchy's equations. These solutions are characterized by three parameters, i.e., the dimensionless frequency W (Womersley number), the size effect parameter λ and the viscosity ratio ε. The latter two parameters present the characteristics of polar fluid.

(2) For high frequencies, the amplitude of velocity becomes small inversely in proportion to the second power of Womersley number, while that of micro-rotation becomes small inversely in proportion to the first power of Womersley number. For low frequencies, the flow of fluid becomes quasi-steady. The phase lag of velocity for pressure diminishes gradually.

(3) A big value of size effect parameter makes the polar fluid get Newtonian, then micro-rotation becomes equal to one-half of vorticity.

(4) When the rate of viscosity becomes small, the polarity of fluid diminishes and micro-rotation reduces to zero.

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