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GEOMETRICAL CONSIDERATION ABOUT A CIRCULAR PIPE OF NON-UNIFORM RADIUS, WITH A TORTUOUS CENTER-LINE

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ABSTRACT

A circular pipe, whose radius is non-uniform, and whose center-line is made up in form of a tortuous space-curve, is taken up. Considering it as a kind of curved surface, we studied its geometrical property, especially its measure of curvatures. Our aim is to give some information about making design and manufacture of such curved pipes, keeping in mind the use of computers for that purposes.

1. Statement of the Problem

It is shown in Fig. 1, a rough sketch of a circular pipe, whose diameter varies along its center-line. The center-line of this pipe is thought to be made up of a tortuous space curve. Considering its middle surface as a kind of curved surface in differential geometry, we wish; (a) to express it in form of an analytical expression suitable for computer use; (b) to examine its geometrical property, especially its measure of curvatures. For this purpose, we consider a spherical surface given by the eq.

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2 \tag{1}$$

where (a, b, c) is the center of sphere, which is thought to travel along a given space-curve C (as shown in Fig. 2). Next we take a plane which passes through the point (a, b, c), and which is perpendicular to tangent line of path C of center (a, b, c), thus;

$$\lambda(x-a)+\mu(y-b)+\nu(z-c)=0. \tag{2}$$

Intersection of (1) and (2) gives us a circular disc of radius R. The locus of

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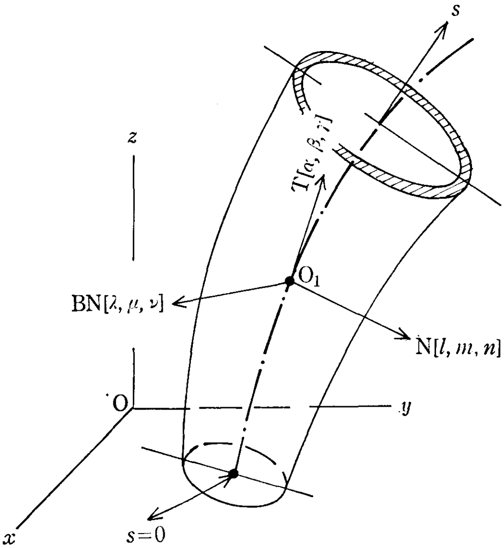

Fig. 1. Sketch of a Circular Pipe of non-uniform Radius, with a tortuous Center-line.

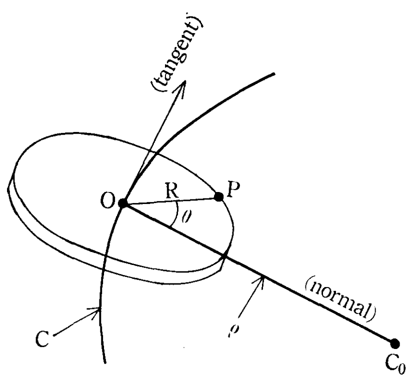

Fig. 2. Circular Disc of Radius R, whose Center O travels along a given Curve C.

center is given in form of eq.

$$a=a(s),\ b=b(s),\ c=c(s) \tag{3}$$

where s is the arc-length of the curve C, as measured from an assigned point on it. The radius R, being thought to vary with s, in a given manner, is regarded to be a function of s, thus;

$$R=R(s).$$

In connection with the curve C, which is the locus of center of circular disc, and being given by the above eq. (3), we have following values of three pairs of direction cosines, namely; tangent (α, β, γ), normal (l, m, n), and binormal (λ, μ, ν). Their values are given by the following formula, in terms of $a(s)$, $b(s)$ and $c(s)$, as is shown in text-books of differential geometry;

$$\alpha=a'(s),\ \beta=b'(s),\ \gamma=c'(s),$$

$$\lambda=\rho(b'c''-c'b''),\ \mu=\rho(c'a''-a'c''),$$

$$\nu=\rho(a'b''-a''b'),$$

$$l=\rho d\alpha/ds,\ m=\rho d\beta/ds,\ n=\rho d\gamma/ds.$$

In these formulae, differentiation ($'$) is to be made with respect to s, the length of curve C, which s we take to be an independent variable. ρ is the radius of curvature of curve C, and is given by,

$$1/\rho^2=(a'')^2+(b'')^2+(c'')^2.$$

Here we notice that, among the three pairs of direction cosines (α, β, γ), (l, m, n) and (λ, μ, ν) there exist following relations.

$$\left.\begin{aligned}&\alpha^2+\beta^2+\gamma^2=1\\&l^2+m^2+n^2=1\\&\lambda^2+\mu^2+\nu^2=1\end{aligned}\right\}\qquad\left.\begin{aligned}&\alpha l+\beta m+\gamma n=0\\&l\lambda+m\mu+n\nu=0\\&\lambda\alpha+\mu\beta+\nu\gamma=0\end{aligned}\right\}$$

$$\left.\begin{aligned}&\alpha=m\nu-n\mu\\&\beta=n\lambda-l\nu\\&\gamma=l\mu-m\lambda\end{aligned}\right\}\qquad\left.\begin{aligned}&\lambda=\beta n-\gamma m\\&\mu=\gamma l-\alpha n\\&\nu=\alpha m-\beta l\end{aligned}\right\}$$

$$\left.\begin{aligned}&l=\mu\gamma-\nu\beta\\&m=\nu\alpha-\lambda\gamma\\&n=\lambda\beta-\mu\alpha\end{aligned}\right\}$$

As to the values of derivatives of direction cosines, α, β, γ, etc., we have by formula of Frenet-Seret;

$$\alpha'=l/\rho,\ \beta'=m/\rho,\ \gamma'=n/\rho,$$

$$l'=-\left(\frac{\alpha}{\rho}+\frac{\lambda}{\tau}\right),\ m'=-\left(\frac{\beta}{\rho}+\frac{\mu}{\tau}\right),$$

$$n'=-\left(\frac{\gamma}{\rho}+\frac{\nu}{\tau}\right),$$

$$\lambda'=l/\tau,\ \mu'=m/\tau,\ \tau'=n/\tau,$$

where $1/\tau$ is the tortuosity of the given curve C. By means of this formula, we obtain following relations;

$$m'\gamma-n'\beta=-l/\tau,\ m'n-n'm=-(\lambda/\rho)+(\alpha/\tau),$$

$$\beta'\gamma-\beta\gamma'=-(\lambda/\rho),$$

$$n\beta'-m\gamma'=0,$$

$$\mu'\nu-\mu\nu'=(\alpha/\tau),$$

$$m'\nu-n'\mu=(l/\rho),$$

$$n\mu'-m\nu'=0.$$

As to second-order derivatives α'', etc., which we require in the following analysis, we obtain, from the above-mentioned formula of Frenet-Seret, thus;

$$\begin{cases} \alpha''=\dfrac{l'}{\rho}-\dfrac{l}{\rho^2}\rho'=-\dfrac{1}{\rho^2}(\alpha+l\rho')-\dfrac{\lambda}{\rho\tau}, \\[2ex] \beta''=-\dfrac{1}{\rho^2}(\beta+m\rho')-\dfrac{\mu}{\rho\tau}, \\[2ex] \gamma''=-\dfrac{1}{\rho^2}(\gamma+n\rho')-\dfrac{\nu}{\rho\tau}, \end{cases}$$

$$\begin{cases} \lambda''=-\dfrac{l'}{\tau'}-\dfrac{l\tau'}{\tau^2}=-\dfrac{1}{\tau^2}(\lambda+l\tau')-\dfrac{\alpha}{\rho\tau}, \\[2ex] \mu''=-\dfrac{1}{\tau^2}(\mu+m\tau')-\dfrac{\beta}{\rho\tau}, \\[2ex] \nu''=-\dfrac{1}{\tau^2}(\nu+n\tau')-\dfrac{\gamma}{\rho\tau}, \end{cases}$$

$$\begin{aligned} l''&=-\left(\frac{\alpha'}{\rho}+\frac{\lambda'}{\rho}\right)+\left(\frac{\alpha\rho'}{\rho^2}+\frac{\lambda\tau'}{\tau^2}\right) \\ &=-\frac{1}{\rho^2}(l-\alpha\rho')-\frac{1}{\tau^2}(l-\lambda\tau'), \end{aligned}$$

$$m''=-\frac{1}{\rho^2}(m-\beta\rho')-\frac{1}{\tau^2}(m-\mu\tau'),$$

$$n''=-\frac{1}{\rho^2}(n-\gamma\rho')-\frac{1}{\tau^2}(n-\nu\tau').$$

The linear element ds along the curve C is given by

$$(ds)^2=(da)^2+(db)^2+(dc)^2,$$

and so we have

$$(a')^2+(b')^2+(c')^2=1$$

2. Analytical Study about geometrical Property of our Curved Surface

Based on these fundamental formulae, we shall study the geometrical property of our curved surface, in what follows.

Coordinates (x, y, z) of any point P, which lies on osculating plane, and also lies on the circle of radius R, may be expressed as follows;

$$x = a + Rl\cos\theta + R\lambda\sin\theta$$
$$y = b + Rm\cos\theta + R\mu\sin\theta$$
$$z = c + Rn\cos\theta + R\nu\sin\theta$$

θ being angular coordinate of point P, which lies on the circumference of the circle. Radius R being a given function of s, we may regard (s, θ) as curvilinear (or, Gaussian) coordinates of the point P, on the surface. Therefore, we have;

$$dx = a'ds + [(lR)'\cos\theta + (\lambda R)'\sin\theta]ds$$
$$+ [-(lR)\sin\theta + (\lambda R)\cos\theta]d\theta$$
$$dy = b'ds + [(mR)'\cos\theta + (\mu R)'\sin\theta]ds$$
$$+ [-mR)\sin\theta + (\mu R)\cos\theta]d\theta$$
$$dz = c'ds + [(nR)'\cos\theta + (\nu R)'\sin\theta]ds$$
$$+ [-(nR)\sin\theta + (\nu R)\cos\theta]d\theta.$$

Hence, the linear element dS of a curve drawn on the surface S is found to be given by;

$$(dS)^2 = (dx)^2 + (dx)^2 + (dz)^2$$
$$= E(ds)^2 + 2F(ds)(d\theta) + G(d\theta)^2. \tag{I}$$

This formula (I) is, what is called the first fundamental expression, in differential geometry, usually expressed in the form of

$$(dS)^2 = E(du)^2 + 2F(du)(dv) + G(dv)^2$$

(in our case, we have $u = s$, $v = \theta$).

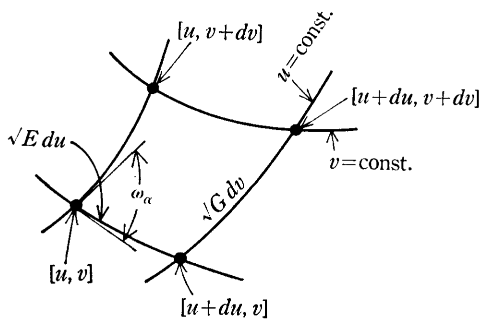

Fig. 3. Curvilinear Coordinates on the Surface ($u = s$, $v = \theta$, for our Case)

The coefficients E, F, and G contained therein have been estimated for the present case, and are found to have following values;

$$E=\left[1-\left(\frac{R}{\rho}\right)\cos\theta\right]^2+\left(\frac{R}{\tau}\right)^2+(R')^2$$

$$F=-\frac{1}{\tau}R^2$$

$$G=R^2.$$

Furthermore, by putting

$$R'=k,\ R/\rho=M,\ R/\tau=N$$

the above formula may be written in simplified form as follows;

$$E=(1-M\cos\theta)^2+N^2+k^2,$$

$$F=-NR,\ G=R^2.$$

Also, we have

$$H^2=GE-F^2=R^2[(1-M\cos\theta)^2+k^2]$$

$$\sin w_\alpha=\frac{\sqrt{EG-F^2}}{\sqrt{EG}}=\frac{H}{\sqrt{EG}}.$$

The second fundamental expression in differential geometry is

$$D_0=D_1(du)^2+2D_2(du)(dv)+D_3(dv)^2 \tag{II}$$

in which we have put

$$D_1=\Sigma X\frac{\partial^2x}{\partial u^2},\ D_2=\Sigma X\frac{\partial^2x}{\partial u\partial v},\ D_3=\Sigma X\frac{\partial^2x}{\partial v^2}$$

with

$$X=\frac{1}{H}\begin{vmatrix}\dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u}\\ \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v}\end{vmatrix}=\frac{D_x}{H}$$

$$Y=\frac{1}{H}\begin{vmatrix}\dfrac{\partial z}{\partial u} & \dfrac{\partial x}{\partial u}\\ \dfrac{\partial z}{\partial v} & \dfrac{\partial x}{\partial v}\end{vmatrix}=\frac{D_y}{H}$$

$$Z=\frac{1}{H}\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u}\\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\frac{D_z}{H}$$

In our case we put $u=s$, $v=0$. The values of D_x, D_y, and D_z are obtained as follows;

$$\frac{\partial x}{\partial u}=a'+[(lR)'\cos\theta+(\lambda R)'\sin\theta]$$

$$\frac{\partial x}{\partial v}=[-(lR)\sin\theta+(\lambda R)\cos\theta]$$

$$\frac{\partial y}{\partial u}=b'+[(mR)'\cos\theta+(\mu R)'\sin\theta]$$

$$\frac{\partial y}{\partial v}=[-(mR)\sin\theta+(\mu R)\cos\theta]$$

$$\frac{\partial z}{\partial u}=c'+[(nR)'\cos\theta+(\nu R)'\sin\theta]$$

$$\frac{\partial z}{\partial v}=[-(nR)\sin\theta+(\nu R)\cos\theta]$$

Using these values, we obtained

$$D_x=-\lambda R\sin\theta-lR\cos\theta+\sin\theta\cos\theta\left(\frac{\lambda}{\rho}\right)R^2$$
$$+\alpha RR'+\cos^2\theta\left(\frac{l}{\rho}\right)R^2$$

$$D_y=-\mu R\sin\theta-mR\cos\theta+\sin\theta\cos\theta\left(\frac{\mu}{\rho}\right)R^2$$
$$+\beta RR'+\cos^2\theta\left(\frac{m}{\rho}\right)R^2$$

$$D_z=-\nu R\sin\theta-nR\cos\theta+\sin\theta\cos\theta\left(\frac{\nu}{\rho}\right)R^2$$
$$+\gamma RR'+\cos^2\theta\left(\frac{n}{\rho}\right)R^2$$

As a next step, we have

$$\frac{\partial^2 x}{\partial u^2}=a''+(lR)''\cos\theta+(\lambda R)''\sin\theta,$$

$$\frac{\partial^2 x}{\partial u\partial v}=-(lR)'\sin\theta+(\lambda R)'\cos\theta,$$

$$\frac{\partial^2 x}{\partial v^2}=-(lR)\cos\theta-(\lambda R)\sin\theta,$$

or,

$$\frac{\partial^2 x}{\partial u^2}=\alpha'+[l''R+2l'R'+lR'']\cos\theta+[\lambda''R+2\lambda'R'+\lambda R'']\sin\theta,$$

$$\frac{\partial^2 x}{\partial u\partial v}=-[l'R+lR']\sin\theta+[\lambda'R+\lambda R']\cos\theta,$$

$$\frac{\partial^2 x}{\partial v^2}=-(lR)\cos\theta-(\lambda R)\sin\theta$$

We obtain similar expression for $\partial^2 y/\partial x^2,\cdots$.

Again, by putting values of $\alpha'', l'', \cdots$, in terms of $\alpha, l, \cdots$, we arrive at the following formula;

$$\begin{aligned}D_1=\cos\theta\Big[&\frac{\rho'}{\rho^2}R^2R'-\frac{2}{\rho}R(R')^2+\frac{R^2}{\rho}R''\\&-\frac{R}{\rho}-\frac{R^3}{\tau^2\rho}\Big]\frac{1}{H}\\&-\sin\theta\Big[\frac{R^2}{\rho\tau}R'\Big]\frac{1}{H}\\&-\cos^2\theta\Big[\frac{R^3}{\rho^3}\Big]\frac{1}{H}\\&-\cos^2\theta\Big[R\Big(\frac{R}{\rho^2}+\frac{R}{\tau^2}-R''\Big)+\Big(\frac{R}{\rho}\Big)^2\Big]\frac{1}{H},\end{aligned}$$

$$D_2=\Big[\frac{R^2}{\rho}R'\sin\theta+\frac{R^3}{\tau\rho}\cos\theta-\frac{R^2}{\tau}\Big]\frac{1}{H},$$

$$D_3=\Big[R^2-\frac{R^3}{\rho}\cos\theta\Big]\frac{1}{H}.$$

Also we have expressions for D_1, D_2 and D_3, in terms of non-dimensional coefficients k, M and N, as shown below;

$$\begin{aligned}D_1=[&(2M^2+N^2-Rk')\cos^2\theta-M^3\cos^3\theta\\&-\{M(k^2+N^2)+R(M'k-Mk')+M\}\cos\theta\\&-MNk\sin\theta]\frac{1}{H},\end{aligned}$$

$$D_2=[kM\sin\theta+MN\cos\theta-N]\frac{R}{H},$$

$$D_3=[1-M\cos\theta]\frac{R^2}{H}$$

with $M=R/\rho<1$, $N=R/\tau<1$, $k=R'$.

Having thus obtained the values of fundamental coefficients, D_1, D_2, and D_3, we can express geometrical properties of our curved surface (giving rise to a form of tortuous circular pipes). Especially, the measure of curvature of them may be given from the known formula, as follows;

$$\frac{\cos \bar{w}}{\rho_n}=\frac{D_1(du)^2+2D_2(du)(dv)+D_3(dv)^2}{E(du)^2+2F(du)(dv)+G(dv)^2}.$$

This value ρ_n is what is called principal radius of normal curvature. As to total (or Gaussian) curvature K_t and mean curvature K_m, we have

$$K_t=\frac{1}{\rho_1\rho_2}=\frac{D_1D_3-(D_2)^2}{H^2}$$

$$K_m=\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{ED_3+GD_1-FD_2}{H^2}.$$

Also, we have for curvatures along the lines $dv=0$ $(d\theta=0)$ and $du=0$ $(ds=0)$ following values

$$\frac{1}{\rho_u}(\text{along the line } dv=0)=\frac{D_1}{E}$$

$$\frac{1}{\rho_v}(\text{along the line } du=0)=\frac{D_3}{G}$$

3. Numerical Examples

In order to show the use of these general formula, we shall give some numerical examples about them. For that purpose, we take up following five cases;

(A) $k=0,\ M=0.50,\ N=0,$

(B) $k=0,\ M=0.25,\ N=0,$

(C) $k=0.20,\ M=0.50,\ N=0,$

(D) $k=0.20,\ M=0.25,\ N=0,$

(E) $k=0.20,\ M=0.25,\ N=0.50.$

The above cases (A) and (B) are most common cases in which there exist no flare, and the center line is bent with curvature of M, but with no tortuosity. The cases (C) and (D) are those cases in which there exist a flare of amount k, and bent in the curvature M but no tortuosity. The last case (E) correspond to the case in which there exist flare k, curvature M, together with the tortuosity of N about its curve of center line. The results of numerical evaluation, for these five cases, are listed as Tables 1 to 6.

Table 1. Value of Curvatures for the Case A. ($k=0$, $M=1/2$, $N=0$)

	$\theta=0$	$\theta=\pm\pi/4$	$\theta=\pm\pi/2$	$\theta=\pm 3\pi/4$	$\theta=\pm\pi$
R/ρ_1	1.000	1.0000	1.0000	1.0000	1.0000
R/ρ_2	−1.0000	−0.54691847	0.0000	0.26120387	0.33333333
R/ρ_s	−1.0000	−0.41789322	0.0000	0.26120387	0.33333333
R/ρ_θ	1.0000	1.0000	1.0000	1.0000	1.0000
R^2K_t	−1.0000	−0.54691900	0.0000	0.26120387	0.33333333
RK_m	0.0000	0.45308256	1.0000	1.26120388	1.33333333

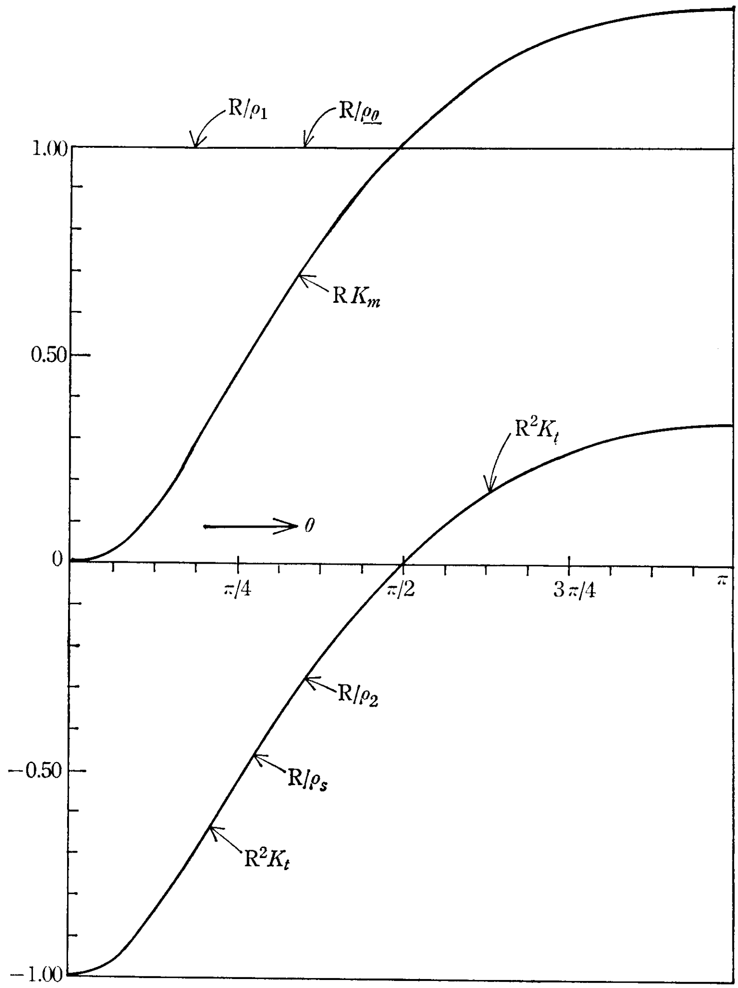

Fig. 4. Value of Curvatures for Case A.

Table 2. Value of Curvatures for the Case B. ($k=0$, $M=1/4$, $N=0$)

	$\theta=0$	$\theta=\pm\pi/4$	$\theta=\pm\pi/2$	$\theta=\pm 3\pi/4$	$\theta=\pm\pi$
R/ρ_1	1.0000	1.0000	1.0000	1.0000	1.0000
R/ρ_2	−0.33333333	−0.21473725	0.0000	0.15022109	0.20000
R/ρ_s	−0.33333333	−0.21473725	0.0000	0.15022109	0.20000
R/ρ_θ	1.0000	1.0000	1.0000	1.0000	1.0000
R^2K_t	−0.33333333	−0.21473724	0.0000	0.15022100	0.20000
RK_m	0.66666667	0.78526275	1.0000	1.15022109	1.20000

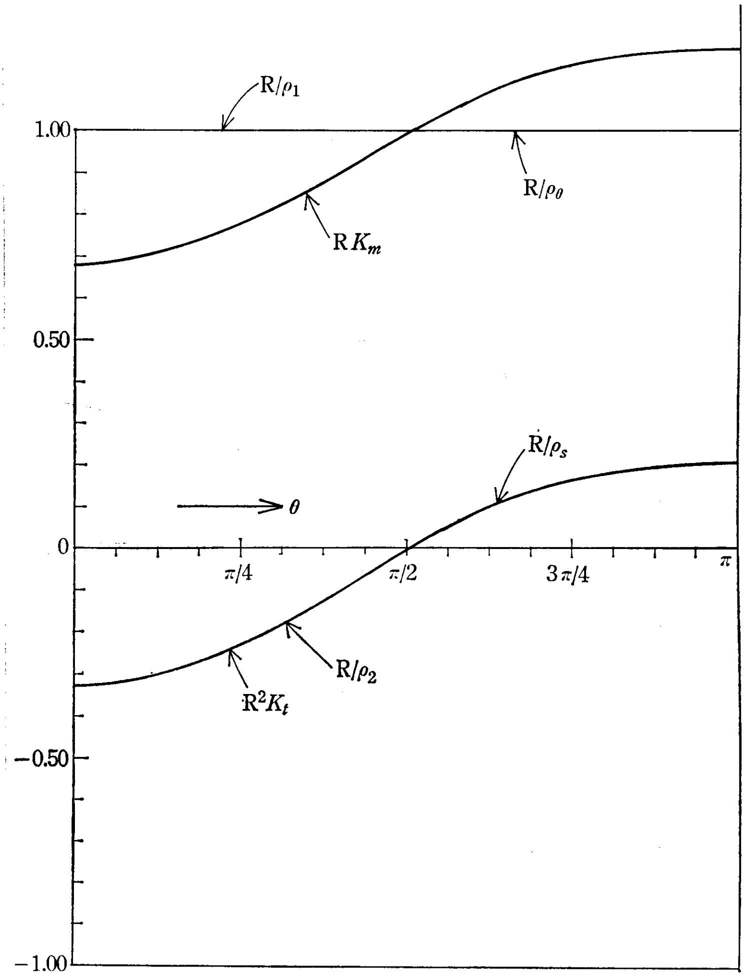

Fig. 5. Value of Curvatures for Case B.

Table 3. Value of Curvatures for the Case C. ($k=0.20$, $M=1/2$, $N=0$)

	$\theta=0$	$\theta=\pm\pi/4$	$\theta=\pm\pi/2$	$\theta=\pm 3\pi/4$	$\theta=\pm\pi$
R/ρ_1	0.92847669	0.97128827	0.98992038	0.99120597	0.99122788
R/ρ_2	−0.92847669	−0.53844846	−0.00933970	0.25645153	0.33040931
R/ρ_s	−0.92847669	−0.52248389	−0.0000	0.25839833	0.33040931
R/ρ_θ	0.92847669	0.95532370	0.98058067	0.98925917	0.99122788
R^2K_t	−0.86206897	−0.52298867	−0.00924556	0.25419629	0.32751092
RK_m	0.0000	0.43283981	0.98058067	1.24765750	1.32163719

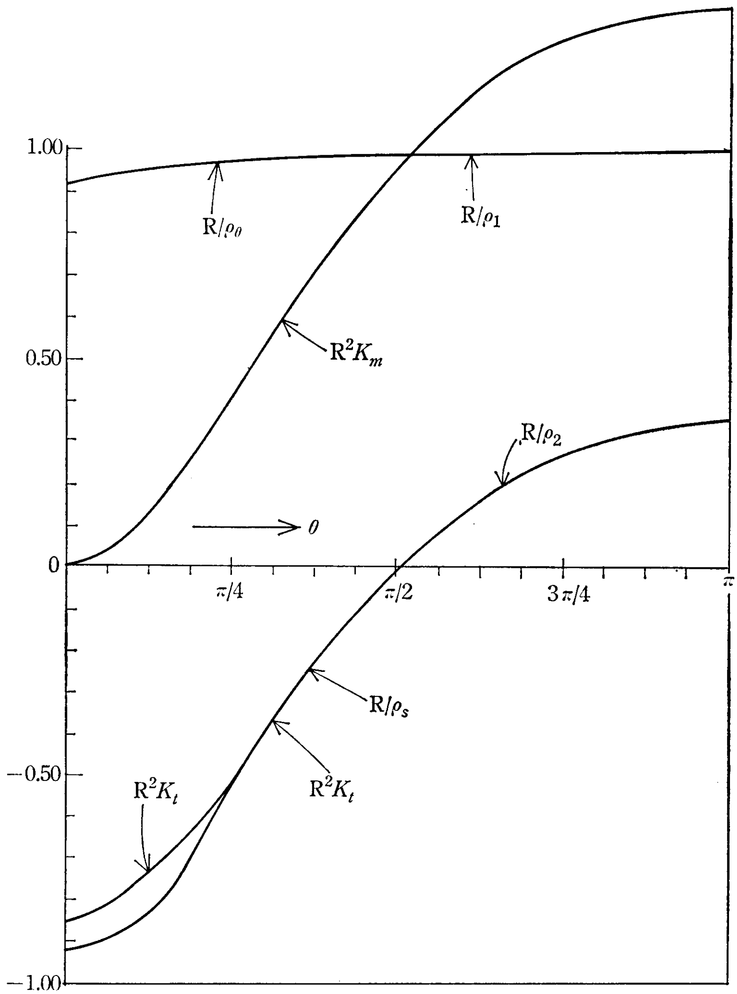

Fig. 6. Value of Curvatures for Case C.

Table 4. Value of Curvatures for the Case D. (k=0.20, M=1/4, N=0)

	$\theta=0$	$\theta=\pm\pi/4$	$\theta=\pm\pi/2$	$\theta=\pm 3\pi/4$	$\theta=\pm\pi$
R/ρ_1	0.96623496	0.97456432	0.98293221	0.98659737	0.98744063
R/ρ_2	−0.32207831	−0.20877899	−0.00235153	0.14736311	0.19748813
R/ρ_s	−0.32207831	−0.20866738	0.0000	0.14809744	0.19748813
R/ρ_θ	0.96623496	0.97173359	0.98058068	0.98586305	0.98744063
R^2K_t	−0.31120332	−0.20611850	−0.00231139	0.14538806	0.19500780
RK_m	0.64415665	0.76306622	0.98058069	1.13396048	1.18492876

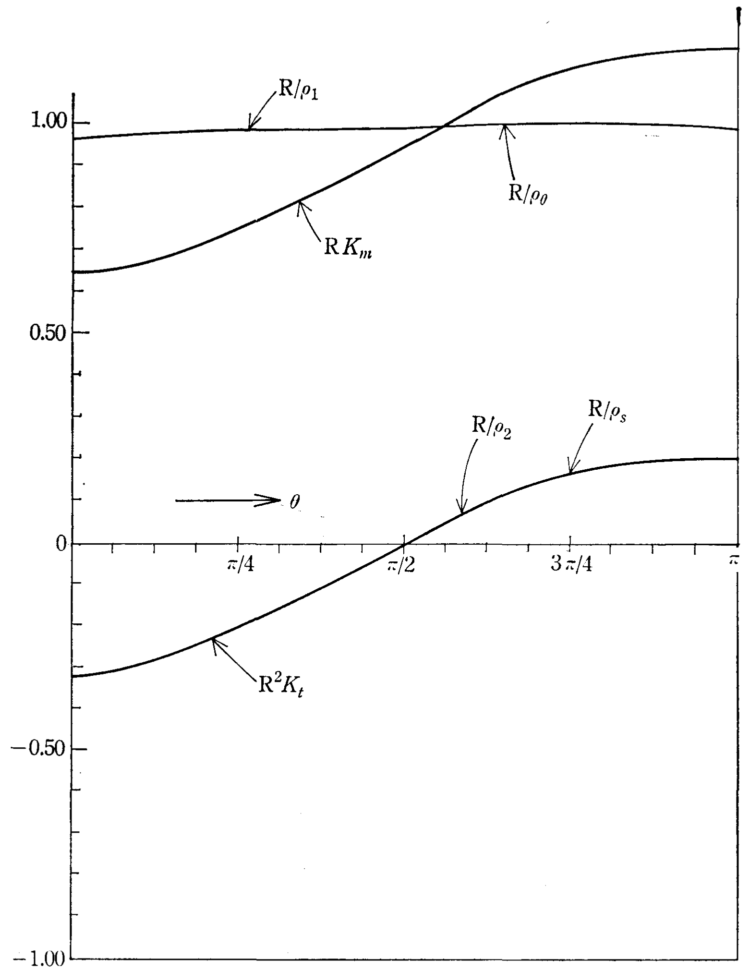

Fig. 7. Value of Curvatures for Case D.

Table 5. Value of Curvatures for the Case E. ($k=0.20$, $M=1/4$, $N=1/2$)

	$\theta=0$	$\theta=+\pi/4$	$\theta=+\pi/2$	$\theta=+3\pi/4$	$\theta=+\pi$
R/ρ_1	0.96623493	0.97351970	0.98251470	0.98750925	0.98744063
R/ρ_2	−0.64415659	−0.38696730	−0.21454569	0.08340864	0.19748813
R/ρ_s	−0.17190103	−0.07775508	−0.01937984	0.20183426	0.34409440
R/ρ_θ	0.96623475	0.97173359	0.98558068	0.98586305	0.98744063
R^2K_t	−0.62240660	−0.37672029	−0.21033654	0.0823668	0.19500780
RK_m	0.32207834	0.58655240	0.76843502	1.07091789	1.18492875

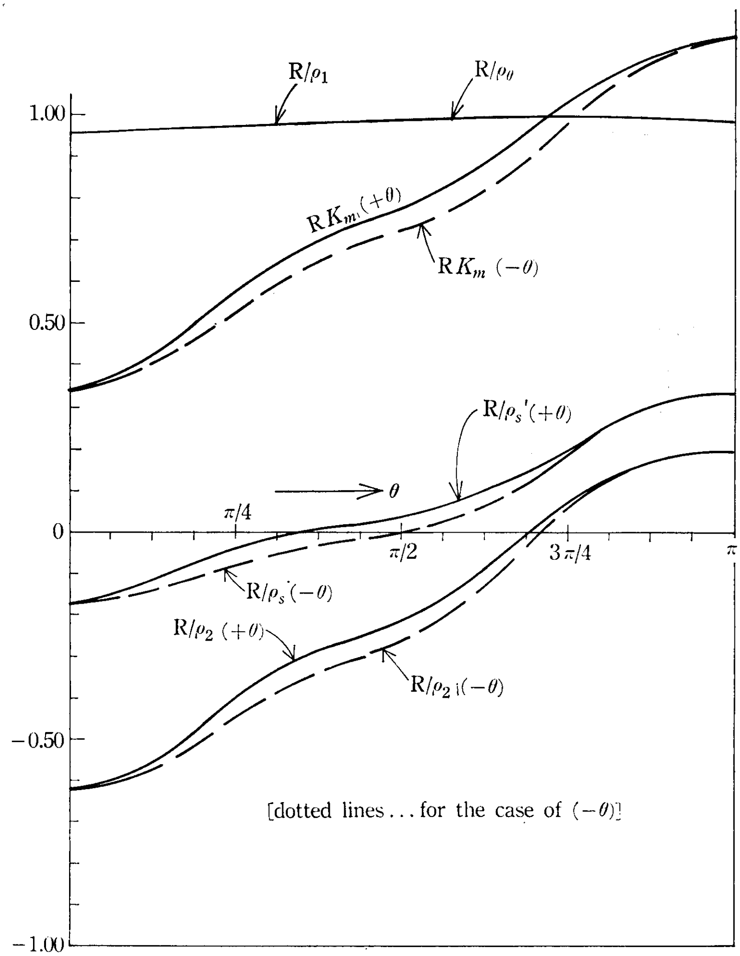

Fig. 8. Value of Curvrtures for Case E.

Table 6. Value of Curvatures for the Case E. ($k=0.20$, $M=1/4$, $N=1/2$)

	$\theta=-\pi/4$	$\theta=-\pi/2$	$\theta=-3\pi/4$
R/ρ_1	0.97344648	0.98244281	0.98653064
R/ρ_2	−0.44504328	−0.26068347	0.06353728
R/ρ_s	−0.03462847	0.01937984	0.21946728
R/ρ_θ	0.97173359	0.98058068	0.98586305
R^2K_t	−0.43322581	−0.25610660	−0.06268147
RK_m	0.52840320	0.72175934	1.05006792

Also, there are shown in Fig. 1 to 8, graphs of these numerical values. It is to be noted that some curves in these graphs lie so closely each other that they cannot be easily distinguishable.

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