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CLASSIFICATION OF THE GENERALIZED HYPERGEOMETRIC FAMILY OF DISTRIBUTIONS

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SUMMARY

Univariate and Multivariate Generalized Hypergeometric distributions are formally defined, and under some conventional rules all possible cases are found and classified.

Univariate distributions on the interval $[0, n]$ or $[0, \infty)$ are clearly defined and classified into five types: A1 and A2 on finite intervals and B1, B2 and B3 on infinite intervals. On the other intervals, shifts or inversions of the distributions of the basic five types are essentially possible.

Bivariate Generalized Hypergeometric distributions are possible on the non-negative or the negative quadrant. Possible types are rather limited irrespective of the general setup. The discussions on the Bivariate distributions cover classification of the Multivariate distributions.

Singular Generalized Hypergeometric distributions are classified in the course of the discussions. Geneses of main Bivariate Generalized Hypergeometric distributions are summarized.

1. Introduction

Scope of the problem

A definition of a Univariate Generalized Hypergeometric (GHg, for short) distribution is a distribution whose probability generating function (or characteristic function) is a hypergeometric function multiplied by a normalization factor;

(1.1) $$F(\alpha, \beta; \gamma; \theta)^{-1}F(\alpha, \beta; \gamma; \theta s),$$

where F is the sum of Gauss hypergeometric series, that is,

(1.2) $$F(\alpha, \beta; \gamma; z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}z^n, \text{ where } (\alpha)_n=\alpha(\alpha+1)\dots(\alpha+n-1), \text{ and } (\alpha)_0=1.$$

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The hypergeometric series can be extended to a ${}_pF_q$ series with p numerator parameters and q denominator parameters, then the GHg family covers a very wide range of distributions. See Kemp (1968), Dacey (1972) and Mathai and Saxena (1973) for studies in this broader scope.

There are two statistical approaches to the GHg family. In one of them, θ is an unknown parameter of main concern. If the other parameters are known, the distribution is a power series distribution (see e.g. Johnson and Kotz (1969)). In the other approach, $\theta=1$ and one or more of the other parameters are unkown. In this paper, we are concerned with the latter case and only the ${}_2F_1$ case and its extension to other directions. That is, we examine distributions with probabilities

$$(1.3)\qquad p(x)=\frac{1}{F(\alpha,\beta;\gamma;1)}\frac{(\alpha)_x(\beta)_x}{(\gamma)_x x!},\quad x=0,1,2,\dots$$

and extend these to distributions on the other domains and to multivariate distributions.

Recall that the Ordinary Hypergeometric (Hg) distribution has probabilities

$$(1.4)\qquad p(x)=\binom{M}{x}\binom{N}{n-x}\Big/\binom{M+N}{n}=\binom{n}{x}\binom{M+N-n}{M-x}\Big/\binom{M+N}{M}$$
$$=\frac{(M+N-n)!\,N!}{(M+N)!\,(N-n)!}\frac{(-M)_x(-n)_x}{(N-n+1)_x x!}$$

One can define GHg distributions by extending binomial coefficient and allowing M, N and n to be real numbers. We prefer the expression (1.3) to (1.4) mainly because the parameters α and β appear symmetrically in (1.3) and this makes classification simpler.

Construction of the paper

In Section 2, we state Shimizu's classification (1968) of (1.3) as distributions on $[0,n]$ or $[0,\infty)$. As shown in Table 1, there are five types of distributions; A1 and A2 on $[0,n]$ and B1, B2 and B3 on $[0,\infty)$. Since the Ordinary Hg (1.4) can be a distribution on positive integers only, this restriction on the range is too severe. To cover even negative intervals, a GHg is redefined by the expressions (3.2) or (3.3) and Convention 3 is assumed. Examining all sossible cases, Theorem 2 states that seven types of distributions in Table 2 are all possible additional distributions. Except for Type C, a very special type, others are shifted or inverted distributions of Types in Table 1, that is, right shift or inversion of A1, or right shift of B1 and B3 on positive intervals; left shift or inversion of A2, or inversion of B3 on negative intervals. Types of distributions in Table 2 are further examined in Section 4 by showing some related distributions outside the table. Possible Singular Bivariate GHg distributions are shown in Table 3 for better understanding of Table 2 and for later discussions.

Bivariate GHg distributions on the nonnegative quadrant are defined by the expression (5.2) or (5.3) and Convention 4. Unexpectedly, there are only several types of distributions since marginal and some related univariate distributions must be GHg as summarized in Table 5. Theorem 4 states that the types of distributions in

Table 4 are all possible ones. Bivariate GHg distributions on the other quadrants are also defined by (5.5) or (5.6). There are two types of distributions on the negative quadrant, as shown in Table 6, which are obtained by inverting two types in Table 4. The types in Tables 4 and 6 are valid for Multivariate GHg distributions if the number of parameters is increased. Singlar Multivariate GHg distributions are shown in Table 7. In the final Section 6, geneses of Bivariate GHg distributions are discussed. These geneses cover essentially univariate and multivariate distributions.

An expository summary of this report will be published separately (Sibuya and Shimizu (1981)).

Previous works

Generalization of the Hg distributions has a long history. It goes back, for example, to Karl Pearson's early work (1895). Studies on moments (Pearson (1924a)), Romanovsky (1925), Ayyangar (1934)), approximation by Pearson curves (Camp (1925), Davies (1934)) and bivariate extension (Pearson (1924b)) followed it. Some unifying efforts appeared more recently. See, for example, Ord (1967a, 1967b) for univariate case, Steyn (1951, 1955), Janardan and Patil (1972), and Janardan (1973) for multivariate case, and Kemp (1968) and Dacey (1972) for a wider scope study.

There have been, however, less efforts to define what GHg distributions are, and to classify them. As the results, descriptions on GHg distributions in statistical literature are often vague, and duplicated works on the distributions are observed.

Classification of Univariate GHg distributions was studied by Davies (1934) and Kemp and Kemp (1956) (the latter result is stated in Johnson and Kotz (1969)), Kemp and Kemp's classification did not cover completely distributions on general intervals and was criticized by Sarkadi (1957) and Shimizu (1968). Shimizu (1968) solved completely classification of Univariate GHg on intervals $[0, n]$ and $[0, \infty)$, indicated a way to examine distributions on the other intervals and pointed out difficulties in studying broader class of GHg distributions, but failed to restrict GHg distributions within a reasonable limit.

Janardan and Patil (1972) covered practically important Multivariate GHg distributions but they did not try to exhaust all possible cases.

2. Univariate Distributions on $[0, n]$ or $[0, \infty)$

Conventions and classification

Let $\mathrm{F}(\alpha, \beta; \gamma)$ denote a GHg distribution (1.3). To cover some distributions on finite interval, we assume the follwing convention.

Convention 1. Let m and n be integers such that $m \geqq n > 0$. We define in (1.3)

$$F(\alpha, -n;\ -m;\ 1) = \sum_{x=0}^{n} \frac{(\alpha)_x(-n)_x}{(-m)_x x!}, \tag{2.1}$$

and regard $\mathrm{F}(\alpha, -n;\ -m)$ a distribution on $[0, n]$. Notice that this distinguishes

F(α, $-n$; $-n$), F(α, -1 ; -1) and F(α, 1 ; 1).

By Gauss theorem, the factor in (1.3) is defined if $\gamma>\alpha+\beta$ and written as

$$\frac{1}{F(\alpha, \beta ; \gamma ; 1)}=\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)},$$

and this expression holds in the case of Convention 1 if another convention is applied;

Convention 2. Let m and n be nonnegative integers. We define formally either

$$\Gamma(-n)=(-1)^n/0.1\ldots n, \tag{2.2}$$

or

$$\Gamma(-n)=(-1)^n\Gamma(0)/n, \tag{2.3}$$

and if a pair of 0's or $\Gamma(0)$'s appears in numerator and denominator of an expression, then we cancel it. In other words,

$$\frac{\Gamma(-n)}{\Gamma(-m)}=\lim_{\varepsilon\to 0}\frac{\Gamma(-n+\varepsilon)}{\Gamma(-m+\varepsilon)}=(-1)^{n-m}\frac{m!}{n!}. \tag{2.4}$$

The value of $1/\Gamma(-n)$ is zero by (2.2) but undefined by (2.3). A suitable one of them can be used in the following discussions.

Theorem 1. (Shimizu (1968), Kemp (1968))

Under Conventions 1 and 2 a GHg distribution F($\alpha, \beta ; \gamma$) with probabilities

$$p(x)=\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}\frac{(\alpha)_x(\beta)_x}{(\gamma)_x x!}, \quad x=0,1,2,\ldots \tag{2.5}$$

can be a probability distribution on $[0, n]$ or $[0, \infty)$, not degenerated at $x=0$, if and only if F($\alpha, \beta ; \gamma$) (or F($\beta, \alpha ; \gamma$) by symmetry) belongs to one of the five types of Table 1.

Proof. Since $p(0)$ and $p(1)$ are positive, $(\alpha)_x(\beta)_x/(\gamma)_x>0$, $x=1,2,\ldots$ This is possible if all of $\alpha+x-1$, $\beta+x-1$ and $\gamma+x-1$ are always positive (Type B3), two of them are always nonpositive and the third is always positive (Types A1 and A2), or two of them change sign from negative to positive at the same point and the third is always positive (Types B1 and B2).

Remarks on Table 1.

We mention briefly some facts on the distributions in Table 1. In Section 6, geneses of Types A1, A2 and B3 are discussed.

Table 1. Classification of the GHg distributions $F(\alpha, \beta; \gamma)$ defined by (2.5) on $[0, n]$ or $[0, \infty)$

range	type	distribution $F(\alpha, \beta: \gamma)$	restriction	name
$[0, n]$	A1	$F(-\xi, -n; \zeta)$	$\xi>n-1$	Positive Hg
	A2	$F(\xi, -n; -\zeta)$	$\zeta>n-1$	Negative Hg, Markov-Pólya, Pólya-Eggenberger, binomial beta
$[0, \infty)$	B1	$F(-n+\varepsilon, -n+\delta; \zeta)$	—	—
	B2	$F(\varepsilon, -n+\delta; -n+\rho)$	$\rho>\varepsilon+\delta$	—
	B3	$F(\xi, \eta; \zeta)$	$\zeta>\xi+\eta$	Inverse Markov-Pólya, inverse Pólya-Eggenberger, generalized Waring, negative binomial bete

n: positive integer,
ξ, η, ζ: positive real numbers,
$\varepsilon, \delta, \rho$: real numbers on the open interval (0, 1).

Type A1

$$(2.6)\quad \frac{\Gamma(\xi+\zeta)\Gamma(\zeta+n)}{\Gamma(\zeta)\Gamma(\xi+\zeta+n)}\frac{(-\xi)_x(-n)_x}{(\zeta)_x x!}=\binom{a}{x}\binom{b}{n-x}\Big/\binom{a+b}{n}$$
$$=\binom{n}{x}\binom{a+b-n}{a-x}\Big/\binom{a+b}{a}=\binom{n}{x}\frac{a^{(x)}b^{(n-x)}}{(a+b)^{(n)}},$$
$$x=0,1,\ldots,n,\quad a^{(x)}=a(a-1)\ldots(a-x+1),$$

where $a=\xi$, $b=\zeta+n-1$ and $a, b>n-1\geqq 0$. This is a Positive Hg distribution, and when ξ and ζ are positive integers this is an Ordinary Hg distribution (1.4).

Type A2

$$(2.7)\quad \frac{\Gamma(-\xi-\zeta)\Gamma(n-\zeta)}{\Gamma(-\zeta)\Gamma(n-\xi-\zeta)}\frac{(\xi)_x(-n)_x}{(-\zeta)_x x!}=\binom{-a}{x}\binom{-b}{n-x}\Big/\binom{-a-b}{n}$$
$$=\binom{a+x-1}{x}\binom{b+n-x-1}{n-x}\Big/\binom{a+b+n-1}{n}$$
$$=\binom{n}{x}\frac{(a)_x(b)_{n-x}}{(a+b)_n},\quad x=0,1,\ldots,n,$$

where $a=\xi$, $b=\zeta-n+1$ and $a, b>0$. This is a Negative Hg distribution, which is also called Markov-Pólya, Pólya-Eggenberger, and so on. The case $b=1$ is included by Convention 1.

Type B1

This has unimodal or monotone decreasing probabilities. Its mean is greater than variance if $n\geqq 2$.

Type B2

The distribution can be unimodal or bimodal and has not finite mean.

Type B3

$$\frac{\Gamma(\zeta-\xi)\Gamma(\zeta-\eta)}{\Gamma(\zeta-\xi-\eta)\Gamma(\zeta)}\frac{(\xi)_x(\eta)_x}{(\zeta)_x x!}=\frac{\Gamma(a+c)\Gamma(b+c)}{\Gamma(c)\Gamma(a+b+c)}\frac{(a)_x(b)_x}{(a+b+c)_x x!}, \tag{2.8}$$

where $a=\xi$, $b=\eta$, $c=\zeta-\xi-\eta$ and $a, b, c>0$. This is called inverse Markov-Pólya, inverse Pólya-Eggenberger, and so on. (See Sibuya (1979) for further discussions of this distribution.)

The following table compares the Table 1 with Kemp and Kemp's classification (1956).

A1	IA (i), (ii)
A2	IIA, IIIA
B1	IB
B2	IIB, IIIB
B3	IV

Their classifiation was based on the binomial coefficient expressions, so the conditions on parameter values were complicated and exhaustiveness of them was not clear. They excluded $F(\alpha, -n; -n)$ of Type A2, the case where $N=0$ and $M>0$ in (1.4), or the case where $b=1$ in (2.7). We include it applying Conventions 1 and 2. In the last expression of (2.7), there is no reason to exclude $b=1$.

Ordinary Hg distribution (1.4) has a positive probability if $\max(0, n-N)\leqq x\leqq \min(M, n)$. That is, it can be a distribution on intervals not including 0, which is not covered by Table 1. It is natural to ask, therefore, what the GHg distuributions on intervals other than $[0, n]$ and $[0, \infty)$ are. Sarkadi (1957) remarked this fact, criticizing Kemp and Kemp's classification, but his comment was not complete. Shimizu (1968) discussed what GHg distributions will be obtained by shifting (i.e. $X\pm p$) a variable X of Table 1. The next Section 3 treats the problem more completely.

Because of its various models and expressions Type A2 Hg, or Negative Hg, got many other names. Since this is obtained by Pólya's contagious urn model which was actually studied by Markov, Eggenberger and Pólya, it is called Markov-Pólya, Pólya, or Pólya-Eggenberger distribution. This is also called binomial beta since this is a compound binomial distribution when its probability parameter is a beta variable. Inverse Hg is another name used by some authors.

Type B3 Hg, or inverse Markov-Pólya, is obtained by inverse sampling (or waiting time) in Pólya's contagious urn model, and inverse Pólya or inverse Pólya-

Eggenberger are used. It is also called negative binomial beta since this is a compound negative binomial distribution when its probability parameter is a beta variable. Another name is generalized Waring since its probabilities are generalization of terms of Waring inverse factorial series.

3. Univariate Distributions on the Other Intervals

Definition of extended probability distributions

Let us study again Ordinary Hg (1.4) with $n>N$. The second expression of (1.4) is positive if $n-N\leqq x\leqq \min(M,n)$. But there is a factor $(-n+N+1)_x$ in denominator of the last expession, which vanishes if $n-N\leqq x$, and there is another factor $(-n+N)!$ which is undefined even applying Convention 1 if $M>n-N$. To cover this case and to define GHg on negative integers, we have to redefine the function form of GHg family of distributions. For this purpose, we extend the factorial function which is expressed as

$$(3.1)\qquad (a)_x=a(a+1)\dots(a+x-1)=\frac{\Gamma(a+x)}{\Gamma(a)}=(-1)^x\frac{\Gamma(1-a)}{\Gamma(1-a-x)},$$

at least for a nonnegative integer x and a noninteger a. Extending the domain of x to all integers and that of a to all real numbers in (3.1), we notice the following facts.

(i) When $a=n\geqq 1$ is a positive integer

$$(n)_x=\begin{cases}\Gamma(n+x)/\Gamma(n)=(n+x-1)!/(n-1)!, & x\geqq -n+1,\\ (-1)^x\Gamma(1-n)/\Gamma(1-n-x), & x\leqq -n,\end{cases}$$

where $\Gamma(1-n)$ is undefined.

(ii) When $a=-n\leqq 0$ is a nonpositive integer

$$(-n)_x=\begin{cases}\Gamma(x-n)/\Gamma(-n), & x\geqq n+1,\\ (-1)^x\Gamma(1+n)/\Gamma(1+n-x), & x\leqq n,\end{cases}$$

where $1/\Gamma(-n)$ is undefied or zero.

(iii) When a is not an integer, the sign of $\Gamma(a+x)/\Gamma(a)$ changes for neighboring integer values of x in $a+x<0$.

Remark that we obtain the last equality of (3.1) for negative integer arguments invoking Convention 2, and that if a is an integer, then $(a)_x$ is definable either for $x>-a$ or for $x\leqq -a$.

Using the first gamma expression of (3.1), we rewrite (2.5) as

$$(3.2)\qquad p(x)=\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha-\beta)\Gamma(\alpha)\Gamma(\beta)}\,\frac{\Gamma(x+\alpha)\Gamma(x+\beta)}{\Gamma(x+1)\Gamma(x+\gamma)},$$

or using also the second gamma expression of (3.1),

$$(3.3)\qquad p(x)=\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(\gamma-\alpha-\beta)\Gamma(x+1)\Gamma(x+\gamma)\Gamma(1-\alpha-x)\Gamma(1-\beta-x)},$$

The expression (3.3) is obtained if we formally rewrite (2.5) in binomial coefficient form;

$$(3.4)\qquad \binom{-\alpha}{x}\binom{\gamma-\beta-1}{-\beta-x}\Big/\binom{\gamma-\alpha-\beta-1}{-\beta}$$

and define

$$\binom{a}{b}=\Gamma(a+1)/\Gamma(b+1)\Gamma(a-b+1)\,.$$

Two expressions, (3.2) and (3.3), are strangely different by factor (-1) for some special parameter values. For example, put $x=-m$ in $\mathrm{F}(\xi, m;\ n+1)$ (see Table 2) of (3.2) to obtain

$$\frac{\Gamma(n+1-\xi)\Gamma(n+1-m)\Gamma(-m+\xi)\Gamma(0)}{\Gamma(n+1-\xi-m)\Gamma(\xi)\Gamma(m)\Gamma(-m+1)\Gamma(n+1-m)}=(-1)^{m+1}\frac{(n-\xi)^{(m)}}{(\xi-1)^{(m)}}=-\frac{(n-\xi)^{(m)}}{(m-\xi)^{(m)}}\,.$$

In $\mathrm{F}(\xi, m;\ n+1)$ of (3.3), however, we obtain

$$\frac{\Gamma(n+1-\xi)\Gamma(n+1-m)\Gamma(1-\xi)\Gamma(1-m)}{\Gamma(n+1-\xi-m)\Gamma(1-m)\Gamma(n+1-m)\Gamma(1-\xi+m)\Gamma(1)}=\frac{(n-\xi)^{(m)}}{(m-\xi)^{(m)}}\,.$$

To show the reason, compare factors in these expressions; starting from the latter

$$\frac{\Gamma(1-m)}{\Gamma(1-m)\Gamma(1)}=(-1)^{m}\frac{\Gamma(0)}{\Gamma(1-m)\Gamma(m)}=-\frac{\Gamma(m)}{\Gamma(1)\Gamma(m)}\,.$$

That is, if gamma functions with nonpositive integer argument in the factor $\Gamma(x+\alpha)\Gamma(x+\beta)/\Gamma(\alpha)\Gamma(\beta)=\Gamma(1-\alpha)\Gamma(1-\beta)/\Gamma(1-\alpha-x)\Gamma(1-\beta-x)$ become definable by applying Convention 2 within this factor, then (3.3) is equal to (3.2). But applying Convention 2 twice to different pairs with a common gamma function, we get factor (-1).

In classification work, (3.2) is more convenient since the gamma functions depending on x appear evenly in both its numerator and denominator while (3.3) will give wider range of distributions as will be seen later.

Convention and classification

Anyhow, we write the distribution defined by (3.2) or (3.3) as $\mathrm{F}(\alpha, \beta;\ \gamma)$, and introduce one more convetion to classify it.

Convention 3. (i) The probability function $p(x)$ is defined on a finite or infinite integer interval which has at least two integers.
(ii) The function $p(x)$ is zero or undefined on the neighboring outside point(s) of the interval. This means that $\mathrm{F}(\alpha, \beta;\ \gamma)$ has an integer parameter or parameters corresponding to the end point(s) of its interval, and some gamma functions are definable (possibly by invoking Convention 2) inside the interval but not outside. Corresponding to the factor $1/x!=1/\Gamma(x+1)$, the value 1 should be regarded as a built-in constant parameter.
(iii) If there are two parameters of the same integer value (including built-in 1),

Table 2. Classification of the Generalized Hypergeometric distributions $F(\alpha, \beta; \gamma)$ defined by (3.2) or (3.3) on intervals not containg 0

type	range	distribution $F(\alpha, \beta; \gamma)$ and restriction	relation with Table 1
$A1^+$	$[m, n]$	$F(-\xi, -n; -m+1)$ $\xi > n-1$	right m shift of A1: $F(-\xi+m, -n+m; 1+m)$ on $[0, n-m]$ or inversion $(Y=n-X)$ of A1: $F(-n, -n+m; \xi-n+1)$ on $[0, n-m]$
$B1^+$	$[m, \infty)$	$F(-m-n+\delta, -m-n+\varepsilon; -m+1)$	right m shift of B1: $F(-n+\delta, -n+\varepsilon; 1+m)$ on $[0, \infty)$
$B3^+$	$[m, \infty)$	$F(\xi-m, \eta-m; -m+1)$ $m+1>\xi+\eta$	right m shift of B3: $F(\xi, \eta; 1+m)$ on $[0, \infty)$
$A2^-$	$[-n, -m]$*	$F(\xi, m; n+1)$ $\xi > n$	left n shift of A2: $F(\xi-n, -n+m; -n+1)$ on $[0, n-m]$ or inversion $(Y=-X-m)$ of A2: $F(m, -n+m; -\xi+m+1)$ on $[0, n-m]$
$B3^-$	$(-\infty, -m]$*	$F(-\zeta+m+1, m; m+1-\xi)$ $\zeta > \xi+m$	inversion $(Y=-X-m)$ of B3: $F(\xi, m; \zeta)$ on $[0, \infty)$
	$(-\infty, -m]$	$F(1, 1; m)$ $m=3, 5, \ldots$	inversion $(Y=-X-m)$ of B3: $F(1, 1; m)$ on $[0, \infty)$
C	$[m, \infty)$	$F(\varepsilon, -m+1; -k+\varepsilon)$; $k=1, 2, 3, \ldots$; $m=k+3, k+5, \ldots$ for (3.2), $m=k+2, k+4, \ldots$ for (3.3). The values of $\varepsilon=\varepsilon(k, m)$, $0<\varepsilon<1$, are given as follows:	

$m-k$ \ k	1	2	3	4	5
2	.56155	.43484	.37228	.33406	.30784
3	.5	.33333	.25	.2	.16667
4	.46293	.27164	.17843	.12568	.09297
5	.43775	.23027	.13328	.08257	.05400
6	.41928	.20052	.10307	.05642	.03279
7	.40499	.17803	.08191	.03992	.02078
8	.39350	.16039	.06655	.02911	.01370
9	.38399	.14614	.05507	.02179	.00935
10	.37594	.13437	.04627	.01669	.00658
11	.36901	.12447	.03939	.01304	.00476

$m, n\, k$: positive integers,
ξ, ζ, η: positive real numbers,
δ, ε: real numbers on (0, 1),
* The form of (3.3) only.

the expression (3.2) or (3.3) might be definable beyond the end point corresponding to the integer value, but we chop off the part beyond it. This overrides the rule (ii) and agrees with Convention 1 for $F(\alpha, -n; -n)$ on $[0, n]$.

Because of Convention 3, possible intervals for the GHg family of distributions are $[m, n]$, $[m, \infty)$, $[-n, -m]$ and $(-\infty, -m]$, where m and n are positive integers $1 \leqq m < n$. Intervals including $[-1, 0]$ are excluded. Surveying all possible values of parameters, we get the following conclusion. Its proof is not difficult but tedious and outlined in Appendix.

Theorem 2.

Under Convention 3, the GHg family of distributions, $F(\alpha, \beta; \gamma)$ of (3.2) or (3.3), of the types in Table 2 are possible in addition to the distributions on $[0, n]$ or $[0, \infty)$ of Table 1, and only the types of Tables 1 and 2 are possible.

Table 2 shows that, except for Type C, essentially the same types as in Table 1 are possible on positive or negative intervals under Convention 3. Type C is exceptional, it is defined only for isolated values of parameters. One might eliminate Type C from the GHg family because of this unnatural property.

Binomial coefficient expressions for Types A1^{+} and A2^{-} are given as follows.

$F(-\xi, -n; -m+1)$:

$$\binom{\xi}{x}\binom{n-m}{n-x}\Big/\binom{\xi+n-m}{n}=\binom{\xi-m}{x-m}\binom{n}{n-x}\Big/\binom{\xi+n-m}{n-m}; \quad m \leqq x \leqq n \quad \text{and} \quad m \leqq n-1 < \xi .$$

$F(\xi, m; n+1)$:

$$\binom{-\xi+n}{n+x}\binom{-m}{-m-x}\Big/\binom{-\xi-m+n}{n-m}=\binom{\xi+x-1}{n+x}\binom{-x-1}{-m-x}\Big/\binom{\xi-1}{n-m};$$
$$-n \leqq x \leqq -m \quad \text{and} \quad n < \xi .$$

It is rather perplexing to find the fact that the set of possible values in Tables 1 and 2 are not completely disjoint. A function of the form (3.2) or (3.3) can be a probability distribution on two intervals. One such a case is $F(\varepsilon, m; m+\rho)$, where $0 < \varepsilon < \rho < 1$ and m is a positive integer, which is of both Type B3 on $[0, \infty)$ and Type B3^{-} on $(-\infty, -m]$ at the same time if the expression (3.3) is assumed. Another is $F(1, 1; m)$, where m is an odd number larger than one, which is also of Types B3 and B3^{-} having probabilities

$$Pr[X=x]=Pr[X=-m-x]=\frac{m-2}{m-1}\frac{x!}{(m)_x}=(m-2)\Big/\binom{x+m-1}{m-2}, \quad x=0, 1, \ldots$$

4. Truncated, Shifted and Inverted Distributions

Truncated distributions

If the function $p(x)$ of (3.2) or (3.3) is positive on an integer interval I but the sum of its values over I is more than one, then it can be a probability distribution on a subinterval $J \subset I$. Two examples are shown below. Convention 3 excludes these cases from our family.

$F(1, -n; -n-2k)$, where n and k are positive integers, is a Type A2 distribution on $[0, n]$, and the $p(x)$ of (3.2) (*not* (3.3)) is positive also on $I=[n+2k+1, \infty)$. The summation over $J=[n+2k+1, 2n+2k+1]$ is equal to one. In fact, when a random variable X is defined by $F(1, -n; -n-2k)$ truncated on J, $Y=2n+2k+1-X$ has the same distribution $F(1, -n; -n-2k)$ on $[0, n]$.

$F(1, n; n+2k)$ is a Type B3 distribution on $[0, \infty)$, $p(x)$ of (3.2) or (3.3) is positive on $I=(-\infty, -n-2k]$, and its summation over $J=(-\infty, -2n-2k+1]$ is one. In fact, when X is defined by $F(1, n; n+2k)$ truncated on J, $Y=-X-2n-2k+1$ has the same distribution $F(1, n; n+2k)$ on $[0, \infty)$. $F(1, 1; 2k+1)$ in Table 2 is a special case of this, where $J=I$.

If we take any interval I on which $p(x)$ of (3.2) or (3.3) is positive and normalize $p(x)$ by its summation over I, then we get a variety of distributions. It is beyond the scope of this paper to discuss such a type of truncated distributions, and the authors do not recommend to include such distributions to the GHg family.

Shifted distributions

In statistical problems, GHg distributions may appear in shifted forms, which are not of (3.2) or (3.3). Distributions in the following examples are shifted Type A2 distributions and describe essentially the same problem. (cf. Sibuya (1978)).

We sample balls, from an urn with b black and w white balls, without replacement until k black balls are found. The number X of sampled white balls, has a Type A2 distribution, $F(k, -w; k-w-b)$. The number $Y=X+k$ of total sampled balls, has the distribution on $[k, k+w]$,

$$(4.1) \qquad Pr[Y=y]=\binom{y-1}{k-1}\frac{b(b-1)\ldots(b-k+1)w(w-1)\ldots(w+k-y+1)}{(b+w)(b+w-1)\ldots(b+w-y+1)},$$

which cannot be reduced generally to the form (3.2) or (3.3). Johnson and Kotz (1978, Section 2.5) call this form a negative hypergeometric distribution.

Get a random sample of size n without replacement from a set of positive integers, $\{1, 2, \ldots, N\}$, and let $X(j)$ be its j-th smallest order statistics. Then, $X(j)-j$ has a Type A2 distribution, $F(j, n-N; j-N)$. But

$$(4.2) \qquad Pr[X(j)=x]=\binom{x-1}{j-1}\binom{N-x}{n-j}\bigg/\binom{N}{n}, \quad j \leqq x \leqq N-n+j,$$

cannot be reduced to the form (3.2) or (3.3).

The "discrete beta" distribution

$$(4.3) \qquad Pr[X=x]=\frac{1}{B(a, b)N^{(a+b-1)}}(x-1)^{(a-1)}(N-x)^{(b-1)}$$

is a distribution on $[a, N-b+1]$ and cannot be reduced to the form (3.2) or (3.3). But $Y=X-a$ has a Type A2 distribution, $F(a, a+b-N-1; a-N)$.

Although shifted distributions are natural in some situations, the standard from

(2.5) makes a unified approach possible.

Inverted or shifted distributions and Singular Bivariate GHg distributions

An inversion $Y=\pm n-X$ and a shift $Y=\pm n+X$ can be regarded as the definition of a singular bivariate distribution of (X, Y) such that $X\pm Y=\pm n$, and the components are GHg variables. In later discussions, Singular Bivariate GHg degenerated on $x+y=\nu$, where ν is a positive or negative integer, will play an important role. This family of distributions is defined by

$$p(x, y)=\frac{\Gamma(\alpha+\beta)\Gamma(\nu+1)\Gamma(\alpha+x)\Gamma(\beta+y)}{\Gamma(\alpha+\beta+\nu)\Gamma(\alpha)\Gamma(\beta)\Gamma(x+1)\Gamma(y+1)}, \tag{4.4}$$

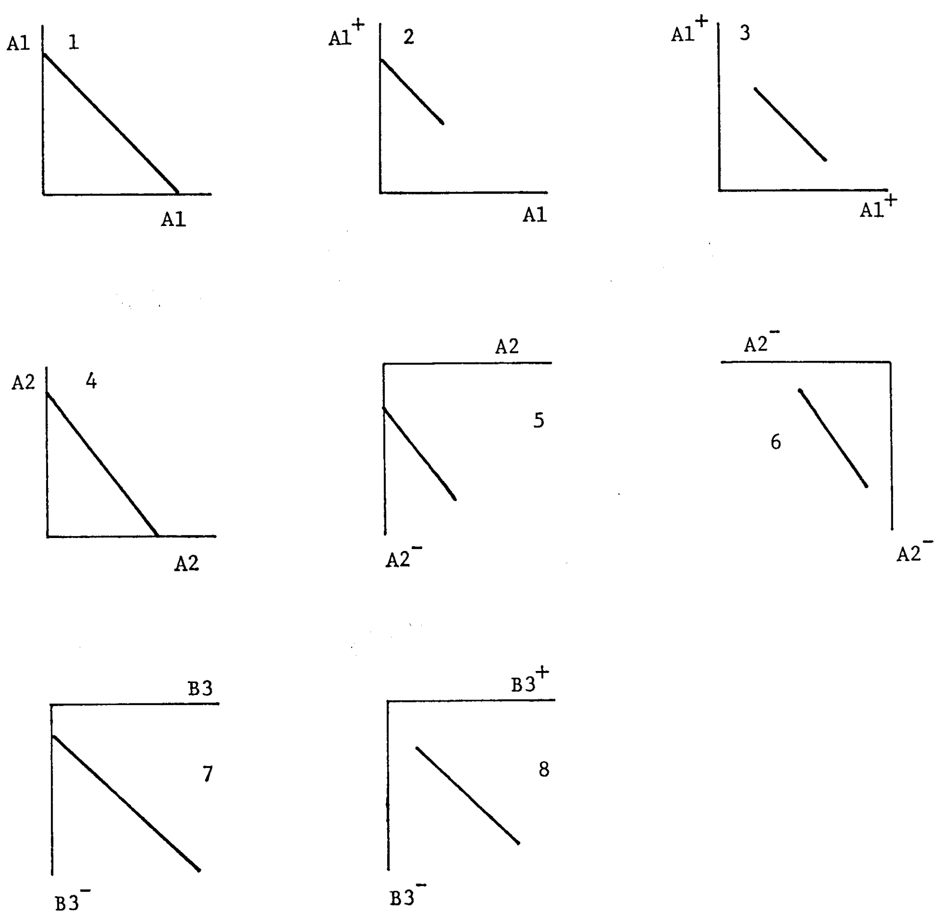

Figure 1. Domains of Singular Bivariate GHg distributions (4.4) or (4.5) of Types in Table 3

or by

$$(4.5)\qquad p(x,y)=\frac{\Gamma(1-\alpha-\beta-\nu)\Gamma(1+\nu)\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(1-\alpha-\beta)\Gamma(1-\alpha-x)\Gamma(1+x)\Gamma(1-\beta-y)\Gamma(1+y)}$$

and its marginal distributions are GHg:

$$(4.6)\qquad X:\ \mathrm{F}(\alpha, -\nu\,;\ -\beta-\nu+1),\quad \text{and}\quad Y:\ \mathrm{F}(\beta, -\nu\,;\ -\alpha-\nu+1).$$

It will be denoted by $\mathrm{SF}(\alpha, \beta\,;\ -\nu)$. All possible Singular Bivariate Hg's in Table 2 are reclassified in Table 3 to show more clearly their domains of distributions.

The expressions (4.4) and (4.5) can be different from each other by the factor (-1) as the difference between (3.2) and (3.3). In either expression, the marginal distributions (4.6) can be expressed in the form (3.2) or (3.3) formally equivalently.

Case numbers 4-6 of Table 3 involving A2 and A2⁻ should be remarked further in relation to Table 2. Let X be an A2 variable $\mathrm{F}(m, -n+m\,;\ -n-k+m+1)$ on $[0, n-m]$. An inversion $W=n-m-X$ is also A2, $\mathrm{F}(k, -n+m\,;\ -n+1)$ on $[0, n-m]$. Another inversion $Y=-m-X$ is an $\mathrm{A2}^-$ variable on $[-n, -m]$, and obtained also by left shift of W; $Y=W-n$. $Z=-k-W-n-k-Y=X-n-k+m$ give an $\mathrm{A2}^-$ on another interval $[-n-k+m, -k]$. This situation is well shown by the following 2×2 like table.

X: $[0, n-m]$	$Y=-X-m$: $[-n, -m]$	$-m$
$W=n-m-X$: $[0, n-m]$	$Z=X-n-k+m$: $[-n-k+m, -k]$	$-k$
$n-m$	$-n-k$	$-m-k$

The pairs (X, W), (X, Y) (or (W, Z)) and (Y, Z) are Cases 4-6 of Table 3 respectively.

Finally, we state a remark relating to the discussions of this section.

Theorem 3.

If X is a GHg random variable of Table 1, and if $\pm X\pm m$ has a probability function of the form (3.2) or (3.3), then $\pm X\pm m$ is one of the shifted or the inverted cases of Table 2.

Proof. The variable $-X+m$ for positive m cannot be in our family since it takes both positive and negative values. The shifted cases are checked within the proof of Theorem 2 in Appendix. The inverted cases are checked in preparing Table 3.

Table 3. Possible Singular Bivariate GHg distributions SF(α, β; $-\nu$) ((4.4) or (4.5)). Distribution of (X, Y), $X+Y=\nu$.

	SF(α, β; $-\nu$)	X: F(α, $-\nu$; $-\beta-\nu+1$)	Y: F(β, $-\nu$; $-\alpha-\nu+1$)
1.	SF($-\xi$, $-\eta$; $-n$) $\xi, \eta>n-1$	F($-\xi$, $-n$; $\eta-n+1$) $[0, n]$ A1	F($-\eta$, $-n$; $\xi-n+1$) $[0, n]$ A1
2.	SF($-k$, $-\eta$; $-n$) $n\geqq k$, $\eta>n-1$	F($-k$, $-n$; $\eta-n+1$) $[0, k]$ A1	F($-\eta$, $-n$; $-n+k+1$) $[n-k, n]$ A1$^+$
3.	SF($-k$, $-l$; $-n$) $n>k, l$	F($-k$, $-n$; $-n+l+1$) $[n-l, n]$ A1$^+$	F($-l$, $-n$; $-n+k+1$) $[n-k, n]$ A1$^+$
4.	SF(ξ, η; $-n$) $\xi, \eta>0$	F(ξ, $-n$; $-\eta-n+1$) $[0, n]$ A2	F(η, $-n$; $-\xi-n+1$) $[0, n]$ A2
5.	SF($-k$, η; n) $\eta>k+n$	F($-k$, n; $-\eta+n+1$) $[0, k]$ A2	F(η, n; $k+n+1$) $[-n-k, -n]$ A2$^-$
6.	SF(k, η; n) $n>l+k$	F(k, n; $-\eta+n+1$) $[-n+l, -k]$ A2$^-$	F(η, n; $-k+n+1$) $[-n+k, -l]$ A2$^-$
7.	SF(ξ, $-\eta$; n) $\xi<\eta+1$	F(ξ, n; $\eta+n+1$) $[0, \infty)$ B3	F($-\eta$, n; $-\xi+n+1$) $(-\infty, -n]$ B3$^-$
8.	SF($\xi-n$, k; $k-n$) $\xi<1+n-k$	F($\xi-n$, $k-n$; $-n+1$) $[n, \infty)$ B3$^+$	F(k, $k-n$; $-\eta+k+1$) $(-\infty, -k]$ B3$^-$

ξ, η: positive numbers,
k, l, n: positive integers,
A1, A2, B3, ...: types of distributions in tables 1 and 2.

5. Bivariate GHg Familyof Distributions

Distribution on the nonnegative quadrant

Some statistical models which will be discussed in Section 6 suggest that a natural extension of (2.4) is a family of distributions with probabilities

$$(5.1)\qquad p(x, y)=\frac{\Gamma(\omega-\alpha-\beta)\Gamma(\omega-\lambda)}{\Gamma(\omega-\alpha-\beta-\lambda)\Gamma(\omega)}\frac{(\lambda)_{x+y}(\alpha)_x(\beta)_y}{(\omega)_{x+y}x\,!\,y\,!},\qquad x, y=0, 1, 2, \dots$$

They suggest also that the condition $p(0, 0)>0$ is too restrictive. Even in Bivariate Ordinary Hg distributions, this limits unnecessarily the range of parameters. From the beginning, therefore, we extend the form (5.1) and introduce a convention similar to the one in Section 3. We define the form of probability function by

$$(5.2)\qquad p(x, y)=\frac{\Gamma(\omega-\alpha-\beta)\Gamma(\omega-\lambda)\Gamma(x+y+\lambda)\Gamma(x+\alpha)\Gamma(x+\beta)}{\Gamma(\omega-\alpha-\beta-\lambda)\Gamma(\lambda)\Gamma(\alpha)\Gamma(\beta)\Gamma(x+y+\omega)\Gamma(x+1)\Gamma(y+1)},$$

or by

Table 4. Classification of the Bivariate Generalized Hypergeometric distributions $F(\alpha, \beta; \lambda; \omega)$ defined by (5.2) or (5.3) on the nonnegative quadrant.

No.	type of distributions				distribution $F(\alpha, \beta; \lambda; \omega)$	restriction	distribution range	name
	$S=X+Y$	$X\|S=s$	X	$Y\|X=x$				
							[finite]	
1.	A1	A1	A1	A1	$F(-\xi, -\eta; -n; \zeta)$	$\xi, \eta>n-1$ or positive integer	$0\leqq x+y\leqq n$	MHg/MP
2.	A1	A1/A1$^+$	A1	A1	$F(-k, -l; -\lambda; \zeta)$	$\lambda>k+l-1$ or positive integer	$0\leqq x\leqq k$, $0\leqq y\leqq l$	—
3.	A1$^+$	A1	A1	A1$^+$/A1	$F(-\xi, -\eta; -n; 1-m)$	$\xi, \eta>n-1$	$m\leqq x+y\leqq n$	MHg/MP
4.	A1$^+$	A1/A1$^+$	A1	A1$^+$/A1	$F(-k, -l; -\lambda; 1-m)$	$\lambda>k+l-1$	$0\leqq x\leqq k$, $0\leqq y\leqq l$, $m\leqq x+y$	—
5.	A2	A2	A2	A2	$F(\xi, \eta; -n; -\zeta)$	$\zeta>n-1$	$0\leqq x+y\leqq n$	MNHg/MP
6.	A2	A1/A1$^+$	A2	A2	$F(-k, -l; \lambda; -\zeta)$	$\zeta>k+l-1$	$0\leqq x\leqq k$, $0\leqq y\leqq l$	MIHg/MIP
							[infinite]	
7.	B2	A2	B2	B2	$F(\delta, \varepsilon; -n+\sigma; -n+\rho)$	$\delta+\varepsilon+\sigma<\rho$	$0\leqq x$, $y<\infty$	—
8.	B3	A2	B3	B3	$F(\xi, \eta; \lambda; \zeta)$	$\xi+\eta+\lambda<\zeta$	$0\leqq x$, $y<\infty$	MNIHg/MIP
9.	B3$^+$	A2	B3	B3$^+$/B3	$F(\rho, \sigma; -m+\delta; -m+1)$	$\rho+\sigma+\delta<1$	$m\leqq x+y<\infty$	—
10.	C	A2	*	*/B3	$F(\rho, \sigma; -m+1; -k+\rho+\sigma)$	$\rho+\sigma=\varepsilon(m, k)$	$m\leqq x+y<\infty$	—

See Table 1 for A1, A2, B2 and B3. See Table 2 for A1$^+$, B3$^+$ and C.
$\xi, \eta, \lambda, \zeta$: positive real numbers; k, l, m, n: positive integers,
$\delta, \varepsilon, \rho, \sigma$: real numbers on (0, 1),
*: See the expressions (5.5) and (5.6),
$\varepsilon(m, k)$: cf. Type C distribution in Table 2.
Names are those in Janardan-Patil (1972).

(5.3)

$$p(x, y)=\frac{\Gamma(\omega-\alpha-\beta)\Gamma(\omega-\lambda)\Gamma(1-\lambda)\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(\omega-\alpha-\beta-\lambda)\Gamma(x+y+\omega)\Gamma(x+1)\Gamma(y+1)\Gamma(1-\lambda-x-y)\Gamma(1-\alpha-x)\Gamma(1-\beta-y)},$$

which shall be denoted by F(α, β ; λ ; ω). We assume the distribution range to be within the nonnegative quadrant at first, and then within the other quadrants in the following subsection.

Convention 4. (i) $p(x, y)$ is positive on a connected region in $\{(x, y) ; x, y=0, 1, 2, \ldots\}$, and is zero or undefined on the region's outside neighboring points because of the corresponding parameter values. Two or more parameters of the same value are treated as Convention 3-(iii).
(ii) $p(x, y)$ is defined for at least two adjacent values of $x+y$, and for at least two adjacent values (x, y) and $(x+1, y-1)$ on one of the $x+y$ values.

The statement (i) means that the distribution range is possibly limited by the condition $x \leqq m_x$, $y \leqq m_y$, $x+y \leqq m$, or $m_0 \leqq x+y$. The statement (ii) means $p(x, y)$ not to degenerate into one-dimensional distribution.

Theorem 4.

Under Convention 4, a Bivariate GHg distribution, F(α, β ; λ ; ω) defined by (5.2) or (5.3), can be a probability distribution on the nonnegative quadrant if and only if F(α, β ; λ ; ω) (or F(β, α ; λ ; ω)) belongs to one of the ten types of Table 4. The distribution of $S=X+Y$, the conditional distributions $X|S=s$, $Y|X=x$, etc., are GHg distributions as summarized in Table 5.

Table 5. Distribution of sum, conditional distributions and marginal distributions of Bivariate Generalized Hypergeometric distributions

(X, Y)	F(α, β; λ; ω)	(5.2) or (5.3)
$S=X+Y$	F($\alpha+\beta$, λ; ω)	
$X\|S=s$	F(α, $-s$; $-\beta-s+1$)	$\binom{-\alpha}{x}\binom{-\beta}{s-x}\Big/\binom{-\alpha-\beta}{s}$
$X\|Y=y$	F(α, $\lambda+y$; $\omega+y$)	
$Y\|X=x$	F(β, $\lambda+x$; $\omega+x$)	
X	F(α, λ; $\omega-\beta$)	
Y	F(β, λ; $\omega-\alpha$)	

(Only the case where S is a type C variate is exceptional.)

Proof. Firstly, we prove the following statements which hold under Convention 4.
Statement 1. If $Pr[X+Y=s]>0$, then the conditional probability of (X, Y) given s is

$$Pr[(X, Y)=(x, y)]=\binom{-\alpha}{x}\binom{-\beta}{y}\Big/\binom{-\alpha-\beta}{s}$$

which must be a Singular Bivariate GHg of Cases 1-4 in Table 3.

In fact, if $Pr[X+Y=s]>0$, then the factor $(\alpha)_x(\beta)_y$ must be always positive or negative on an integer interval where $p(x,y)>0$ and $x+y=s$. If the interval is $[0,s]$, then it is shown that (i) α and β are positive or (ii) α and β are negative and $\alpha,\beta<-s+1$. If the interval is such that $0<m\leqq x$ and/or $x\leqq n<s$, then Convention 4 requires that $\beta=-s+m$ and/or $\alpha=-n$, and $\alpha+\beta<-s+1$ must hold. In all of these cases

$$\sum_{x+y=s}\frac{(\alpha)_x}{x\,!}\frac{(\beta)_y}{y\,!}=\frac{(\alpha+\beta)_s}{s\,!}=(-1)^s\binom{-\alpha-\beta}{s},$$

and the conditional distribution is a Singular Bivariate GHg on the nonnegative quadrant, namely Cases 1-4 in Table 3.

Statement 2. The distribution of $S=X+Y$ is $F(\alpha+\beta,\lambda;\omega)$ which must be one of Types in Table 1 or A1$^+$, B1$^+$ or B3$^+$ in Table 2.

Because of Statement 1,

$$Pr[X+Y=s]=\frac{\Gamma(\omega-\alpha-\beta)\Gamma(\omega-\lambda)}{\Gamma(\omega-\alpha-\beta-\lambda)\Gamma(\omega)}\frac{(\lambda)_s(\alpha+\beta)_s}{(\omega)_s s\,!},$$

which is $F(\alpha+\beta,\lambda;\omega)$. Since Convention 4 reduces to Convention 3 with regard to the distribution of $S=X+Y$, and since $s\geq 0$, this must be a distribution in Table 1 or 2 on a nonnegative interval.

If the distribution region of $F(\alpha,\beta;\lambda;\omega)$ is infinite, then S is distributed on an infinite interval. Then, α and β must be positive since otherwise $\alpha+\beta<-s+1$ must hold for $s\to\infty$. Thus, in the infinite case, the distribution of S is either B2, B3, B3$^+$ or C, and the conditional distribution of (X,Y) given $S=s$ is Case 4 of Table 3. If the distribution region is finite, then S is distributed as type A1, A1$^+$ or A2. The conditional distribution of (X,Y) given $X+Y=s$ must be one of Cases 1-4 in Table 3.

Statement 3. If $p(x,0)>0$, then the conditional distribution of Y given $X=x$ is $F(\beta,\lambda+x;\omega+x)$ which must be one of distribution types in Table 1.

From the assumption of the statement, $Pr[X=x]>0$ and the conditional distribution must have a range $[0,n]$ or $[0,\infty)$. Since (5.1) can be written as

$$p(x,y)=C(\alpha,\beta,\lambda,\omega,x)\frac{(\lambda+x)_y(\beta)_y}{(\omega+x)_y y\,!},$$

the conditional distribution is $F(\lambda+\omega,\beta;\omega+x)$ which must be one of Types in Table 1.

We cannot extend Statement 3 to the case where $Pr[X=x]>0$ but $p(x,0)=0$ using Table 2 instead of Table 1. Because, in constructing Table 2, the function form of the probabilities was fixed. Here, however, the normalization factor is undetermined.

Now, we are ready to prove Theorem 4. For each possible type of distribution $F(\alpha+\beta,\lambda;\omega)$ of S, we examine the possibility of the bivariate distribution $F(\alpha,\beta;\lambda;\omega)$. Firstly, the case where S is finite is analyzed, that is the case where S is of Type A1, Type A1$^+$ or Type A2.

S: Type A1 or A1$^+$, $F(-\lambda,-n;\gamma)$, $\gamma=\zeta$ or $1-m\,(n>m)$ and $\lambda>n-1$. (#1-#4 of Table 4)

The conditional distribution of $(X, Y)|S=s$ should be Cases 1-3 of Table 2. There are two choices for the parameter of the conditional distribution, that is, $\alpha+\beta=\lambda$ or n. In the first case, the distribution of (X, Y) is $F(-\xi, -\eta; -n; \gamma)$ rewriting λ as $\xi+\eta$, which can be named as Multivariate Positive Hg distributions. In the second case, the distribution of (X, Y) is $F(-k, -l; -\lambda; \gamma)$, $k+l=n$. The parameters k and l must be integers, because otherwise k and l should be larger

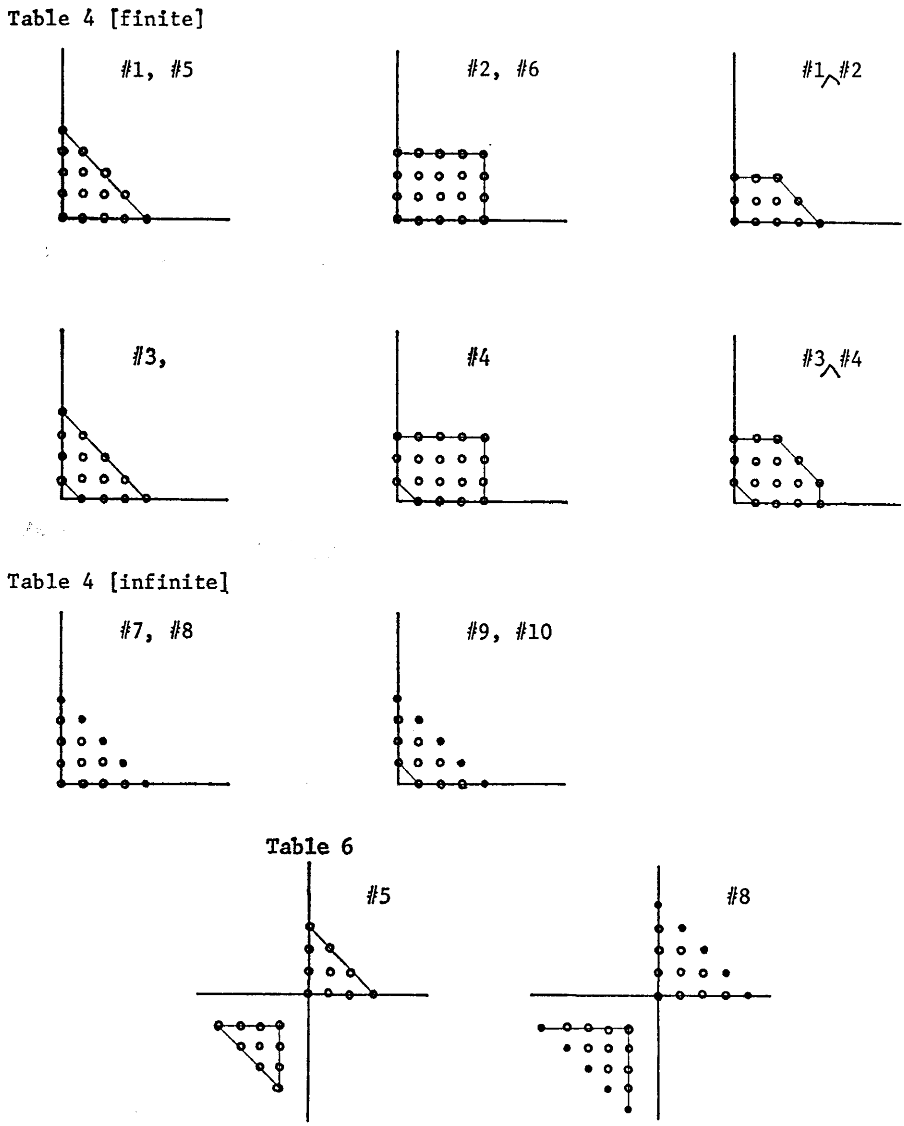

Figure 2. Domains of Bivariate GHg distributions (5.1), (5.5) or (5.6) of Types in Tables 4 and 6

than $s-1$, $s=1,\ldots,n$, which contradicts the condition $k+l=n$. The difference between the two cases is the shape of distribution range: a triangle in the first case and a rectangle in the second case assuming S to be A1. Multivariate Ordinary Hg distributions with all integer parameters are common to both the first and the second cases.

S: Type A2, $F(-\lambda, n; -\zeta)$, $\zeta > n-1$. (#5 and #6 of Table 4)

The distribution of $X|S=s$ can be either Negative Hg with $\alpha+\beta=-\lambda$ or Ordinary Hg with $\alpha+\beta=n$. In the first case, (X, Y) is distributed as $F(\xi, \eta; -n; \zeta)$, which is a Multivariate Negative Hg distribution, or a Multivariate Pólya-Eggenberger distribution, and the distribution range is $0 \leqq x+y \leqq n$. In the second case, (X, Y) is distributed as $F(-k, -l; \lambda, \zeta)$, which is called a Multivariate Inverse Hg in Janardan and Patil (1972), and the distribution range is $0 \leqq x \leqq k$ and $0 \leqq y \leqq l$.

Secondly, the case where S is distributed on $[0, \infty)$ or $[m, \infty)$ can be analyzed as follows.

S: Type B2, $F(\delta+\varepsilon, -n+\sigma; -n+\rho)$, $\rho > \varepsilon+\delta+\sigma$. (#7 in Table 4)

The distribution of (X, Y) is $F(\delta, \varepsilon; -n+\sigma; -n+\rho)$. The conditional distribution $Y|X=x$ is $F(\varepsilon, -n+x+\sigma; -n+x+\rho)$ which is of Type B2 if $x<n$ and of Type B3 if $x \geqq n$. The marginal distributions are of Type B2: $F(\varepsilon, -n+\sigma; -n+\rho+\delta)$ for X.

S: Type B3, $F(\xi+\eta, \zeta; \chi)$, $\chi > \xi+\eta+\zeta$. (#8 in Table 4)

The distribution of (X, Y) is $F(\xi, \eta; \zeta; \chi)$, the conditional distribution of $Y|X=x$ is $F(\xi, \zeta+x; \chi+x)$ of Type B3, the marginal distribution of X is $F(\xi, \zeta; \chi+\eta)$ of Type B3.

S: Type B3$^+$, $F(\xi-m, \eta-m; -m+1)$, where $\xi+\eta < m+1$. (#9 in Table 4)

At least one of $\xi-m$ and $\eta-m$ must be negative, and if both are negative, then $Y|X=x$ cannot be a GHg when $x \to \infty$. If $0 < \xi-m$, then $0 < \varepsilon = \xi-m < 1-\eta = 1-\delta < 1$. The distribution of (X, Y) is $F(\rho, \sigma; -m+\delta; -m+1)$, $\rho+\sigma+\delta=\varepsilon+\delta<1$. The conditional distribution of $Y|X=x$ is $F(\sigma, -m+x+\delta; -m+x+1)$, which is Type B3 if $x \geqq m$, and Type B3$^+$ (right shifted by $m-x$) if $x<m$. The marginal distributions are $F(\rho, -m+\delta; -m+1-\sigma)$ of X and $F(\sigma, -m+\delta; -m+1-\rho)$ of Y, and both are of Type B2.

S: Type C, $F(\varepsilon, -m+1; -k+\varepsilon)$. (#10 in Table 4)

The distribution of (X, Y) is $F(\rho, \sigma; -m+1; -k+\rho+\sigma)$, where $\rho+\sigma=\varepsilon$ is a value in Table 2. The conditional distribution of $Y|X=x \geqq m$ is $F(\sigma, x-m+1; x-k+\rho+\sigma)$, which is a Type B3 distribution. The conditional distribution when $x<m$ (or $y<m$) and the marginal distributions are not GHg distributions. Using the function introduced in Appendix,

$$f(\xi, \zeta; m) = \sum_{y=0}^{\infty} \frac{1}{(1+y)_m} \frac{(\xi)_y}{(\zeta)_y}, \tag{5.4}$$

$$Pr[Y=y|X=x<m] = \frac{(m-x+\sigma)_v}{(1+v)_{m-x}(m-k+\rho+\sigma)_v} \Big/ f(m-x+\sigma, m-k+\rho+\sigma; m-x), \tag{5.5}$$

where $y=v+m-x$. The marginal distribution of X is

$$(5.6)\qquad Pr[X=x]=\begin{cases}\dfrac{(\sigma)_{m-x}}{(m-k-1+\varepsilon)}\dfrac{(m-1)!}{(m-k-2)!\,k!}\dbinom{\rho+x-1}{x} \\ \quad\times f(m-x+\sigma, m-k+\rho+\sigma\,;\,m-x)\,, & x<m\,, \\ \dfrac{\Gamma(-k)\Gamma(m-k+\rho-1)}{\Gamma(m-k-1)\Gamma(-m+1)\Gamma(\rho)}\dfrac{\Gamma(\rho+x)\Gamma(-m+1+x)}{\Gamma(-k+\rho+x)x\,!}\,, & x\geqq m\,,\end{cases}$$

The proof is now completed.

Remarks. In Table 4, we have classified distributions into ten types, but #1 and #2, or #3 and #4 are not mutually exclusive, and each pair can be combined into one type. If we classify them by the Types of S and $X|S=s$ (if necessary) and regard pairs A1 and A1$^+$, and B3 and B3$^+$ essentially of the same type, we have six types: A1 (#1-#4), A2-A2 (#5), A2-A1 (#6), B2 (#7), B3 (#8, #9), C (#10).

One might imagine that Bivariate GHg with marginal distributions of different types like (A1, B1) or (A2, B2) can be defined. But the above discussions show that these are impossible. Classification by marginal distribution type mixes up A2-A2 (#5) and A2-A1 (#6). However, these two should be distinguished.

Types #2 and #4 F$(-k, -l\,;\,-\lambda\,;\,\gamma)$ are not well studied in literature. Their probability functions are written as

$$\binom{k}{x}\binom{l}{y}\frac{\lambda^{(x+y)}(k+l+\gamma)^{(k+l-x-y)}}{(\lambda+k+l+\gamma)^{(k+l)}}\,,$$

where $\gamma=\zeta>0$ (#2) or $\gamma=1-m$ (#4). This expression cannot be reduced to the form $\binom{a}{x}\binom{b}{y}\binom{c}{n-x-y}\Big/\binom{a+b+c}{n}$, where n is a positive integer, unless λ is a positive integer. An appropriate name is required.

Bivariate Ordinary Hg (#1$\wedge$#2 or #3$\wedge$#4, the intersection of #1 and #2 or #3 and #4) is best understood by a 2×3 table:

$N-x$	$M-y$	$L-z$	m
x	y	z	n
N	M	L	$L+M+N=m+n$

Values of any two entries determine the value of the other entries. Any two of x, y and z, or any two of $N-x$, $M-y$ and $L-z$ are Types #1$\wedge$#2 or #3$\wedge$#4 variables. Two entries of the different rows like $(N-x, y)$ are not GHg variables. Usually some pairs are #1$\wedge$#2 while others are #3$\wedge$#4. In this case, Types #3$\wedge$#4 variables are transformed to Types #1$\wedge$#2 by shift and inversion. If, however, $\min(m, n)>\max(L, M, N)$, then all pairs are Types #3$\wedge$#4, and $p(0, 0)$ remains zero by shift or inversion, showing the necessity of our approach based on Convention 4.

To distinguish Types #5 and #6, names negative and inverse are used by Janardan and Patil (1972). But they are rather confusing.

Distributions on the other quadrant

The distributions of the form (5.2) or (5.3) can be those outside the nonnegative quadrant. Because of the "built-in parameter 1", the distribution range cannot include points with $x=0$ and $x=-1$ or $y=0$ and $y=-1$. Therefore, the range is confined in a quadrant, the axes $x=0$ and $y=0$ being included in the positive side.

Theorem 5.

A Bivariate GHg distribution, $F(\alpha, \beta; \lambda; \omega)$ defined by (5.2) or (5.3) can be a probability distribution outside the nonnegative quadrant under Convention 4 (the range being modified) if and only if it is one of two types of distributions on the negative quadrant of Table 6.

Proof. On negative finite intervals, only Type $A2^-$ distributions are defined as

Table 6. Two types of Bivariate GHg distributions (5.2) or (5.3) on the negative quadrant

Two types of distributions on the negative quadrant	Corresponding distributions on the positive quadrant (cf. Table 4)
(X, Y): $F(m, k; \xi; n+1)$ $(-m-X', -k-Y')$, $x \leqq -m$, $y \leqq -k$, $-n \leqq x+y$	$(X' Y')$: $F(m, k; -n+m+k; -\xi+m+k+1)$ #5 $0 \leqq x'+y' \leqq n-m-k$, $m+k<n<\infty$
S: $F(m+k, \xi; n+1)$ $-m-k-S'$, $A2^-$, $[-n, -m-k]$	S': $F(m+k, -n+m+k; -\xi+m+k+1)$ A2, $[0, n-m-k]$
$X\|S=s$: $F(m, -s; -s-k+1)$ $\binom{-m}{x}\binom{-k}{s-x}/\binom{-m-k}{s}$, $A2^-$, $[s+k, -m]$	$X'\|S'=s'$: $F(m, -s'; -s'-k+1)$ $\binom{-m}{x'}\binom{-k}{s'-x'}/\binom{-m-k}{s'}$, A2, $[0, s']$
X: $F(m, \xi; n-k+1)$ $-m-X'$, $A2^-$, $[-n+k, -m]$	X': $F(m, -n+m+k; -\xi+m+1)$ A2, $[0, n-m-k]$
$X\|Y=y$: $F(m, \xi+y; n+y+1)$ $A2^-$, $[-n-y, -m]$	$X'\|Y'=y'$: $F(m, -n-y'+k, -\xi-y'+k+1)$ A2, $[0, n-y'-k]$
(X, Y): $F(m, k; -\zeta+m+k+1; -\xi+m+k+1)$ $(-m-X', -k-Y')$, $x \leqq -m$, $y \leqq -k$	(X', Y'): $F(m, k; \xi; \zeta)$ #8 $x', y'=0, 1, 2, \ldots$
S: $F(m+k; -\zeta+m+k+1; -\xi+m+k+1)$ $-m-k-S'$, $B3^-$, $(-\infty, -m-k]$	S': $F(m+k, \xi; \zeta)$ B3, $[0, \infty)$
$X\|S=s$: $F(m, -s; -s-k+1)$ $\binom{-m}{x}\binom{-k}{s-x}/\binom{-m-k}{s}$, $A2^-$, $[s+k, -m]$	$X'\|S'=s'$: $F(m, -s'; -s'-k+1)$ $\binom{-m}{x'}\binom{-k}{s'-x'}/\binom{-m-k}{s'}$, A2, $[0, s']$
X: $F(m, -\zeta+m+k+1; -\xi+m+1)$ $-m-X'$, $B3^-$, $(-\infty, -m]$	X': $F(m, \xi; \zeta-k)$ B3, $[0, \infty)$
$X\|Y=y$: $F(m, -\zeta+m+k+y+1; -\xi+m+k+y+1)$ $B3^-$, $(-\infty, -m]$	$X'\|Y'=y'$: $F(m, \xi+y'; \zeta+y')$ B3, $[0, \infty)$

shown in Table 2. Since the distribution of $S=X+Y$ is a GHg, it cannot take both positive and negative values, and the distribution region of (X, Y) is limited to $x+y\geqq 0$ or $x+y<0$. If $S=s>0$, then the conditional distribution $X|S=s$ must be Type A1 or A1$^+$ and this means that both components, X and Y, are nonnegative. Therefore, S must be negative and of Type either A2$^-$ or B3$^-$.

As shown in Table 3, there are two cases of Singular Bivariate GHg such that $X+Y<0$ and X and Y are finite; the types of X and Y are A2$^-$ and A2$^-$ or A2 and A2$^-$. The first case gives Bivariate GHg distributions on the negative quadrant as shown in Table 6.

Let (X, Y) be a Type A2-A2$^-$ variable with the distribution

$$Pr[(X, Y)=(x, y)|X+Y=s]=\binom{k}{x}\binom{-m-k}{y}\Big/\binom{-m}{s}, \quad 0\leqq x\leqq k, \quad s-k\leqq y\leqq s<0.$$

If S is a Type A2$^-$ variable with $\mathrm{F}(\xi, m; n+1)$ on $[-n, -m]$, then we can cenceive formally a bivariate distribution $\mathrm{F}(-k, m+k; \xi; n+1)$. However, in this function form, the original restriction $s\leqq -m$ is lost and it is replaced by $x\leqq k$ and $y\leqq -m-k$. So the intended distribution has not the form of (5.5) or (5.6). Similar trouble occurs when S is a Type B3$^-$ variable, and the proof is completed.

As shown in Table 6, the two types of distributions are obtained by inverting two types of Table 4. Each component can be inverted into different intervals.

Janardan and Patil (1972) discussed the case where λ is an integer. (They discussed Multivariate GHg in general.) Examining Tables 4 and 6, one may find the case too restrictive. This restriction is weak enough for defining Singular Multivariate GHg, which, howevre, cannot be defined naturally from this restriction only.

Singular Multivariate GHg distributions

Singular Bivariate Hg distributions of (4.4) or (4.5) classified in Table 3 can be extended to Singular Multivariate Hg distributions. Their probabilites are expressed by

$$p(x_0, x_1, \ldots, x_q)=\frac{\Gamma(\alpha')\Gamma(1+\nu)}{\Gamma(\alpha'+\nu)}\prod_{i=0}^{q}\frac{\Gamma(\alpha_i+x_i)}{\Gamma(\alpha_i)\Gamma(1+x_i)}, \tag{5.7}$$

where $\alpha'=\alpha_0+\alpha=\sum_{i=0}^{q}\alpha_i$ and $\nu=\sum_{i=0}^{q}x_i$ is a positive or a negative integer, or by

$$p(x_0, x_1, \ldots, x_q)=\frac{\Gamma(1-\alpha'-\nu)\Gamma(1+\nu)}{\Gamma(1-\alpha')}\prod_{i=0}^{q}\frac{\Gamma(1-\alpha_i)}{\Gamma(1-\alpha_i-x_i)\Gamma(1+x_i)}, \tag{5.8}$$

and we assume the simultaneous distribution of $(X_0, \sum_{i=1}^{q}X_i)$ to be Singular Bivariate GHg $\mathrm{SF}(\alpha_0, \alpha; -\nu)$ and the marginal distribution (X_i, X_j) to be Bivariate GHg, $0\leqq i<j\leqq q$.

The possible cases of Singular Multivariate GHg correspond to those of Singular Bivariate GHg and shown in Table 7. The possibility is easily checked, and the impossibility of the other cases are checked by inspecting the bivariate marginal distributions.

Table 7. Possible Singular Multivariate Hg distribution (5.7) or (5.8)

Type of $(X_1+\dots+X_q, X_0)$ in Table 3	$(X_1, \dots, X_q, X_0)$: SF$(\alpha_i, \alpha_0; -\nu)$	(X_1, X_2): Type in Table 4 F$(\alpha_1, \alpha_2; -\nu; -\alpha'+\alpha_1+\alpha_2-\nu+1)$
1-3	SF$(-\xi_i, -\xi_0; -n)$	#1-#4 F$(-\xi_1, -\xi_2; -n; \xi'-\xi_1-\xi_2-n+1)$
	$\left[\begin{array}{l} 0\leqq x_i\leqq n, \text{ if } \xi_i>n-1, \text{ and} \\ 0\leqq x_i\leqq k_i, \text{ if } \xi_i=k_i<n; \\ n-c_i\leqq x_i \text{ if } \xi_i=k_i \text{ is an integer and } \sum_{j\neq i} k_j=c_i<n; \\ \text{for } i=0, 1, \dots, q. \end{array}\right]$	
4	SF$(\xi_i, \xi_0; -n)$ $\xi_i, \xi_0>n-1$; $0\leqq x_i\leqq n$,	#5, $0\leqq x_1+x_2\leqq n$ F$(\xi_1, \xi_2; -n; -\xi'+\xi_1+\xi_2-n+1)$
5	SF$(-k_i, \lambda; n)$ $\eta>k+1$; $0\leqq x_i\leqq k_i$.	#6 $(\lambda=n)$, $0\leqq x_i\leqq k_i$ F$(-k_1-k_2; n; -\eta+k-k_1-k_2+n+1)$
6	SF$(k_i, k_0; n)$ $n>k'$; $-n+k'-k_i\leqq x_i\leqq k_i$.	Table 6 $(\xi=n)$, $-n+k'-k_1-k_2\leqq x_1+x_2\leqq -k_1-k_2$ F$(k_1, k_2; n; -k'+k_1+k_2+n+1)$
7	SF$(\xi_i, -\zeta; n)$ $\xi>\zeta+1$; $0\leqq x_i<\infty$, $-\infty<x_0\leqq -n$.	#8 $(\lambda=n)$, $0\leqq x_1<\infty$ F$(\xi_1, \xi_2; n; \zeta-\xi+\xi_1+\xi_2+n+1)$
8	SF$(k_i, \xi-n; k-n)$ $\xi<1+n-k$; $-\infty<x_i\leqq -k_i$, $n<x_0<\infty$.	Table 6 $(\zeta=n+1)$, $-\infty<x_i\leqq -k_i$ F$(k_1, k_2; -n+k; -\xi+k_1+k_2+1)$

ξ_i, ξ_0, ζ: positive numbers, $\xi'=\xi_0+\xi_1+\dots+\xi_q$.
k_i, k_0, n: positive integers, $k'=k_0+k=k_0+k_1+\dots+k_q$.

$(X_1, \dots, X_q, X_0)$ has a Singular Multivariate Hg distribution SF$(\alpha_1, \dots, \alpha_q; \alpha_0; -\nu)$, abbreviated as SF$(\alpha_i, \alpha_0; -\nu)$ in the second column of the table, which is a distribution on the discrete hyperplane $x_0+x_1+\dots+x_q=\nu$, where ν is a positive or a negative integer.

$(X_1+\dots+X_q, X_0)$ is a Singular Bivariate GHg distribution SF$(\alpha_1+\dots+\alpha_q, \alpha_0; -\nu)$, and (X_1, X_2) is a Bivariate GHg distribution F$(\alpha_1, \alpha_2; -\nu; -\alpha'+\alpha_1+\alpha_2-\nu+1)$, where $\alpha'=\alpha_0+\alpha_1+\dots+\alpha_q$.

The above definition excludes the distributions like

$$(5.9)\qquad Pr[(X, Y, Z)=(x, y, z)]=\binom{-\xi}{x}\binom{-\eta}{y}\binom{-\zeta+n-1}{n+k+z}\Big/\binom{-\xi-\eta-\zeta+n-1}{n},$$

where $x+y+z=-k$, $0\leqq x+y\leqq n$, $-k-n\leqq z\leqq -k$ and $\xi+\eta=k$. The distribution of $(X+Y, Z)$ is SF$(-n, \zeta+k+1; k)$, Case 5 Table 3, and the marginal distribution of (X, Y) is F$(\xi, \eta; -n; -\zeta)$, Case 5 Table 4. But the marginal distribution of (X, Z) is not a GHg.

Multivariate GHg distributions

The discussions on Bivariate GHg distributions of this section can be extended to Multivariate GHg. The distributions are defined by

$$(5.10) \qquad p(x)=\frac{\Gamma(\omega-\Sigma\alpha_i)\Gamma(\omega-\lambda)\Gamma(\lambda+\Sigma x_i)}{\Gamma(\omega-\Sigma\alpha_i-\lambda)\Gamma(\lambda)\Gamma(\omega+\Sigma x_i)}\prod_{i=1}^{q}\frac{\Gamma(x_i+\alpha_i)}{\Gamma(\alpha_i)\Gamma(x_i+1)},$$

where $x=(x_1, \ldots, x_q)$, or

$$(5.11) \qquad p(x)=\frac{\Gamma(\omega-\Sigma\alpha_i)\Gamma(\omega-\lambda)\Gamma(1-\lambda)}{\Gamma(\omega-\Sigma\alpha_i-\lambda)\Gamma(\omega+\Sigma x_i)\Gamma(1-\lambda-\Sigma x_i)}\prod_{i=1}^{q}\frac{\Gamma(1-\alpha_i)}{\Gamma(1-\alpha_i-x_i)\Gamma(x_i+1)},$$

which will be denoted by $\mathrm{F}(\alpha\,;\lambda\,;\omega)$.

Theorem 6.

The function of (5.10) or (5.11) can be, under a modification of Convention 4, a probability distribution on the nonnegative orthant if the conditions of Table 4 (with the number of parameters increased) are satisfied, and a distribution on the negative orthant if the conditions of Tabe 6 (with the number of parameters increased) are satisfied, and only these types of distributions are possible.
Proof. The possibility of the types is easily checked for each one. The impossibility of other cases is proved by inspecting the conditional bivariate distribution of a pair of components when the values of other components are given, which must be one of the types in Table 4 or 6.

6. Geneses of Bivariate GHg Distributions

In this section, we list models which generate Bivariate GHg distributions of Types #1-#6 and #8. The authors believe several models to be new. Well known ones are included to show the relationship between parameter values in our notation $\mathrm{F}(\alpha, \beta\,;\lambda\,;\omega)$ and those in alternative expressions of probability functions. Janardan (1973) studied many of them, and there are many publications on Univariate GHg geneses (e.g. Guenther (1975) and Kemp and Kemp (1975)).

The models cover essentially geneses of Multivariate and Univariate distributions. It should be noticed, however, that the differences between #1 and #2 or #5 and #6 do not appear in Univariate case. Models leading to distributions of specific parameter values are not included. Unfortunately, no model leading to Type B1, B2 or C distributions has yet been found.

The discussions in Section 5 show a common genesis of Bivariate GHg. If a parameter of an Ordinary Hg or a Negative Hg distribution is a GHg variable, then a Bivariate GHg distribution is obtained. This Compound Hg or Negative Hg distribution is obvious from Tables 3 and 5, and will not be included in the following models.

In a couple of models we use the occupancy distribution of b indistinguishable balls allocated randomly into c cells. All possible $\binom{b+c-1}{b}$ configurations are assumed to be equiprobable (Bose-Einstein statistics). It should be remarked that the distribution is a special case of Singular c-variate Negative Hg

$$\prod_i \frac{\Gamma(c_i+x_i)}{\Gamma(c_i)x_i\,!} \Big/ \frac{\Gamma(c+b)}{\Gamma(c)b\,!}, \qquad \Sigma c_i=c\,;\,\Sigma b_i=b\,;$$

where $c_1=c_2=\ldots=1$. (S. Kunte (1977) missed to note this fact.) Thus the follow ing geneses based on the occupancy are related to other more popular models.

#1 $F(-\xi, -\eta\,;\,-n\,;\,\zeta)$ or #3 $F(-\xi, -\eta\,;\,-n\,;\,1-m)$.

(i) Pólya's urn model (negative contagion). An urn contains b black, r red and w white balls. We sample at random a ball from the urn, and observing its color take out further $c-1$ balls of the same color. Repeating the procedure n times without replacement, we observe (at random sampling of a single ball) X black balls and Y red balls and therefore $n-X-Y$ white balls with the probability

$$Pr[(X,\,Y)=(x,\,y)]=\frac{n\,!}{x\,!\,y\,!\,(n-x-y)!}\;\frac{\prod_{i=0}^{x-1}(b-ic)\prod_{j=0}^{y-1}(r-jc)\prod_{k=0}^{n-x-y-1}(w-kc)}{\prod_{m=0}^{n-1}(b+r+w-mc)},$$

which is $F(-bc, -rc\,;\,-n\,;\,(w/c)-n+1)$. The model is valid if b, r and w are larger than $(n-1)c$, or are multiples of c. If w/c is an integer less than n, then this is of Type #3. If all b, r and w are multiples of c, then this corresponds to the case $c=1$, the Bivariate Ordinary Hg, and belongs to #1$\wedge$#2 or #3$\wedge$#4 according to the value of w.

(ii) Positive Hg compound Ordinary Hg. Let (X, Y) be a Bivariate Ordinary Hg variable;

$$Pr[(X,\,Y)=(x,\,y)]=\binom{k}{x}\binom{l}{y}\binom{m-k-l}{n-x-y}\Big/\binom{m}{n}.$$

If (k, l) is a Positive Hg variable with probabilities

$$\binom{a}{k}\binom{b}{l}\binom{c-a-b}{m-k-l}\Big/\binom{c}{m},$$

then (X, Y) is a Positive Hg variable with the probabilities

$$\binom{a}{x}\binom{b}{y}\binom{c-a-b}{n-x-y}\Big/\binom{c}{n},$$

and the conditional distribution of $(k-x, l-y)$ given $(X, Y)=(x, y)$ is also Positive Hg with the probabilities

$$\binom{a-x}{k-x}\binom{b-y}{l-y}\binom{c-a-b-n+x+y}{m-k-l-n+x+y}\Big/\binom{c-n}{m-n}.$$

#1$\wedge$#2 F$(-k, -l; -n; m)$ or #3$\wedge$#4 F$(-k, -l; -n; 1-m)$.

(i) 2×3 table. The Bivariate Ordinary Hg is best illustrated by a 2×3 table, which arises in many situations.

x	y	$n-x-y$	n
$b-x$	$r-y$	$w-n+x+y$	m
b	r	w	$b+r+w=n+m$

This is a random division of $b+r+w$ balls into two groups of sizes n and m, or a random division of $n+m$ items of two kinds into three parts of size b, r and w. This table also arises at random matching of items of two sets of $b+r+w=n+m$ items, where one is categorized in two groups and the other in three. The table shows F$(-b, -r; -n; w-n+1)$.

(ii) 2×3 table (contiuned). In a 2×3 table, let the probability of an entry (i, j) be $r_{ij}=p_i q_j$, where $p_1+p_2=q_1+q_2+q_3=1$. Let n_{ij} be the number of observations of the entry (i, j) in N observations. The joint distribution of n_{ij}'s is

$$N! \prod_i \prod_j r_{ij}^{n_{ij}}/n_{ij}! = N!(\prod_i p_i^{n_{i\cdot}})(\prod_j q_j^{n_{\cdot j}})/\prod_i \prod_j n_{ij}!,$$

where $n_{i\cdot}=\Sigma_j n_{ij}$ and $n_{\cdot j}=\Sigma_i n_{ij}$. Then the conditional distribution of $X=n_{11}$ and $Y=n_{12}$ when all the marginal frequencies are given ($n_{1\cdot}=n, n_{2\cdot}=m; n_{\cdot 1}=b, n_{\cdot 2}=r, n_{\cdot 3}=w$) is the same as the cases of finite sampling of (i). The following two models are essentially the same as this, but sampling procedures are different.

(iii) Conditional trinomial distributions. Let (X_i, Y_i), $i=1, 2$, be independent trinomial variables having the same probabilities:

$$Pr[(X_i, Y_i)=(x_i, y_i)]=\frac{x_i!\, y_i!\,(n_i-x_i-y_i)!}{n_i!} p^{x_i} q^{y_i} (1-p-q)^{n_i-x_i-y_i}.$$

The conditional distribution of (X_1, Y_1), given $X_1+X_2=s$ and $Y_1+Y_2=t$, is

$$Pr[(X_1, Y_1)=(x, y)|X_1+X_2=s, Y_1+Y_2=t]=\binom{s}{x}\binom{t}{y}\binom{n_1+n_2-s-t}{n_1-x-y}\bigg/\binom{n_1+n_2}{n_1},$$

which is F$(-s, -t; -n_1; n_2-s-t+1)$.

(iv) Conditional three binomial distributions. Let X_i, $i=1, 2, 3$, be independent binomial variables of the same probability parameter;

$$Pr[X_i=x_i]=\binom{n_i}{x_i} p^{x_i}(1-p)^{n_i-x_i}.$$

The conditional distribution of (X_1, X_2), given $X_1+X_2+X_3=s$, is

$$Pr[(X_1, X_2)=(x, y)|X_1+X_2+X_3=s]=\binom{n_1}{x}\binom{n_2}{y}\binom{n_3}{s-x-y}\Big/\binom{n_1+n_2+n_3}{s},$$

(v) Exceedance. Let $(V_1^{(k)}, \ldots, V_{n_k}^{(k)})$ be three independent random samples of size n_k $(k=1, 2, 3)$ from a common continuous distribution function $F(v)$. Mix all of them and rearrange them in ascending order, and let ξ_{k+1} be the $(k+1)$-st smallest value. The numbers X and Y of V's, which are less than ξ_{k+1} in the first and the second samples respectively, have the probabilities

$$Pr[(X, Y)=(x, y)]=\binom{n_1}{x}\binom{n_2}{y}\binom{n_3}{k-x-y}\Big/\binom{n_1+n_2+n_3}{k}.$$

#2 $F(-k, -l; -\lambda; \eta)$ or #4 $F(-k, -l; -\lambda; 1-m)$.

(i) Urn model. An urn contains b black and w white balls. Sample a ball at random from the urn, observe its color and take out further $c-1$ balls of the same color. Repeat the trial $k+l$ times, and let X and Y be the number of black balls in the first k and the last l trials respectively.

$$Pr[(X, Y)=(x, y)]=\binom{k}{x}\binom{l}{y}\frac{\prod_{i=0}^{x+y}(b-ic)\prod_{i=0}^{k+l-x-y}(w-jc)}{\prod_{m=0}^{k+l}(b+w-mc)},$$

which is $F(-k, -l; -b/c; (w/c)-k-l+1)$. If w/c is an integer less than $k+l$, then this is of Type #4. If b/c is an integer, then this is also of Type #1.

(ii) Two stage lottery. An urn contains g green balls and w white balls. From it $b+r$ persons, one by one, sample a ball and take out further $c-1$ balls of the same color. Let S be the number of persons having taken out green balls. These S persons sample again a ball without replacement from another urn with b black and r red balls. If X persons take a black ball, and Y a red ball $(X+Y=S)$, then

$$Pr[(X, Y)=(x, y)]=\left\{\binom{g/c}{s}\binom{w/c}{b+r-s}\Big/\binom{(g+w)/c}{b+r}\right\}\times\left\{\binom{b}{x}\binom{r}{y}\Big/\binom{b+r}{x}\right\},$$

which is $F(-b, -r; -g/c; (w/c)-b-r+1)$. If w/c is an integer less than $b+r$, then this is of Type #4. When g is a multiple of c, this belongs to $\#1 \wedge \#2$ or $\#3 \wedge \#4$.

#5 $F(\xi, \eta; -n; -\zeta)$. (Sibuya, Yoshimura and Shimizu (1964) mentioned briefly the following models (i)-(iv), (vi) and (viii).)

(i) Pólya's urn model (Positive contagion). From an urn with b black, r red and w white balls, sample a ball at random. Observing its color, replace it with c balls of the same color. Repeating the procedure n times, we observe X black balls and Y red balls $(n-X-Y$ white balls) with the probability

$$Pr[(X,\ Y)=(x,y)]=\frac{x\,!\,y\,!(n-x-y)\,!}{n\,!}\ \frac{\prod_{i=0}^{x-1}(b+ic)\prod_{j=0}^{y-1}(r+jc)\prod_{k=0}^{n-x-y-1}(w+kc)}{\prod_{m=0}^{n-1}(b+r+w+mc)},$$

which is $F(b/c, r/c\,;\ -n\,;\ -(w/c)-n+1)$.

(ii) Doubly inverse sampling. From an urn with b black balls and w white balls, sample balls at random without replacement until $k+1$ white balls are obtained and let S be the number of black balls drawn out. Next, mix up the $k+S$ drawn balls except for the last white ball, and repeat sampling from these without replacement until c $(<k)$ white balls are observed, and let X be the number of black balls taken out at the second stage.

The distribution of S is

$$Pr[S=s]=\binom{-k-1}{s}\binom{-w+k}{b-s}\Big/\binom{-w-1}{b},$$

and under the condition $S=s$, the distribution of X is

$$Pr[X=x|S=s]=\binom{-c}{x}\binom{-k+c-1}{s-x}\Big/\binom{-k-1}{s}.$$

Therefore, the joint distribution of X and $Y=S-X$ is

$$Pr[(X,\ Y)=(x,y)]=\binom{-c}{x}\binom{-k+c-1}{y}\binom{-w+k}{b-x-y}\Big/\binom{-w-1}{b}.$$

which is $F(c, k-c+1\,;\ -b\,;\ -w+k-b+1)$.

(iii) Dirichlet compound multinomial distribution. Compound trinomial distribution

$$\frac{n\,!}{x\,!\,y\,!(n-x-y)\,!}p^x q^y(1-p-q)^{n-x-y}$$

by a Dirichlet distribution

$$\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}p^{a-1}q^{b-1}(1-p-q)^{c-1}$$

to obtain

$$\frac{\Gamma(a+c+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}\ \frac{\Gamma(a+x)\Gamma(b+y)\Gamma(c+n-x-y)}{\Gamma(a+b+c+n-x-y)}\ \frac{n\,!}{x\,!\,y\,!(n-x-y)\,!},$$

which is $F(a,b\,;\ -n\,;\ -c-n+1)$. (Ishii and Hayakawa (1960), Mosimann (1962).)

(iv) Conditional distribution of negative binomials. Let x_i, $i=1,2,3$, be independent negative binomial variables with the same probability parameter;

$$Pr[X_i=x_i]=\frac{\Gamma(k_i+x_i)}{\Gamma(k_i)x_i\,!}p^{k_i}q^{x_i}.$$

The probability distribution of (X_1, X_2) under the given $X_1+X_2+X_3=s$, is

$$Pr[(X_1, Y_2)=(x, y)|X_1+X_2+X_3=s]$$
$$=\frac{\Gamma(k_1+x)}{\Gamma(k_1)x\,!}\ \frac{\Gamma(k_2+y)}{\Gamma(k_2)y\,!}\ \frac{\Gamma(k_3+s-x-y)}{\Gamma(k_3)(s-x-y)\,!}\bigg/\frac{\Gamma(k_1+k_2+k_3+s)}{\Gamma(k_1+k_2+k_3)s\,!},$$

which is $\mathrm{F}(k_1, k_2\,; -s\,; -k_3-s+1)$.

(v) Type #5 GHg compound of Ordinary Hg. Let (X, Y) be a Bivariate Ordinary variable;

$$Pr[(X, Y)=(x, y)]=\binom{k}{x}\binom{l}{y}\binom{m-k-l}{m-x-y}\bigg/\binom{m}{n}.$$

If (k, l) is a Type #5 Hg variable with the probabilities

$$\binom{-a}{k}\binom{-b}{l}\binom{-c+a+b}{m-k-l}\bigg/\binom{-c}{m},$$

then (X, Y) is a Type #5 Hg variable with the probabilities

$$\binom{-a}{x}\binom{-b}{y}\binom{-c+a+b}{n-x-y}\bigg/\binom{-c}{n},$$

and the conditional distribution of $(k-x, l-y)$ given $(X, Y)=(x, y)$ is also Type #5 Hg with the probabilities

$$\binom{-a-x}{k-x}\binom{-b-y}{l-y}\binom{-c+a+b-n+x+y}{m-k-l-n+x+y}\bigg/\binom{-c-n}{m-n}.$$

See Sarndal (1964) and Hoadley (1969).

(vi) Exceedance. Let $(V_1, V_2, \ldots, V_k)$ and $(W_1, W_2, \ldots, W_n)$ be random samples from the same population with a continuous distribution function, and $V_{(1)}\leqq V_{(2)}\leqq\ldots\leqq V_{(k)}$ be order statistics of V's. Let X and Y be the number of W's which are smaller than $V_{(i)}$ and between $V_{(i)}$ and $V_{(j)}$ respectively.

$$Pr[(X, Y)=(x, y)]=\binom{x+i-1}{x}\binom{y+j-i-1}{y}\binom{n-x-y+k-j}{n-x-y}\bigg/\binom{k+n}{n},$$

which is $\mathrm{F}(i, j-i\,; -n\,; -k+j-n)$. (Guenther, 1975).

(vii) Discrete order statistics. From a finite population $(1, 2, \ldots, N)$, take a sample of size n and let $X_{(1)}\leqq X_{(2)}\leqq\ldots\leqq X_{(n)}$ be its order statistics.

$$Pr[(X_{(j)}, X_{(k)})=(x, y)]=\binom{x-1}{j-1}\binom{y-x-1}{k-j-1}\binom{N-y}{k-j-1}\bigg/\binom{N}{n},$$

and

$$Pr[(X_{(j)}-j, X_{(k)}-X_{(j)}-k+j)=(u, v)]=\binom{-j}{u}\binom{-k+j}{v}\binom{-n+k-1}{N-n-u-v}\bigg/\binom{-n-1}{N-n},$$

which is F($j, k-j$; $-N+n$; $-N+k$).

(viii) Occupancy. Allocate b indistinguishable balls at random into c cells (Bose-Einstein statistics), which are grouped into three families of k_1, k_2 and k_3 cells ($k_1+k_2+k_3=c$), and let the numbers of balls in the first and the second families be X and Y respectively.

$$Pr[(X, Y)=(x, y)]=\binom{k_1+x-1}{x}\binom{k_2+y-1}{y}\binom{k_3+b-x-y-1}{b-x-y}\Big/\binom{k_1+k_2+k_3+b-1}{b}$$

which is F(k_1, k_2 ; $-b$; $-k_3-b+1$). The above two models are equivalent to this.

(ix) Conditional distribution in occupancy. There are r red balls and w white balls. They are allocated into $c+1$ cells as if they are indistinguishable (Bose-Einstein statistics). Let the number of red and white balls in the cells be ($X_0, X_1, \ldots, X_c$) and ($Y_0, Y_1, \ldots, Y_c$) respectively. The joint distribution of these is

$$\prod_{i=0}^{c}\binom{x_i+y_i}{x_i}\Big/\frac{(r+w+c)\,!}{r\,!\,w\,!\,c\,!},$$

and the $\binom{r+w+c}{c}$ configurations of possible marginal ($c+1$)-tuples (X_0+Y_0, $X_1+Y_1, \ldots, X_c+Y_c$) are equiprobable. The conditional distribution of X's, when $Y_i=y_i, i=0, 1, \ldots, c$, are given, is

$$\prod_{i=0}^{c}\binom{x_i+y_i}{x_i}\Big/\binom{r+w+c}{r},$$

and when $c=2$, (X_1, X_2) has F(y_1+1, y_2+1 ; $-r$; $-w+y_1+y_2-r$). The above joint distribution arises in an extension of the law of succession. See Sarndal (1965) and Janardan (1968).

#6 F($-k, -l$; λ ; $-\zeta$) (Janardan and Patil (1972) discussed the following (i)-(iii).)

(i) Inverse sampling in Pólya's urn model (negative contagion). From an urn with b black, r red and w white balls, we sample a ball at random without replacement until k white balls are taken out. Let X and Y denote the numbers of black and red balls taken out respectively. Then

$$Pr[(X, Y)=(x, y)]=\frac{(k+x+y-1)\,!}{(k-1)\,!\,x\,!\,y\,!}\;\frac{w^{(k)}b^{(x)}r^{(y)}}{(w+b+r)^{(k+x+y)}},$$

which is F($-b, -r$; k ; $-w-b-r+k$). If further $c-1$ balls of the same color as that of the drawn ball are deleted, and b/c and r/c are integers, then (X, Y) is F($-b/c$, $-r/c$; k ; $-(w+b+r)/c+k$). Thus, only the last parameter can be real in this extension. For white balls $w\geq c(k-1)+1$ is necessary and sufficient. (Steyn (1951)).

(ii) Beta compound of binomials. Compound a simultaneous distribution of two independent binomial distributions with the same probability,

$$\binom{m}{x}p^x q^{m-x}\binom{n}{y}p^y q^{n-y}$$

by a beta distribution

$$\frac{\Gamma(\lambda+\omega)}{\Gamma(\lambda)\Gamma(\omega)}p^{\lambda-1}q^{\omega-1}$$

to get

$$\binom{m}{x}\binom{n}{y}\frac{\Gamma(\lambda+\omega)}{\Gamma(\lambda)\Gamma(\omega)}\frac{\Gamma(\lambda+x+y)\Gamma(\omega+m+n-x-y)}{\Gamma(\lambda+\omega+m+n)},$$

which is $F(-m, -n; \lambda; -\omega-m-n+1)$.

(iii) Conditional distribution of negative trinomials. Let (X_i, Y_i), $i=1,2$, be independent negative trinomial variables of the same probability parameter,

$$Pr[(X_i, Y_i)=(x_i, y_i)]=\frac{\Gamma(k_i+x_i+y_i)}{\Gamma(k_i)x_i!\,y_i!}p^{x_i}q^{y_i}(1-p_i-q_i)^{k_i}.$$

The probability distribution of (X_1, Y_1) under the condition that $X_1+X_2=s$ and $Y_1+Y_2=t$ are given, is

$$Pr[(X_1, Y_1)=(x, y)|X_1+X_2=s, Y_1+Y_2=t]$$
$$=\frac{\Gamma(k_1+x+y)}{\Gamma(k_1)x!\,y!}\frac{\Gamma(k_2+s+t-x-y)}{\Gamma(k_2)(s-x)!\,(t-y)!}\frac{\Gamma(k_1+k_2)s!\,t!}{\Gamma(k_1+k_2+s+t)},$$

which is $F(-s, -t; k_1; -k_2-s-t+1)$.

(iv) Type #6 GHg compound Type #6 GHg. Let (X, Y) be a Type #6 $F(-s, -t; \lambda; -\omega-s-t+1)$ variable;

$$Pr[(X, Y)=(x, y)]=\binom{s}{x}\binom{t}{y}\frac{(\lambda)_{x+y}(\omega)_{s+t-x-y}}{(\lambda+\omega)_{s+t}}.$$

If (s, t) is also a Type #6 variable with the probabilities

$$\binom{u}{s}\binom{v}{t}\frac{(\xi)_{s+t}(\eta)_{u+v-s-t}}{(\xi+\eta)_{u+v}},$$

where $\xi=\lambda+\omega$, then (X, Y) is again a Type #6 variable with the probabilities

$$\binom{u}{x}\binom{v}{y}\frac{(\lambda)_{x+y}(\eta+\omega)_{u+v-x-y}}{(\xi+\eta)_{u+v}},$$

and the conditional distribution of $(s-x, t-y)$ given $(X, Y)=(x, y)$ is also of Type #6

$$\binom{u-x}{s-x}\binom{v-y}{t-y}\frac{(\omega)_{s+t-x-y}(\eta)_{u+v-s-t}}{(\omega+\eta)_{u+v-x-y}},$$

(v) Exceedance. Let $(V_1, \ldots, V_k)$, $(W_1, \ldots, W_m)$ and $(Z_1, \ldots, Z_n)$ be random samples from the same population with a continuous distribution function, and $V_{(1)}\leqq \ldots \leqq V_{(k)}$ be order statistics of V's. Let X and Y be numbers of W's and Z's which are smaller than $V_{(j)}$ respectively.

$$Pr[(W,Z)=(x,y)]=\frac{k!}{(j-1)!\,(k-i)!}\binom{m}{x}\binom{n}{y}\Big/\frac{(k+m+n)!}{(j+x+y-1)!\,(k-j+m+n-x-y)!},$$

with is $\mathrm{F}(-m, -n; j; -k+j-m-n)$.

(vi) Occupancy. There are c cells and b balls which consist of r red balls and w white balls ($b=r+w$). All the b balls are allocated at random indistinguishably into the cells (Bose-Einstein statistics), and X red balls and Y white balls are put into specified k cells ($k<c$) with the probabilities,

$$Pr[(X,Y)=(x,y)]=\left\{\binom{r}{x}\binom{w}{y}\Big/\binom{r+w}{s}\right\}\left\{\binom{k+s-1}{s}\binom{c-k+b-s-1}{b-s}\Big/\binom{c+b-1}{b}\right\},$$

where $s=x+y$. This is $\mathrm{F}(-r, -w; -c+k-b+1)$, and equivalent to the exceedance model (v).

♯8. $\mathrm{F}(\xi, \eta; \lambda; \zeta)$ (The following (i)-(iii) are discussed by Janardan and Patil, 1971.)

(i) Inverse sampling in Pólya's urn model. From an urn with b black, r red and w whith balls, we sample a ball at random and replace it with c balls of the same color. We repeat the procedure until n white balls are observed. Let X and Y be the number of black and red balls observed respectively, then

$$Pr[(X,Y)=(x,y)]=\frac{(n+x+y-1)!}{x!\,y!\,(n-1)!}\,\frac{\prod_{i=0}^{x-1}(b+ic)\prod_{j=0}^{y-1}(r+jc)\prod_{k=0}^{n-1}(w+kc)}{\prod_{m=0}^{n+x+y-1}(b+r+w+mc)}$$

which is $\mathrm{F}(b/c, r/c; n; (b+r+w)/c+n)$.

(ii) Dirichlet compound negative trinomial. Compound a negative multinomial distribution

$$\frac{\Gamma(k+x+y)}{\Gamma(k)x!\,y!}p^x q^y (1-p-q)^{k-1}$$

by a Dirichlet distribution

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}p^{a-1}q^{b-1}(1-p-q)^{c-1}$$

to obtain

$$\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}\,\frac{\Gamma(b+x)\Gamma(a+y)\Gamma(c+k)}{\Gamma(a+b+c+k+x+y)}\,\frac{\Gamma(k+x+y)}{\Gamma(k)x!\,y!},$$

which is $\mathrm{F}(a, b; k; a+b+c+k)$. (Mosimann 1963)

(iii) Beta compound of two negative binomials. Compound a simultaneous distribution of two negative binomial distributions of the same probability parameter;

$$\frac{\Gamma(a+x)}{\Gamma(a)!}p^a q^x \frac{\Gamma(b+y)}{\Gamma(b)y!}p^b q^y$$

by a beta distribution

$$\frac{\Gamma(\omega+\lambda)}{\Gamma(\omega)\Gamma(\lambda)}p^{\omega-1}q^{\lambda-1}$$

to obtain

$$\frac{\Gamma(a+x)\Gamma(b+y)}{\Gamma(a)x\,!\,\Gamma(b)y\,!}\;\frac{\Gamma(\omega+\lambda)}{\Gamma(\omega)\Gamma(\lambda)}\;\frac{\Gamma(\lambda+x+y)\Gamma(\omega+a+b)}{\Gamma(\lambda+\omega+a+b+x+y)},$$

which is $F(a, b\,;\,\lambda\,;\,\lambda+\omega+a+b)$.

(iv) Type #8 GHg compound of Type #6 GHg. Let (X, Y) be a Type #6 F$(-s, -t\,;\,b\,;\,-c+s+t-1)$ variable;

$$Pr[(X, Y)=(x, y)]=\frac{B(b+x+y, c+s+t-x-y)}{B(b, c)}\binom{s}{x}\binom{t}{y}.$$

If (s, t) is a Type #8 F$(\xi, \eta\,;\,\lambda\,;\,\xi+\eta+\lambda+\zeta)$ variable with the probabilities

$$\frac{B(\xi+\eta+\zeta, \lambda+s+t)}{B(\zeta, \lambda)}\;\frac{\Gamma(\xi+s)}{\Gamma(\xi)s\,!}\;\frac{\Gamma(\eta+t)}{\Gamma(\eta)t\,!},$$

where $\lambda=b+c$, then (X, Y) is a Type #8 variable with the probabilities

$$\frac{B(\xi+\eta+\zeta, b+x+y)}{B(\zeta, b)}\;\frac{\Gamma(\xi+x)}{\Gamma(\xi)x\,!}\;\frac{\Gamma(\eta+y)}{\Gamma(\eta)y\,!},$$

which is of Type #8 F$(\xi, \eta\,;\,b\,;\,\xi+\eta+b+\zeta)$, and the conditional probabilities of $(s-x, t-y)$ given $(X, Y)=(x, y)$ are

$$\frac{B(\xi+\eta+x+y+b+\zeta, c+s+t-x-y)}{B(b+\zeta, c)}\;\frac{\Gamma(\xi+s)}{\Gamma(\xi+x)(s-x)!}\;\frac{\Gamma(\eta+t)}{\Gamma(\eta+y)(t-y)!},$$

which is of Type #8 F$(\xi+x, \eta+y\,;\,c\,;\,\xi+\eta+x+y+c+b+\zeta)$, (Janardan (1973).)

(v) Exceedance. In the exceedance model for F$(\xi, \eta\,;\,-n\,;\,\zeta)$, #5 (vi), let the sample size of W_j's be indefinite and continue the observations until the number of W_j's larger than $V_{(j)}$ becomes k.

Let the number of W_j's smaller than $V_{(i)}$ be X, and let the number between $V_{(i)}$ and $V_{(j)}$ be Y. Then

$$Pr[(X, Y)=(x, y)]=\frac{(k+x+y-1)!}{(k-1)\,!\,x\,!\,y\,!}p^x q^y(1-p-q)^k,$$

where $p=F(V_{(i)})$ and $q=F(V_{(j)})-F(V_{(i)})$ follows the Dirichlet distribution

$$\frac{m\,!}{(i-1)\,!\,(j-i-1)\,!\,(m-j)\,!}p^{i-1}q^{j-i-1}(1-p-q)^{m-j}.$$

The distribution of (X, Y) is

$$\frac{m\,!\,(m-j+k)\,!}{(m-j)\,!\,(m+k)\,!}\;\frac{(k-1+x+y)\,!\,(m+k)\,!}{(k-1)\,!\,(m+k+x+y)\,!}\;\frac{(x+i-1)\,!}{(i-1)\,!\,x\,!}\;\frac{(y+j-i-1)\,!}{(j-i-1)\,!\,y\,!},$$

which is $F(i, j; k; m+k+1)$.

(vi) Compounding independent Poisson distributions by a multivariate gamma product-ratio distribution. Let V_α, V_β, V_λ and V_ω be independent gamma variables, all with scale parameter 1 and with shape parameters α, β, λ and ω respectively. The distribution of $(\xi, \eta)=(V_\alpha V_\lambda/V_\omega, V_\beta V_\lambda/V_\omega)$ can be named multivariate (bivariate) gamma product-ratio distribution MGPR$(\alpha, \beta; \lambda; \omega)$. Its probability density is

$$\frac{\Gamma(\lambda+\omega)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\omega)}\xi^{\alpha-1}\eta^{\beta-1}\int_0^\infty e^{-(\xi+\eta)/s}\frac{s^{\lambda-\alpha-\beta-1}}{(1+s)^{\lambda+\omega}}ds$$
$$=\frac{\Gamma(\lambda+\omega)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\omega)}\xi^{\alpha-1}\eta^{\beta-1}\int_0^\infty e^{-(\xi+\eta)t}\frac{t^{\alpha+\beta+\omega-1}}{(1+t)^{\lambda+\omega}}dt\,.$$

If X and Y are independent Poisson variables with expectation ξ and η respectively, and if (ξ, η) is a random vector following MGPR$(\alpha, \beta; \lambda; \omega)$, then a compound Poisson (X, Y) is a $F(\alpha, \beta; \lambda; \alpha+\beta+\lambda+\omega)$ variable. The models (ii) and (iii) can be regarded as other versions of this model. (Sibuya (1980))

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Appendix. Proof of Theorem 2

Case [1]. Distributions on $[m, n]$ or $[m, \infty)$, $1 \leqq m < n$.

There must be a parameter with the value $-m+1$. We partition this into two cases;
[1.1] $F(\alpha, \beta; -m+1)$, and
[1.2] $F(\alpha, -m+1; \gamma)$.
In Case [1.1], (3.2) becomes

$$p(x)=\frac{\Gamma(-m+1-\alpha)\Gamma(-m+1-\beta)\Gamma(x+\alpha)\Gamma(x+\beta)}{\Gamma(-m+1-\alpha-\beta)\Gamma(\alpha)\Gamma(\beta)\Gamma(x+1)\Gamma(x-m+1)}$$

$$=\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(y+\alpha+m)\Gamma(y+\beta+m)}{\Gamma(1-\alpha-\beta-m)\Gamma(\alpha+m)\Gamma(\beta+m)\Gamma(y+m+1)\Gamma(y+1)},$$

where $y=x-m$. This can be a probability distribution $F(\alpha+m, \beta+m; 1+m)$ of $Y=X-m$ on $[0, n-m]$ or $[0, \infty)$ if and only if one of the following three conditions holds because of Theorem 1.

[1.11] $F(-\xi+m, -n+m; 1+m)$, $\xi>n-1$. This is a Type A1 distribution on $[0, n-m]$.
[1.12] $F(-n+\delta, -n+\varepsilon; 1+m)$. This is a Type B1 distribution on $[0, \infty)$.
[1.13] $F(\alpha+m, \beta+m; 1+m)$, $\alpha, \beta>-m>\alpha+\beta-1$. This is a Type B3 distribution on $[0, \infty)$.

$F(-\xi, -n; -m+1)$, which is obtained by right m shift of $F(-\xi+m, -n+m; 1+m)$ of Case [1.11], is also obtained by inversion, $Y=n-X$, of $F(-n, -n+m; \xi-n+1)$, $\xi>n-1$, a Type A1 distribution on $[0, n-m]$.

Case [1.2] is partitioned further as follows.
[1.21] $F(-n, -m+1; -\xi)$ on $[m, n]$, $\xi>n-1$.
[1.22] $F(-\xi, -m+1; -n)$ on $[m, n]$, $\xi>n-1$.
[1.23] $F(-n+\delta, -m+1; -n+\varepsilon)$ on $[m, \infty)$, $0<\delta, \varepsilon<1$.
[1.24] $F(\alpha, -m+1; \gamma)$ on $[m, \infty)$, $\alpha, \gamma>-m$.

Other parameter values do not define a distribution on these intervals because of the property of factorial function discussed in Section 3. In Case [1.21], $p(x)$ is definable if ξ is an integer such that $n \leqq \xi \leqq m+n-2$. Then, however,

$$p(m)=\frac{(n-2)!}{(-\xi+m+n-2)!\,(\xi-n)!\,(n-m)!}\,\frac{n(n-1)}{m(\xi-m+1)}>1.$$

In Case [1.22], $p(x)$ is definable if $\xi=n$ and $m \geqq 2$. Then, however,

$$p(m)=\frac{n!}{m(m-2)!\,(m+1-n)!}>1.$$

In Case [1.23], $p(x)$ is definable only when $\delta=\varepsilon$, and it reduces to a special case of [1.24]. In Case [1.24], $p(x)$ is definable if $\alpha-\gamma$ is an integer such that $m-2 \geqq \alpha-\gamma \geqq 0$ and α is not a nonnegative integer. Then

$$p(m)=(-1)^{\alpha+\gamma+m+1}\frac{\Gamma(\gamma+m-1)}{\Gamma(\gamma-\alpha+m-1)\Gamma(\alpha)}\,\frac{1}{m(\alpha-\gamma)!}\,\frac{\Gamma(m+\alpha)}{\Gamma(m+\gamma)}$$

$$=(-1)^{\alpha+\gamma+m+1}\frac{(\alpha+m-k-2)^{(m-k-1)}}{m(m-k-2)!}\,\frac{(\alpha+m-1)^{(k)}}{k!}$$

where $k=\alpha-\gamma$. (The first factor is $(-1)^{\alpha+\gamma+m}$ if starting from (3.3).) This cannot be between 0 and 1 if $\alpha \geqq 1$ and $k \geqq 1$.

When $\alpha=\gamma$,

$$\sum_{y=0}^{\infty} p(m+y)=(-1)^{m+1}(m-1)(\alpha+m-2)^{(m-1)}\sum_{y=0}^{\infty}\frac{1}{(1+y)_m}$$

$$=(-1)^{m+1}\frac{(\alpha+m-2)^{(m-1)}}{(m-1)!},$$

which is equal to one if $\alpha=\gamma=-m+1$ or $\alpha=\gamma=1$ and m is odd. The case $\alpha=\gamma=-m+1$ is covered by Case [1.1]. F(1, $-m+1$; 1) is, except for the factor $(-1)^{m+1}$, a right shift of a Type B3 distribution F(1, 1; $1+m$) if $m\geqq 2$. However, this is essentially the same as F($1-m$, $1-m$; $1-m$), a special case of [1.13]. The factor $(-1)^{m+1}$ does not appear in the form of the expression (3.3).

Now, for $\alpha>1$, $k=1, 2, 3, \ldots$ and $m=k+3, k+5, \ldots$ ($m=k+2, k+4, \ldots$ for (3.3))

$$\sum_{y=0}^{\infty} p(m+y)=\frac{(\alpha+m-k-2)^{(m-k-1)}}{(m-k-2)!}\frac{(\alpha+k-1)^{(k)}}{m\cdot k!}f(\alpha, \alpha-k; m),$$

where

$$f(\xi, \zeta; m)=\sum_{y=0}^{\infty}\frac{1}{(1+y)_m}\frac{(\xi)_y}{(\zeta)_y},$$

is equal to one for a special value of α determined by a pair (k, m). The values of such α, denoted by $\varepsilon(k, m)$, are numerically computed and listed in Table 2.

The function $f(\alpha, \alpha-k; m)$ is expressed as

$$f(\alpha, \alpha-k; m)=(\alpha-k-1)\sum_{n=0}^{k}\frac{k^{(n)}}{(m-n-1)!\,(m-1)^{(n+1)}(\alpha-1)^{(n+1)}},$$

and

$$\sum_{y=0}^{\infty} p(m+y)=\frac{1}{(m-k-2)!}\sum_{n=0}^{k}\frac{(\alpha-m)_{m-n-1}}{(m-k-1)(n-k)!}.$$

It is shown that the value of α for which this summation is equal to one is uniquely determined in the interval (0,1). It is conjectured that α does not satisfy the condition if α is a negative noninteger, but this is not proved.

Case [2]. Distribution on $[-n, -m]$ or $(-\infty, -m]$, $1\leqq m<n$.

There must be a parameter with the value m. We partition this into two cases;

[2.1] F(α, m; γ), and

[2.2] F(α, β; m).

In Case [2.1],

$$\begin{aligned} p(x)&=\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-m)}{\Gamma(\gamma-\alpha-m)\Gamma(\alpha)\Gamma(m)}\frac{\Gamma(x+\alpha)\Gamma(x+m)}{\Gamma(x+1)\Gamma(x+\gamma)}\\ &=\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-m)}{\Gamma(\gamma-\alpha-m)\Gamma(\alpha)\Gamma(m)}(-1)^{m+1}\frac{\Gamma(m-\alpha+1)}{\Gamma(m-\gamma+1)}\frac{\Gamma(w+m)\Gamma(w+m-\gamma+1)}{\Gamma(w+1)\Gamma(w+m-\alpha+1)}\\ &=-\frac{\Gamma(\gamma-\alpha)\Gamma(1-\alpha)}{\Gamma(\gamma-\alpha-m)\Gamma(m)\Gamma(m-\gamma+1)}\frac{\Gamma(w+m)\Gamma(w+m-\gamma+1)}{\Gamma(w+1)\Gamma(w+m-\alpha+1)}\end{aligned}$$

where $x=-m-w$. Except for the factor (-1), this can be a distribution F(m, $m-\gamma+1$; $m-\alpha+1$) of Type A2 or B3:

[2.11] F(m, $-n+m$; $-\xi+m+1$), $\xi>n>m$ ($\alpha=\xi$, $\gamma=n+1$), Type A2 on $[0, n-m]$.

[2.12] F(m, ξ; ζ), $\zeta>\xi+m$ ($\alpha=1+m-\zeta$, $\gamma=1+m-\xi$), Type B3 on $[0, \infty)$.

The factor (-1) does not appear if we deal with the form of the expression (3.3). The inversion of [2.11], F(ξ, m; $n+1$) on $[-n, -m]$ is also obtained by left shift:

Consider F(α, β; $n+1$), which is another expression of Case [2.2], but we are considering here the interval $[-n, -m]$.

$$\begin{aligned} p(x)&=\frac{\Gamma(n+1-\alpha)\Gamma(n+1-\beta)}{\Gamma(n+1-\alpha-\beta)\Gamma(\alpha-n)\Gamma(\beta)}\frac{\Gamma(x+\alpha)\Gamma(x+\beta)}{\Gamma(x+1)\Gamma(x+n+1)}\\ &=-\frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(1-\alpha-\beta-n)\Gamma(\alpha-n)\Gamma(\beta-n)}\frac{\Gamma(y+\alpha-n)\Gamma(y+\beta-n)}{\Gamma(y+1)\Gamma(y+1-n)},\end{aligned}$$

where $x=-n+y$. Except for the factor (-1), this can be a distribution F($\alpha-n$, $\beta-n$; $1-n$) of Type A2:

$F(\xi-n,\ -n+m;\ 1-n)$, $\xi>n$, $n>m$ on $[0,\ n-m]$.

The factor (-1) does not appear if we deal with the expression (3.3).

In Case [2.2],

$$p(-m)=\frac{\Gamma(m-\alpha)\Gamma(m-\beta)}{\Gamma(m-\alpha-\beta)\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha-m)\Gamma(\beta-m)}{\Gamma(1-m)\Gamma(0)}.$$

This can be definable only if both α and β are integers. To be a distribution on $(-\infty,\ -m]$, the condition $m>\alpha$ and $m>\beta$ must be satisfied. From this, $m-\alpha-\beta\geqq 1$, $\alpha\geqq 1$ and $\beta\geqq 1$ must be satisfied so that $p(-m)$ be definable. Further, $0<p(-m)<1$ is satisfied just by $F(1,\ 1;\ m)$ with $m=3, 5, 7, \ldots$ It is interesting that this is an inversion of $F(1,\ 1;\ m)$ itself, a Type B3 distribution on $[0,\ \infty)$, by $Y=-X-m$ (see Section 4).

Case [3]. There remains a possibility to get a distribution on $(-\infty,\ -1]$ by the following sets of parameters.

[3.1] $F(\alpha,\ \beta;\ \gamma)$, $\alpha,\ \beta,\ \gamma<1$,

[3.2] $F(n+\delta,\ n+\varepsilon;\ \gamma)$, $\gamma<1$, and

[3.3] $F(n+\delta,\ \beta;\ n+\varepsilon)$, $\beta<1$.

Checking $p(-1)$, we see that the distribution is definable only for the parameter values which are covered by Cases [2.1] and [2.2].