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ON THE GENERALIZED HILBERT TRANSFORMS OF FUNCTIONS OF TWO VARIABLES

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ABSTRACT

On the basis of the generalized harmonic analysis of functions of two variables under a restricted limit process, the spectral analysis of the generalized Hilbert transforms of functions of two variables is shown.

§ 1. Introduction

Following the preceding paper of Matsuoka [7], the purpose of this paper is to develop the generalized harmonic analysis of the generalized Hilbert transforms of functions of two variables.

We shall establish the spectral relation between a given function and its generalized Hilbert transforms, and see that the properties of the given function are reflected on those of its generalized Hilbert transforms. Several related theorems are proved.

The method of discussion can be used along the lines of Koizumi [6] in the case of functions of one variable. As for the modified Hilbert transforms, we should also refer to Achieser [1] and Kober [3].

§ 2. Preliminaries

In this section, we show some facts which will be useful later.

First, in a way similar to Koizumi [4], we introduce the generalized Hilbert transforms taken in the "x_1" variable, taken in the "x_2" variable and the double generalized Hilbert transform, defined on functions of two variables as follows:

$$(2.1)\qquad \tilde{f}^{(1)}(x_1, x_2)=\lim_{\varepsilon_1\to 0}\frac{x_1+i}{\pi}\int_{0<\varepsilon_1\leqq|x_1-s|}\frac{f(s, x_2)}{s+i}\frac{ds}{x_1-s},$$

$$(2.2)\qquad \tilde{f}^{(2)}(x_1, x_2)=\lim_{\varepsilon_2\to 0}\frac{x_2+i}{\pi}\int_{0<\varepsilon_2\leqq|x_2-t|}\frac{f(x_1, t)}{t+i}\frac{dt}{x_2-t}$$

and

$$(2.3)\qquad \tilde{f}(x_1, x_2)=\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{(x_1+i)(x_2+i)}{\pi^2}\int_{\substack{0<\varepsilon_1\leq|x_1-s|\\ 0<\varepsilon_2\leq|x_2-t|}}\frac{f(s,t)}{(s+i)(t+i)}\frac{dsdt}{(x_1-s)(x_2-t)},$$

respectively. Here, $\tilde{f}(x_1, x_2)$ is often easy to deal with, because of the obvious formula

$$(2.4)\qquad \tilde{f}(x_1, x_2)=(\tilde{f}^{(1)})^{\sim(2)}(x_1, x_2)=(\tilde{f}^{(2)})^{\sim(1)}(x_1, x_2).$$

Now, we can state

THEOREM 1. If $f(x_1, x_2)/\{(x_1+i)(x_2+i)\}$ belongs to the class $L^2(R^2)$, then $\tilde{f}^{(1)}(x_1, x_2)/\{(x_1+i)(x_2+i)\}$, $\tilde{f}^{(2)}(x_1, x_2)/\{(x_1+i)(x_2+i)\}$ and $\tilde{f}(x_1, x_2)/\{(x_1+i)(x_2+i)\}$ also belong to the same class, and

$$(2.5)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\tilde{f}^{(1)}(s,t)|^2}{(1+s^2)(1+t^2)}dsdt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(s,t)|^2}{(1+s^2)(1+t^2)}dsdt,$$

$$(2.6)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\tilde{f}^{(2)}(s,t)|^2}{(1+s^2)(1+t^2)}dsdt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(s,t)|^2}{(1+s^2)(1+t^2)}dsdt$$

and

$$(2.7)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\tilde{f}(s,t)|^2}{(1+s^2)(1+t^2)}dsdt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(s,t)|^2}{(1+s^2)(1+t^2)}dsdt.$$

As for details, see Cotlar [2] and Sokol-Sokolowski [8].

Next, we introduce several integrals of the Cauchy type, the Poisson type and the conjugate Poisson type. These are defined by the formulas

$$\begin{cases}(2.8)\qquad (C^{(1)}f)(z, x_2)=\dfrac{z+i}{2\pi i}\displaystyle\int_{-\infty}^{\infty}\frac{f(s,x_2)}{s+i}\frac{ds}{s-z},\\ (2.9)\qquad (C^{(2)}f)(x_1, w)=\dfrac{w+i}{2\pi i}\displaystyle\int_{-\infty}^{\infty}\frac{f(x_1,t)}{t+i}\frac{dt}{t-w},\\ (2.10)\qquad (Cf)(z,w)=(C^{(1)}C^{(2)}f)(z,w)=(C^{(2)}C^{(1)}f)(z,w),\end{cases}$$

$$\begin{cases}(2.11)\qquad (P^{(1)}f)(z, x_2)=\dfrac{z+i}{\pi}\displaystyle\int_{-\infty}^{\infty}\frac{f(s,x_2)}{s+i}\frac{y_1ds}{(s-x_1)^2+y_1^2},\\ (2.12)\qquad (P^{(2)}f)(x_1, w)=\dfrac{w+i}{\pi}\displaystyle\int_{-\infty}^{\infty}\frac{f(x_1,t)}{t+i}\frac{y_2dt}{(t-x_2)^2+y_2^2},\\ (2.13)\qquad (Pf)(z,w)=(P^{(1)}P^{(2)}f)(z,w)=(P^{(2)}P^{(1)}f)(z,w),\end{cases}$$

and

$$\begin{cases} (Q^{(1)}f)(z, x_2) = -\dfrac{z+i}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{f(s, x_2)}{s+i} \frac{(s-x_1)ds}{(s-x_1)^2+y_1^2}, & (2.14) \\ (Q^{(2)}f)(x_1, w) = -\dfrac{w+i}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{f(x_1, t)}{t+i} \frac{(t-x_2)dt}{(t-x_2)^2+y_2^2}, & (2.15) \\ (Qf)(z, w) = (Q^{(1)}Q^{(2)}f)(z, w) = (Q^{(2)}Q^{(1)}f)(z, w), & (2.16) \end{cases}$$

where $z=x_1+iy_1, y_1>0$ and $w=x_2+iy_2, y_2>0$. Moreover, among these integrals, the following relations are valid:

$$(2.17) \qquad 2(C^{(1)}f)(z, x_2) = (P^{(1)}f)(z, x_2) + i(Q^{(1)}f)(z, x_2),$$

$$(2.18) \qquad 2(C^{(2)}f)(x_1, w) = (P^{(2)}f)(x_1, w) + i(Q^{(2)}f)(x_1, w)$$

and

$$(2.19) \qquad 4(Cf)(z, w) = (Pf)(z, w) - (Qf)(z, w) + i\{(Q^{(1)}P^{(2)}f)(z, w) + (P^{(1)}Q^{(2)}f)(z, w)\}.$$

Combining these integrals and the generalized Hilbert transforms, we have the properties in the following two theorems. Let us write $C_+ = \{z=x+iy : y>0\}$ and $C_+^2 = \{(z, w) = (x_1+iy_1, x_2+iy_2) : y_1, y_2>0\}$.

THEOREM 2. If $f(x_1, x_2)/\{(x_1+i)(x_2+i)\}$ belongs to the class $L^2(R^2)$, then

$$(2.20) \qquad (Q^{(1)}P^{(2)}f)(z, w) = (P\tilde{f}^{(1)})(z, w),$$

$$(2.21) \qquad (P^{(1)}Q^{(2)}f)(z, w) = (P\tilde{f}^{(2)})(z, w)$$

and

$$(2.22) \qquad (Qf)(z, w) = (P\tilde{f})(z, w)$$

for every $(z, w) \in C_+^2$.

Proof. These are consequences of the following well-known equalities:

$$(2.23) \qquad (Q^{(1)}f)(z, x_2) = (P^{(1)}\tilde{f}^{(1)})(z, x_2)$$

for every $z \in C_+$ and almost every $x_2 \in (-\infty, \infty)$, and

$$(2.24) \qquad (Q^{(2)}f)(x_1, w) = (P^{(2)}\tilde{f}^{(2)})(x_1, w)$$

for almost every $x_1 \in (-\infty, \infty)$ and every $w \in C_+$.

THEOREM 3. If $f(x_1, x_2)/\{(x_1+i)(x_2+i)\}$ belongs to the class $L^2(R^2)$, then

$$(2.25) \qquad (C^{(1)}P^{(2)}f)(z, w) = (C^{(1)}P^{(2)}(i\tilde{f}^{(1)}))(z, w),$$

$$(2.26) \qquad (P^{(1)}C^{(2)}f)(z, w) = (P^{(1)}C^{(2)}(i\tilde{f}^{(2)}))(z, w)$$

and

$$(2.27) \qquad (Cf)(z, w) = (C(-\tilde{f}))(z, w)$$

for every $(z, w)\in C_+^2$.

Proof. These are consequences of the following equalities:

$$(C^{(1)}f)(z, x_2)=(C^{(1)}(i\tilde{f}^{(1)}))(z, x_2) \tag{2.28}$$

for every $z\in C_+$ and almost every $x_2\in(-\infty, \infty)$, and

$$(C^{(2)}f)(x_1, w)=(C^{(2)}(i\tilde{f}^{(2)}))(x_1, w) \tag{2.29}$$

for almost every $x_1\in(-\infty, \infty)$ and every $w\in C_+$ (see Theorem 37 of Koizumi [6]).

THEOREM 4. If $f(x_1, x_2)/\{(x_1+i)(x_2+i)\}$ belongs to the class $L^2(R^2)$, then

$$(S)\text{-}\lim_{C_+^2\ni(z,w)\to(x_1,x_2)}(Pf)(z, w)=f(x_1, x_2) \tag{2.30}$$

for almost every $(x_1, x_2)\in R^2$, where notation (S) indicates the limit in the sense of Stoltz (i. e., as a nontangential limit), and

$$\lim_{y_1,y_2\to+0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|(Pf)(x_1+iy_1, x_2+iy_2)-f(x_1, x_2)|^2}{(1+x_1^2)(1+x_2^2)}dx_1dx_2=0\,. \tag{2.31}$$

Proof. (2.30) is well known. As for (2.31), letting $z=x_1+iy_1, y_1>0$ and $w=x_2+iy_2, y_2>0$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|(Pf)(z, w)-f(x_1, x_2)|^2}{(1+x_1^2)(1+x_2^2)}dx_1dx_2$$

$$\leqq 4^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx_1dx_2}{(1+x_1^2)(1+x_2^2)}\left|\frac{(x_1+i)(x_2+i)}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(s,t)}{(s+i)(t+i)}\right.$$

$$\left.\cdot\frac{y_1ds}{(s-x_1)^2+y_1^2}\,\frac{y_2dt}{(t-x_2)^2+y_2^2}-f(x_1, x_2)\right|^2$$

$$+4^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|(z+i)-(x_1+i)|^2}{1+x_1^2}$$

$$\cdot\left|\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(s,t)}{(s+i)(t+i)}\,\frac{y_1ds}{(s-x_1)^2+y_1^2}\,\frac{y_2dt}{(t-x_2)^2+y_2^2}\right|^2dx_1dx_2$$

$$+4^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|(w+i)-(x_2+i)|^2}{1+x_2^2}\left|\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\,{\prime\prime}\,)dsdt\right|^2dx_1dx_2$$

$$+4^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|(z+i)-(x_1+i)|^2|(w+i)-(x_2+i)|^2}{(1+x_1^2)(1+x_2^2)}\left|\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\,{\prime\prime}\,)dsdt\right|^2dx_1dx_2$$

$=P_1+P_2+P_3+P_4$, say.

Now, by Theorem 3.22 of Stein and Weiss [9, p. 60],

$$P_1=o(1)\quad\text{as}\quad y_1, y_2\to+0\,.$$

By Jensen's inequality and Fubini's theorem,

$$P_2 \leqq 4^2 y_1^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|f(s,t)|^2}{(1+s^2)(1+t^2)} dsdt \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{1+x_1^2} \cdot \frac{y_1 dx_1}{(s-x_1)^2+y_1^2} \frac{y_2 dx_2}{(t-x_2)^2+y_2^2}$$

$$\leqq 4^2 y_1^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|f(s,t)|^2}{(1+s^2)(1+t^2)} dsdt$$

$$= o(1) \quad \text{as} \quad y_1, y_2 \to +0 .$$

Similarly,

$$P_3 \leqq 4^2 y_2^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|f(s,t)|^2}{(1+s^2)(1+t^2)} dsdt$$

$$= o(1) \quad \text{as} \quad y_1, y_2 \to +0 ,$$

and

$$P_4 \leqq 4^2 y_1^2 y_2^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|f(s,t)|^2}{(1+s^2)(1+t^2)} dsdt$$

$$= o(1) \quad \text{as} \quad y_1, y_2 \to +0 .$$

Thus, (2.31) is established.

Therefore, from Theorems 2 and 4, we can easily obtain

THEOREM 5. If $f(x_1, x_2)/\{(x_1+i)(x_2+i)\}$ belongs to the class $L^2(R^2)$, then

$$(2.32) \qquad (S)\text{-}\lim_{C_+^2 \ni (z,w) \to (x_1,x_2)} (Q^{(1)}P^{(2)}f)(z,w) = \tilde{f}^{(1)}(x_1, x_2) ,$$

$$(2.33) \qquad (S)\text{-}\lim_{C_+^2 \ni (z,w) \to (x_1,x_2)} (P^{(1)}Q^{(2)}f)(z,w) = \tilde{f}^{(2)}(x_1, x_2) ,$$

$$(2.34) \qquad (S)\text{-}\lim_{C_+^2 \ni (z,w) \to (x_1,x_2)} (Qf)(z,w) = \tilde{f}(x_1, x_2)$$

for almost every $(x_1, x_2) \in R^2$, and

$$(2.35) \qquad \lim_{y_1,y_2 \to +0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|(Q^{(1)}P^{(2)}f)(x_1+iy_1, x_2+iy_2) - \tilde{f}^{(1)}(x_1,x_2)|^2}{(1+x_1^2)(1+x_2^2)} dx_1 dx_2 = 0 ,$$

$$(2.36) \qquad \lim_{y_1,y_2 \to +0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|(P^{(1)}Q^{(2)}f)(x_1+iy_1, x_2+iy_2) - \tilde{f}^{(2)}(x_1,x_2)|^2}{(1+x_1^2)(1+x_2^2)} dx_1 dx_2 = 0 ,$$

$$(2.37) \qquad \lim_{y_1,y_2 \to +0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|(Qf)(x_1+iy_1, x_2+iy_2) - \tilde{f}(x_1,x_2)|^2}{(1+x_1^2)(1+x_2^2)} dx_1 dx_2 = 0 .$$

Furthermore, from Theorems 3, 4 and 5, the following theorem holds.

THEOREM 6. If $f(x_1, x_2)/\{(x_1+i)(x_2+i)\}$ belongs to the class $L^2(R^2)$, then

$$(\tilde{f}^{(1)})^{\sim(1)}(x_1, x_2) = -f(x_1, x_2), \tag{2.38}$$

$$(\tilde{f}^{(2)})^{\sim(2)}(x_1, x_2) = -f(x_1, x_2) \tag{2.39}$$

and

$$(\tilde{f})^{\sim}(x_1, x_2) = f(x_1, x_2) \tag{2.40}$$

for almost every $(x_1, x_2) \in R^2$.

Finally, from these theorems, we can show the following principal result of this section.

THEOREM 7. Suppose that $f(x_1, x_2)/\{(x_1+i)(x_2+i)\}$ belongs to the class $L^2(R^2)$ and put

$$h^{(1)}(z, w) = 2(C^{(1)}P^{(2)}f)(z, w), \tag{2.41}$$

$$h^{(2)}(z, w) = 2(P^{(1)}C^{(2)}f)(z, w), \tag{2.42}$$

$$h(z, w) = 4(Cf)(z, w) \tag{2.43}$$

for every $(z, w) \in C_+^2$. Then

$$(S)\text{-}\lim_{C_+^2 \ni (z,w) \to (x_1, x_2)} h^{(1)}(z, w) = f(x_1, x_2) + i\tilde{f}^{(1)}(x_1, x_2) \equiv h^{(1)}(x_1, x_2), \tag{2.44}$$

$$(S)\text{-}\lim_{C_+^2 \ni (z,w) \to (x_1, x_2)} h^{(2)}(z, w) = f(x_1, x_2) + i\tilde{f}^{(2)}(x_1, x_2) \equiv h^{(2)}(x_1, x_2), \tag{2.45}$$

$$\begin{aligned}(S)\text{-}\lim_{C_+^2 \ni (z,w) \to (x_1, x_2)} h(z, w) &= f(x_1, x_2) - \tilde{f}(x_1, x_2) \\ &\quad + i\{\tilde{f}^{(1)}(x_1, x_2) + \tilde{f}^{(2)}(x_1, x_2)\} \equiv h(x_1, x_2)\end{aligned} \tag{2.46}$$

for almost every $(x_1, x_2) \in R^2$,

$$\lim_{y_1, y_2 \to +0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|h^{(1)}(x_1+iy_1, x_2+iy_2) - h^{(1)}(x_1, x_2)|^2}{(1+x_1^2)(1+x_2^2)} dx_1 dx_2 = 0, \tag{2.47}$$

$$\lim_{y_1, y_2 \to +0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|h^{(2)}(x_1+iy_1, x_2+iy_2) - h^{(2)}(x_1, x_2)|^2}{(1+x_1^2)(1+x_2^2)} dx_1 dx_2 = 0, \tag{2.48}$$

$$\lim_{y_1, y_2 \to +0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|h(x_1+iy_1, x_2+iy_2) - h(x_1, x_2)|^2}{(1+x_1^2)(1+x_2^2)} dx_1 dx_2 = 0, \tag{2.49}$$

and

$$h^{(1)}(z, w) = (C^{(1)}P^{(2)}h^{(1)})(z, w) = (Ph^{(1)})(z, w), \tag{5.50}$$

$$h^{(2)}(z, w) = (P^{(1)}C^{(2)}h^{(2)})(z, w) = (Ph^{(2)})(z, w), \tag{2.51}$$

$$h(z, w) = (Ch)(z, w) = (Ph)(z, w) \tag{2.52}$$

for every $(z, w) \in C_+^2$.

§ 3. The spectral analysis of the generalized Hilbert transforms

We begin with the following notations, which will be used in what follows (see Matsuoka [7]):

(a) $W(R^2)=\left\{f(x_1, x_2)\in L^2_{loc}(R^2):\ \sup_{0<S,T<\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2 dsdt<\infty\right\}$;

(b) The double generalized Fourier transform.

$$\begin{aligned} s(u,v;f) &= \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\left[\int_1^A+\int_{-A}^{-1}\right]\left[\int_1^A+\int_{-A}^{-1}\right] f(s,t)\frac{e^{-ius}}{-is}\frac{e^{-ivt}}{-it}dsdt \\ &+ \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\left[\int_1^A+\int_{-A}^{-1}\right]\int_{-1}^{1} f(s,t)\frac{e^{-ius}-1}{-is}\frac{e^{-ivt}}{-it}dsdt \\ &+ \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-1}^{1}\left[\int_1^A+\int_{-A}^{-1}\right] f(s,t)\frac{e^{-ius}}{-is}\frac{e^{-ivt}-1}{-it}dsdt \\ &+ \frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1} f(s,t)\frac{e^{-ius}-1}{-is}\frac{e^{-ivt}-1}{-it}dsdt; \end{aligned}$$

(c) $\Delta_{\varepsilon,\eta}s(u,v;f)=s(u+\varepsilon, v+\eta;f)-s(u-\varepsilon, v+\eta;f)$
$\qquad -s(u+\varepsilon, v-\eta;f)+s(u-\varepsilon, v-\eta;f)$;

(d) The notations $\mathcal{R}_1$ and $\mathcal{R}_2$ mean that a limit exists and has the same limit for every positive constant C whenever S and T tend to infinity in such a way that $S=CT$ and whenever ε and η tend to zero in such a way that $\eta=C\varepsilon$, respectively;

(e) $\phi(x_1, x_2;f)=\mathcal{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}f(x_1+s, x_2+t)\overline{f(s,t)}dsdt$;

(f) $S(R^2)=\{f(x_1, x_2)\in W(R^2):\ \phi(x_1, x_2;f)$ exists for all $(x_1, x_2)\in R^2\}$;

(g) $S'(R^2)=\{f(x_1, x_2)\in S(R^2):\ \phi(x_1, x_2;f)$ is continuous on $R^2\}$.

In this section, we shall determine the spectral relation between any given function which belongs to the class $W(R^2)$ and its generalized Hilbert transforms.

It is to be noted that whenever $f(x_1, x_2)$ belongs to the class $W(R^2)$, by Theorem 1 of Matsuoka [7], the generalized Hilbert transforms $\tilde{f}^{(1)}(x_1, x_2)$, $\tilde{f}^{(2)}(x_1, x_2)$ and $\tilde{f}(x_1, x_2)$ are defined, and the double generalized Fourier transforms of these are also defined.

First, we state the following two fundamental theorems, the proofs of which are given in § 4.

THEOREM 8. If $f(x_1, x_2)$ belongs to the class $W(R^2)$, then for any given positive numbers ε, η,

(i) when $|u|>\varepsilon$,

$$\Delta_{\varepsilon,\eta}s(u, v; \tilde{f}^{(1)})=(-i \operatorname{sgn} u)\Delta_{\varepsilon,\eta}s(u, v; f); \tag{3.1}$$

(ii) when $|u|\leqq\varepsilon$,

$$\Delta_{\varepsilon,\eta}s(u, v; \tilde{f}^{(1)})=i\Delta_{\varepsilon,\eta}s(u, v; f)+2r_1(u+\varepsilon, v; f)+2r_2(u+\varepsilon, v; f), \tag{3.2}$$

and

(i) when $|v|>\eta$,

$$\Delta_{\varepsilon,\eta}s(u, v; \tilde{f}^{(2)})=(-i \operatorname{sgn} v)\Delta_{\varepsilon,\eta}s(u, v; f); \tag{3.3}$$

(ii) when $|v|\leqq\eta$,

$$\Delta_{\varepsilon,\eta}s(u, v; \tilde{f}^{(2)})=i\Delta_{\varepsilon,\eta}s(u, v; f)+2r_3(u, v+\eta; f)+2r_4(u, v+\eta; f), \tag{3.4}$$

where

$$r_1(u, v; f)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s, t)}{s+i}\frac{e^{-ius}-1}{-is}\frac{2\sin \eta t}{t}e^{-ivt}dsdt, \tag{3.5}$$

$$r_2(u, v; f)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s, t)}{s+i}\frac{2\sin \eta t}{t}e^{-i(us+vt)}dsdt, \tag{3.6}$$

$$r_3(u, v; f)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s, t)}{t+i}\frac{e^{-ivt}-1}{-it}\frac{2\sin \varepsilon s}{s}e^{-ius}dsdt, \tag{3.7}$$

$$r_4(u, v; f)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s, t)}{t+i}\frac{2\sin \varepsilon s}{t}e^{-i(us+vt)}dsdt. \tag{3.8}$$

THEOREM 9. If $f(x_1, x_2)$ belongs to the class $W(R^2)$, then for any given positive numbers ε, η,

(i) when $|u|>\varepsilon$, $|v|>\eta$,

$$\Delta_{\varepsilon,\eta}s(u, v; \tilde{f})=(-i \operatorname{sgn} u)(-i \operatorname{sgn} v)\Delta_{\varepsilon,\eta}s(u, v; f); \tag{3.9}$$

(ii) when $|u|\leqq\varepsilon$, $|v|>\eta$,

$$\Delta_{\varepsilon,\eta}s(u, v; \tilde{f})=(-i \operatorname{sgn} v)\{i\Delta_{\varepsilon,\eta}s(u, v; f)+2r_1(u+\varepsilon, v; f)+2r_2(u+\varepsilon, v; f)\}; \tag{3.10}$$

(iii) when $|u|>\varepsilon$, $|v|\leqq\eta$,

$$\Delta_{\varepsilon,\eta}s(u, v; \tilde{f})=(-i \operatorname{sgn} u)\{i\Delta_{\varepsilon,\eta}s(u, v; f)+2r_3(u, v+\eta; f)+2r_4(u, v+\eta; f)\}; \tag{3.11}$$

(iv) when $|u|\leqq\varepsilon$, $|v|\leqq\eta$,

$$\begin{aligned}\Delta_{\varepsilon,\eta}s(u, v; \tilde{f})=&-\Delta_{\varepsilon,\eta}s(u, v; f)\\&+2i\{r_1(u+\varepsilon, v; f)+r_2(u+\varepsilon, v; f)+r_3(u, v+\eta; f)+r_4(u, v+\eta; f)\}\\&+4\{r_5(u+\varepsilon, v+\eta; f)+r_6(u+\varepsilon, v+\eta; f)+r_7(u+\varepsilon, v+\eta; f)+r_8(u+\varepsilon, v+\eta; f)\},\end{aligned} \tag{3.12}$$

where

$$(3.13)\qquad r_5(u,v\,;f)=\underset{B\to\infty}{\text{l.i.m.}}\,\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{(s+i)(t+i)}\,\frac{e^{-ius}-1}{-is}\,\frac{e^{-ivt}-1}{-it}\,dsdt\,,$$

$$(3.14)\qquad r_6(u,v\,;f)=\underset{B\to\infty}{\text{l.i.m.}}\,\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{(s+i)(t+i)}\,\frac{e^{-ivt}-1}{-it}\,e^{-ius}dsdt\,,$$

$$(3.15)\qquad r_7(u,v\,;f)=\underset{B\to\infty}{\text{l.i.m.}}\,\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{(s+i)(t+i)}\,\frac{e^{-ius}-1}{-is}\,e^{-ivt}dsdt\,,$$

$$(3.16)\qquad r_8(u,v\,;f)=\underset{B\to\infty}{\text{l.i.m.}}\,\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{(s+i)(t+i)}\,e^{-i(us+vt)}dsdt\,.$$

Next, we introduce the following class of functions: By $S_0(R^2)$, we denote the class of functions such that $f(x_1,x_2)\in W(R^2)$ and

$$(3.17)\qquad \mathcal{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt$$

exists. Then we shall show the following two theorems.

THEOREM 10. Let $f(x_1,x_2)$ belong to the class $S_0(R^2)$. If we assume that

$$(\mathrm{M}_1)\qquad \mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to0}\frac{1}{8\pi\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\varepsilon}^{\varepsilon}|\Delta_{\varepsilon,\eta}s(u,v\,;f)|^2dudv=0$$

and that

(M_2) there exists a constant k_1^f such that

$$\mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to0}\frac{1}{8\pi\varepsilon\eta}\int_{-\infty}^{\infty}\int_{0}^{2\varepsilon}\left|\underset{B\to\infty}{\text{l.i.m.}}\,\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{s+i}\,\frac{2\sin\eta t}{t}\right.$$
$$\left.\cdot e^{-i(us+vt)}dsdt-\pi k_1^f\chi_\eta(v)\right|^2dudv=0\,,^*$$

then the generalized Hilbert transform $\tilde{f}^{(1)}(x_1,x_2)$ also belongs to the same class $S_0(R^2)$ and

$$(3.18)\qquad \mathcal{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|\tilde{f}^{(1)}(s,t)|^2dsdt$$
$$=\mathcal{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt+|k_1^f|^2\,.$$

And, if we assume that

$$(\mathrm{M}_3)\qquad \mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to0}\frac{1}{8\pi\varepsilon\eta}\int_{-\eta}^{\eta}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s(u,v\,;f)|^2dudv=0$$

and that

* Hereafter, by χ_a, we denote the characteristic function of an interval $(-a,a)$.

(M_4) there exists a constant k_2^f such that

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{8\pi\varepsilon\eta}\int_0^{2\eta}\int_{-\infty}^{\infty}\left|\underset{B\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{t+i}\,\frac{2\sin\varepsilon s}{s}\right.$$
$$\left.\cdot e^{-i(us+vt)}dsdt-\pi k_2^f\chi_\varepsilon(u)\right|^2 dudv=0\,,$$

then the generalized Hilbert transform $\tilde{f}^{(2)}(x_1, x_2)$ also belongs to the same class $S_0(R^2)$ and

(3.19)
$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|\tilde{f}^{(2)}(s,t)|^2dsdt$$
$$=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt+|k_2^f|^2\,.$$

THEOREM 11. Let $f(x_1, x_2)$ belong to the class $S_0(R^2)$. If we assume the conditions (M_1)-(M_4), and, in addition, that

(M_5)
$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{4\varepsilon\eta}\int_{-\eta}^{\eta}\int_{0}^{2\varepsilon}\left|\underset{B\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{s+i}\,\frac{2\sin\eta t}{t}\right.$$
$$\left.\cdot e^{-i(us+vt)}dsdt\right|^2 dudv=0\,,$$

(M_6)
$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{4\varepsilon\eta}\int_{0}^{2\eta}\int_{-\varepsilon}^{\varepsilon}\left|\underset{B\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{t+i}\,\frac{2\sin\varepsilon s}{s}\right.$$
$$\left.\cdot e^{-i(us+vt)}dsdt\right|^2 dudv=0$$

and that

(M_7) there exists a constant k_3^f such that

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{4\varepsilon\eta}\int_{0}^{2\eta}\int_{0}^{2\varepsilon}\left|\underset{B\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{(s+i)(t+i)}\right.$$
$$\left.\cdot e^{-i(us+vt)}dsdt-\frac{\pi}{2}k_3^f\right|^2 dudv=0\,,$$

then the double generalized Hilbert transform $\tilde{f}(x_1, x_2)$ also belongs to the same class $S_0(R^2)$ and

(3.20)
$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|\tilde{f}(s,t)|^2dsdt$$
$$=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt+|k_1^f|^2+|k_2^f|^2+|k_3^f|^2\,.$$

By Theorem 3 of Matsuoka [7], in order to prove Theorems 10 and 11, it suffices to verify the following two theorems, respectively.

THEOREM 12. If $f(x_1, x_2)$ satisfies the hypotheses of Theorem 10, then we have

$$\mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Delta_{\varepsilon,\eta}s(u, v\,;\tilde{f}^{(1)})|^2 du dv \tag{3.21}$$
$$= \mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Delta_{\varepsilon,\eta}s(u, v\,;f)|^2 du dv + |k_1^f|^2$$

and

$$\mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Delta_{\varepsilon,\eta}s(u, v\,;\tilde{f}^{(2)})|^2 du dv \tag{3.22}$$
$$= \mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Delta_{\varepsilon,\eta}s(u, v\,;f)|^2 du dv + |k_2^f|^2 .$$

THEOREM 13. If $f(x_1, x_2)$ satisfies the hypotheses of Theorem 11, then we have

$$\mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Delta_{\varepsilon,\eta}s(u, v\,;\tilde{f})|^2 du dv \tag{3.23}$$
$$= \mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Delta_{\varepsilon,\eta}s(u, v\,;f)|^2 du dv + |k_1^f|^2 + |k_2^f|^2 + |k_3^f|^2 .$$

It is clear that

$$S'(R^2) \subset S(R^2) \subset S_0(R^2) \subset W(R^2) . \tag{3.24}$$

Then we shall show the following two theorems.

THEOREM 14. Let $f(x_1, x_2)$ belong to the class $S(R^2)$. If we assume the conditions (M_1) and (M_2), then the generalized Hilbert transform $\tilde{f}^{(1)}(x_1, x_2)$ also belongs to the same class $S(R^2)$ and

$$\phi(x_1, x_2\,;\tilde{f}^{(1)}) = \phi(x_1, x_2\,;f) + |k_1^f|^2 . \tag{3.25}$$

And, if we assume the conditions (M_3) and (M_4), then the generalized Hilbert transform $\tilde{f}^{(2)}(x_1, x_2)$ also belongs to the same class $S(R^2)$ and

$$\phi(x_1, x_2\,;\tilde{f}^{(2)}) = \phi(x_1, x_2\,;f) + |k_2^f|^2 . \tag{3.26}$$

THEOREM 15. Let $f(x_1, x_2)$ belong to the class $S(R^2)$. If we assume the conditions (M_1)-(M_7), then the double generalized Hilbert transform $\tilde{f}(x_1, x_2)$ also belongs to the same class $S(R^2)$ and

$$\phi(x_1, x_2\,;\tilde{f}) = \phi(x_1, x_2\,;f) + |k_1^f|^2 + |k_2^f|^2 + |k_3^f|^2 . \tag{3.27}$$

Finally, by Theorems 14 and 15, we obtain the following two theorems, respectively.

THEOREM 16. Let $f(x_1, x_2)$ belong to the class $S'(R^2)$. If we assume the conditions (M_1) and (M_2), then the generalized Hilbert transform $\tilde{f}^{(1)}(x_1, x_2)$ also belongs to the same class $S'(R^2)$ and (3.25) is true. And, if we assume the conditions (M_3) and (M_4), then the generalized Hilbert transform $\tilde{f}^{(2)}(x_1, x_2)$ also belongs to the same class $S'(R^2)$ and (3.26) is true.

THEOREM 17. Let $f(x_1, x_2)$ belong to the class $S'(R^2)$. If we assume the conditions (M_1)-(M_7), then the double generalized Hilbert transform $\tilde{f}(x_1, x_2)$ also belongs to the same class $S'(R^2)$ and (3.27) is true.

The above theorems are proved in the following section.

§4. Proofs of the theorems of §3

First, in order to prove Theorems 8 and 9, we show several lemmas which are due to Koizumi [6].

LEMMA 1. If $f(x_1, x_2)$ belongs to the class $W(R^2)$, then

$$\text{(4.1)} \quad \underset{y_1,y_2\to+0}{\text{l.i.m.}}\ \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}(C^{(1)}P^{(2)}f)(x_1+iy_1, x_2+iy_2) \cdot\frac{2\sin\varepsilon x_1}{x_1}\frac{2\sin\eta x_2}{x_2}e^{-i(ux_1+vx_2)}dx_1dx_2 = \frac{1}{2}\{\Delta_{\varepsilon,\eta}s(u,v\,;f)+i\Delta_{\varepsilon,\eta}s(u,v\,;\tilde{f}^{(1)})\},$$

$$\text{(4.2)} \quad \underset{y_1,y_2\to+0}{\text{l.i.m.}}\ \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}(P^{(1)}C^{(2)}f)(x_1+iy_1, x_2+iy_2) \cdot\frac{2\sin\varepsilon x_1}{x_1}\frac{2\sin\eta x_2}{x_2}e^{-i(ux_1+vx_2)}dx_1dx_2 = \frac{1}{2}\{\Delta_{\varepsilon,\eta}s(u,v\,;f)+i\Delta_{\varepsilon,\eta}s(u,v\,;\tilde{f}^{(2)})\}$$

and

$$\text{(4.3)} \quad \underset{y_1,y_2\to+0}{\text{l.i.m.}}\ \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}(Cf)(x_1+iy_1, x_2+iy_2) \cdot\frac{2\sin\varepsilon x_1}{x_1}\frac{2\sin\eta x_2}{x_2}e^{-i(ux_1+vx_2)}dx_1dx_2 = \frac{1}{4}[\Delta_{\varepsilon,\eta}s(u,v\,;f)-\Delta_{\varepsilon,\eta}s(u,v\,;\tilde{f}) +i\{\Delta_{\varepsilon,\eta}s(u,v\,;\tilde{f}^{(1)})+\Delta_{\varepsilon,\eta}s(u,v\,;\tilde{f}^{(2)})\}].$$

Proof. These follow immediately from Theorem 7 and the Plancherel theorem.

LEMMA 2. We have for $y>0$,

$$\text{(4.4)}\qquad \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon x}{x}\,\frac{e^{-iux}}{s-(x+iy)}dx$$

$$=\begin{cases}\sqrt{2\pi}ie^{-i(s-iy)u}\dfrac{e^{i(s-iy)\varepsilon}-e^{-i(s-iy)\varepsilon}}{i(s-iy)} & (u>\varepsilon)\\[2ex] \sqrt{2\pi}ie^{-i(s-iy)u}\dfrac{e^{i(s-iy)u}-e^{-i(s-iy)\varepsilon}}{i(s-iy)} & (-\varepsilon\leqq u\leqq\varepsilon)\\[2ex] 0 & (u<-\varepsilon)\end{cases}$$

and

$$\text{(4.5)}\qquad \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon x}{x}\,\frac{y}{(s-x)^2+y^2}e^{-iux}dx$$

$$=\begin{cases}\sqrt{\dfrac{\pi}{2}}e^{-i(s-iy)u}\dfrac{e^{i(s-iy)\varepsilon}-e^{-i(s-iy)\varepsilon}}{i(s-iy)} & (u>\varepsilon)\\[2ex] \sqrt{\dfrac{\pi}{2}}\left\{e^{-i(s-iy)u}\dfrac{e^{i(s-iy)u}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}\right. \\[2ex] \qquad\left.+e^{-i(s+iy)u}\dfrac{e^{i(s+iy)\varepsilon}-e^{i(s+iy)u}}{i(s+iy)}\right\} & (-\varepsilon\leqq u\leqq\varepsilon)\\[2ex] \sqrt{\dfrac{\pi}{2}}e^{-i(s+iy)u}\dfrac{e^{i(s+iy)\varepsilon}-e^{-i(s+iy)\varepsilon}}{i(s+iy)} & (u<-\varepsilon).\end{cases}$$

Proof. From the well-known formulas

$$\text{(4.6)}\qquad \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon x}{x}e^{-iux}dx=\sqrt{2\pi}\chi_\varepsilon(u),$$

$$\text{(4.7)}\qquad \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{e^{-iux}}{s-(x+iy)}dx=i\frac{1+\operatorname{sgn}u}{2}\sqrt{2\pi}e^{-i(s-iy)u}\qquad (y>0)$$

and

$$\text{(4.8)}\qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{y}{(s-x)^2+y^2}e^{-iux}dx=\sqrt{\frac{\pi}{2}}e^{-i(su-iy|u|)}\qquad (y>0),$$

we have (4.4) and (4.5).

LEMMA 3. We have for $0<y<1$ and $-\varepsilon\leqq u\leqq\varepsilon$,

$$\text{(4.9)}\qquad \left|\frac{e^{i(s-iy)\varepsilon}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}\right|\leqq A_1\left\{\left|\frac{\sin\varepsilon s}{s}\right|+\frac{1}{1+|s|}\right\}$$

and

$$(4.10)\qquad \left.\begin{array}{l}\left|\dfrac{e^{i(s-iy)u}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}-\dfrac{e^{isu}-e^{-is\varepsilon}}{is}\right| \\ \left|\dfrac{e^{i(s+iy)\varepsilon}-e^{i(s+iy)u}}{i(s+iy)}-\dfrac{e^{is\varepsilon}-e^{isu}}{is}\right|\end{array}\right\}\leqq A_2\left\{\frac{1}{(1+|s|)^2}+\frac{1}{1+|s|}\right\},$$

where the constants A_1 and A_2 depend only on ε (as for detailed calculations, refer to Lemma 49_5 of Koizumi [6]).

Proof of THEOREM 8. Suppose that $f(x_1, x_2)\in W(R^2)$, $B>0$, and $z=x_1+iy_1$, $y_1>0$ and $w=x_2+iy_2$, $y_2>0$. For each $(x_1, x_2)\in R^2$, let

$$(4.11)\qquad f_B(x_1, x_2)=\begin{cases} f(x_1, x_2) & (|x_1|\leqq B, |x_2|\leqq B) \\ 0 & (\text{elsewhere}). \end{cases}$$

Then, using Fubini's theorem and (4.6), we have

$$(4.12)\qquad \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}(C^{(1)}P^{(2)}f_B)(z, w)\frac{2\sin\varepsilon x_1}{x_1}\frac{2\sin\eta x_2}{x_2}e^{-i(ux_1+vx_2)}dx_1dx_2$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_B(s, t)dsdt\left\{\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi i}\int_{-A}^{A}\frac{2\sin\varepsilon x_1}{x_1}\frac{e^{-iux_1}}{s-z}dx_1\right.$$

$$\left.\cdot\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\pi}\int_{-A}^{A}\frac{2\sin\eta x_2}{x_2}\frac{w+i}{t+i}\frac{y_2}{(t-x_2)^2+y_2^2}e^{-ivx_2}dx_2\right\}$$

$$+\frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f_B(s, t)}{s+i}dsdt\,\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\pi}\int_{-A}^{A}\frac{2\sin\eta x_2}{x_2}$$

$$\cdot\frac{w+i}{t+i}\frac{y_2}{(t-x_2)^2+y_2^2}e^{-ivx_2}dx_2\cdot\chi_\varepsilon(u).$$

Now, we prove the first part.

In the case of (i), by (4.12) and Lemma 2, we have immediately

$$\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}(C^{(1)}P^{(2)}f_B)(z, w)\frac{2\sin\varepsilon x_1}{x_1}\frac{2\sin\eta x_2}{x_2}e^{-i(ux_1+vx_2)}dx_1dx_2$$

$$=\frac{1+\operatorname{sgn} u}{2}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_B(s, t)\frac{e^{i(s-iy_1)\varepsilon}-e^{-i(s-iy_1)\varepsilon}}{i(s-iy_1)}$$

$$\cdot\frac{e^{i(t-iy_2)\eta}-e^{-i(t-iy_2)\eta}}{i(t-iy_2)}e^{-(y_1u+y_2v)}e^{-i(us+vt)}dsdt$$

$$+o(1)\quad\text{as}\quad y_1, y_2\to+0\,(L^2(R^2)).$$

While, from Lemma 3,

$$\underset{y_1,y_2\to+0}{\text{l.i.m.}}\ f_B(s, t)\frac{e^{i(s-iy_1)\varepsilon}-e^{-i(s-iy_1)\varepsilon}}{i(s-iy_1)}\frac{e^{i(t-iy_2)\eta}-e^{-i(t-iy_2)\eta}}{i(t-iy_2)}$$

$$=f_B(s,t)\frac{2\sin\varepsilon s}{s}\frac{2\sin\eta t}{t}\,.$$

Consequently, using the Plancherel theorem and (4.1) of Lemma 1, we obtain

$$\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}\frac{1}{2}\{f_B(x_1,x_2)+i\tilde{f}_B^{(1)}(x_1,x_2)\}$$

$$\cdot\frac{2\sin\varepsilon x_1}{x_1}\frac{2\sin\eta x_2}{x_2}e^{-i(ux_1+vx_2)}dx_1dx_2$$

$$=\frac{1+\operatorname{sgn}u}{2}\cdot\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}f(s,t)\frac{2\sin\varepsilon s}{s}\frac{2\sin\eta t}{t}e^{-i(us+vt)}dsdt\,.$$

It follows from the Plancherel theorem and Theorem 1, therefore, that

$$\frac{1}{2}\{\Delta_{\varepsilon,\eta}s(u,v\,;f)+i\Delta_{\varepsilon,\eta}s(u,v\,;\tilde{f}^{(1)})\}=\frac{1+\operatorname{sgn}u}{2}\Delta_{\varepsilon,\eta}s(u,v\,;f)\,,$$

which implies (3.1).

In the case of (ii), by the same argument as in (i), we obtain

$$\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}\frac{1}{2}\{f_B(x_1,x_2)+i\tilde{f}_B^{(1)}(x_1,x_2)\}$$

$$\cdot\frac{2\sin\varepsilon x_1}{x_1}\frac{2\sin\eta x_2}{x_2}e^{-i(ux_1+vx_2)}dx_1dx_2$$

$$=\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}f(s,t)\frac{e^{-i(u+\varepsilon)s}-1}{-is}\frac{2\sin\eta t}{t}e^{-ivt}dsdt$$

$$+\frac{i}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{s+i}\frac{2\sin\eta t}{t}e^{-ivt}dsdt$$

$$=\frac{i}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{s+i}\frac{e^{-i(u+\varepsilon)s}-1}{-is}\frac{2\sin\eta t}{t}e^{-ivt}dsdt$$

$$+\frac{i}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{s+i}\frac{2\sin\eta t}{t}e^{-i\{(u+\varepsilon)s+vt\}}dsdt\,.$$

Therefore,

$$\frac{1}{2}\{\Delta_{\varepsilon,\eta}s(u,v\,;f)+i\Delta_{\varepsilon,\eta}s(u,v\,;\tilde{f}^{(1)})\}=ir_1(u+\varepsilon,v\,;f)+ir_2(u+\varepsilon,v\,;f)\,,$$

which implies (3.2). Thus, the first part of Theorem 8 is proved.

Similarly, the second part of Theorem 8 is proved by (4.2) of Lemma 1.

Proof of THEOREM 9. In a way similar to the proof of Theorem 8, we prove the theorem.

Suppose that $f(x_1,x_2)\in W(R^2)$, $B>0$, and $z=x_1+iy_1$, $y_1>0$ and $w=x_2+iy_2$, $y_2>0$.

For each $(x_1, x_2)\in R^2$, let $f_B(x_1, x_2)$ be defined by (4.11). Then, using Fubini's theorem and (4.6),

$$\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}(Cf_B)(z,w)\frac{2\sin\varepsilon x_1}{x_1}\frac{2\sin\eta x_2}{x_2}e^{-i(ux_1+vx_2)}dx_1dx_2$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_B(s,t)dsdt\,\underset{A\to\infty}{\mathrm{l.i.m.}}\left(\frac{1}{2\pi i}\right)^2\int_{-A}^{A}\int_{-A}^{A}\frac{2\sin\varepsilon x_1}{x_1}\frac{2\sin\eta x_2}{x_2}\cdot\frac{e^{-i(ux_1+vx_2)}}{(s-z)(t-w)}dx_1dx_2$$

$$+\frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f_B(s,t)}{s+i}dsdt\,\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{2\pi i}\int_{-A}^{A}\frac{2\sin\eta x_2}{x_2}\frac{e^{-ivx_2}}{t-w}dx_2\cdot\chi_\varepsilon(u)$$

$$+\frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f_B(s,t)}{t+i}dsdt\,\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{2\pi i}\int_{-A}^{A}\frac{2\sin\varepsilon x_1}{x_1}\frac{e^{-iux_1}}{s-z}dx_1\cdot\chi_\eta(v)$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f_B(s,t)}{(s+i)(t+i)}dsdt\cdot\chi_\varepsilon(u)\chi_\eta(v)\,.$$

Therefore, by the Plancherel theorem, Theorem 1, (4.3) of Lemma 1, and Lemmas 2 and 3, we obtain

$$\frac{1}{4}[\Delta_{\varepsilon,\eta}s(u,v\,;f)-\Delta_{\varepsilon,\eta}s(u,v\,;\tilde{f})+i\{\Delta_{\varepsilon,\eta}s(u,v\,;\tilde{f}^{(1)})+\Delta_{\varepsilon,\eta}s(u,v\,;\tilde{f}^{(2)})\}]$$

$$=\begin{cases}\dfrac{(1+\mathrm{sgn}\,u)(1+\mathrm{sgn}\,v)}{4}\Delta_{\varepsilon,\eta}s(u,v\,;f) & (|u|>\varepsilon,\ |v|>\eta)\\[2mm] i\dfrac{1+\mathrm{sgn}\,v}{2}\{r_1(u+\varepsilon,v\,;f)+r_2(u+\varepsilon,v\,;f)\} & (|u|\leqq\varepsilon,\ |v|>\eta)\\[2mm] i\dfrac{1+\mathrm{sgn}\,u}{2}\{r_3(u,v+\eta\,;f)+r_4(u,v+\eta\,;f)\} & (|u|>\varepsilon,\ |v|\leqq\eta)\\[2mm] -\{r_5(u+\varepsilon,v+\eta\,;f)+r_6(u+\varepsilon,v+\eta\,;f)+r_7(u+\varepsilon,v+\eta\,;f)+r_8(u+\varepsilon,v+\eta\,;f)\} & \\ \qquad (|u|\leqq\varepsilon,\ |v|\leqq\eta)\,, & \end{cases}$$

which implies (3.9)-(3.12) by Theorem 8. This completes the proof of Theorem 9.

Next, we prove Theorems 12 and 13. Before proving these, we observe the following lemma.

LEMMA 4. Let $f(x_1, x_2)$ belong to the class $W(R^2)$. If we assume the conditions (M_2) and (M_4), then we have respectively

$$\mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{8\pi\varepsilon\eta}\int_{-\infty}^{\infty}\int_{0}^{2\varepsilon}|r_1(u,v\,;f)|^2dudv=0 \tag{4.13}$$

and

(4.14) $$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{8\pi\varepsilon\eta}\int_0^{2\eta}\int_{-\infty}^{\infty}|r_3(u,v;f)|^2dudv=0\,.$$

And, if we assume the condition (M_7), then we have

(4.15) $$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{4\varepsilon\eta}\int_0^{2\eta}\int_0^{2\varepsilon}|r_5(u,v;f)|^2dudv=0\,,$$

(4.16) $$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{4\varepsilon\eta}\int_0^{2\eta}\int_0^{2\varepsilon}|r_6(u,v;f)|^2dudv=0$$

and

(4.17) $$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{4\varepsilon\eta}\int_0^{2\eta}\int_0^{2\varepsilon}|r_7(u,v;f)|^2dudv=0\,.$$

(For $r_1(u,v;f)$-$r_8(u,v;f)$ in the present lemma and its proof, see Theorems 12 and 13.)

Proof. By the Schwartz inequality, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\int_0^{2\varepsilon}&|r_1(u,v;f)|^2dudv\\&=\int_{-\infty}^{\infty}\int_0^{2\varepsilon}\left|\int_0^u r_2(\xi,v;f)d\xi\right|^2dudv\\&\leqq\int_{-\infty}^{\infty}\int_0^{2\varepsilon}u\left\{\int_0^u|r_2(\xi,v;f)|^2d\xi\right\}dudv\\&=\int_{-\infty}^{\infty}\left\{\int_0^{2\varepsilon}|r_2(\xi,v;f)|^2d\xi\int_\xi^{2\varepsilon}udu\right\}dv\\&\leqq 2\varepsilon^2\int_{-\infty}^{\infty}\int_0^{2\varepsilon}|r_2(u,v;f)|^2dudv\,.\end{aligned}$$

Similarly, we obtain

$$\int_0^{2\eta}\int_{-\infty}^{\infty}|r_3(u,v;f)|^2dudv\leqq 2\eta^2\int_0^{2\eta}\int_{-\infty}^{\infty}|r_4(u,v;f)|^2dudv$$

and also

$$\int_0^{2\eta}\int_0^{2\varepsilon}|r_5(u,v;f)|^2dudv\leqq(2\varepsilon^2)(2\eta^2)\int_0^{2\eta}\int_0^{2\varepsilon}|r_8(u,v;f)|^2dudv\,,$$

$$\int_0^{2\eta}\int_0^{2\varepsilon}|r_6(u,v;f)|^2dudv\leqq 2\eta^2\int_0^{2\eta}\int_0^{2\varepsilon}|r_8(u,v;f)|^2dudv\,,$$

$$\int_0^{2\eta}\int_0^{2\varepsilon}|r_7(u,v;f)|^2dudv\leqq 2\varepsilon^2\int_0^{2\eta}\int_0^{2\varepsilon}|r_8(u,v;f)|^2dudv\,.$$

Thus, the lemma follows easily by using the conditions (M_2), (M_4) and (M_7).

Proof of THEOREM 12. By Theorem 8, (4.13) of Lemma 4, and the conditions (M_1) and (M_2), we have immediately

$$\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s(u,v\,;\tilde{f}^{(1)})|^2dudv$$

$$=\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{|u|>\varepsilon}|\Delta_{\varepsilon,\eta}s(u,v\,;f)|^2dudv$$

$$+\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{|u|\leqq\varepsilon}|\Delta_{\varepsilon,\eta}s(u,v\,;f)$$

$$-2ir_1(u+\varepsilon,v\,;f)-2ir_2(u+\varepsilon,v\,;f)|^2dudv$$

$$=\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s(u,v\,;f)|^2dudv$$

$$+\frac{1}{4\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{0}^{2\varepsilon}|r_2(u,v\,;f)|^2dudv+o(1)$$

$$=\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s(u,v\,;f)|^2dudv+|k_1^f|^2$$

$$+o(1) \quad \text{as} \quad \varepsilon,\eta\to 0\ (\mathscr{R}_2),$$

which gives us (3.21). Similarly, we have (3.22) by (4.14) of Lemma 4, and the conditions (M_3) and (M_4).

Proof of THEOREM 13. By applying the same argument as in the proof of Theorem 12, it then follows from Theorem 9, Lemma 4 and the conditions (M_1)-(M_7) that

$$\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s(u,v\,;\tilde{f})|^2dudv$$

$$=\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s(u,v\,;f)|^2dudv+|k_1^f|^2+|k_2^f|^2+|k_3^f|^2$$

$$+o(1) \quad \text{as} \quad \varepsilon,\eta\to 0\ (\mathscr{R}_2).$$

This concludes the proof of Theorem 13.

Finally, we prove Theorems 14 and 15.

Proof of THEOREM 14. Let w be any real or complex number such as $|w|=1$. Then, from Theorem 8, (4.13) of Lemma 4, and the conditions (M_1) and (M_2), it is easy to see that

$$(4.18)\qquad \frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{2+we^{-i(ux_1+vx_2)}+\bar{w}e^{i(ux_1+vx_2)}\}$$

$$\cdot|\Delta_{\varepsilon,\eta}s(u,v;\tilde{f}^{(1)})|^2dudv$$

$$=\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{2+we^{-i(ux_1+vx_2)}+\bar{w}e^{i(ux_1+vx_2)}\}$$

$$\cdot|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv$$

$$+|k_1^f|^2\cdot\frac{1}{4\varepsilon\eta}\int_{-\eta}^{\eta}\int_{-\varepsilon}^{\varepsilon}\{2+we^{-i(ux_1+vx_2)}+\bar{w}e^{i(ux_1+vx_2)}\}dudv$$

$$+o(1)\quad\text{as}\quad \varepsilon,\eta\to 0\,(\mathcal{R}_2).$$

Hence, taking w in (4.18) successively to equal ± 1, $\pm i$, and combining four expressions, we have

$$\mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(ux_1+vx_2)}|\Delta_{\varepsilon,\eta}s(u,v;\tilde{f}^{(1)})|^2dudv$$

$$=\mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(ux_1+vx_2)}|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv+|k_1^f|^2.$$

Thus, by Theorem 6 of Matsuoka [7], (3.25) is proved. Similarly, using (4.14) of Lemma 4, and the conditions (M_3) and (M_4), (3.26) is proved. This completes the proof of theorem.

Applying the same argument that was used in the proof of Theorem 14, we can also prove Theorem 15 by Theorem 9, Lemma 4 and the conditions (M_1)-(M_7).

Remark. In this paper, in order to study the spectral analysis of the generalized Hilbert transforms of functions of two variables, we used the generalized harmonic analysis of functions of two variables which was established by Matsuoka [7]. Its limit process, therefore, also depended on the limit process involved in the above generalized harmonic analysis. On the other hand, the generalized harmonic analysis of functions of two variables is also obtained under the unrestricted rectangular mean concerning the double limit process. Thus, if we use this generalized harmonic analysis, then the spectral analysis of the generalized Hilbert transforms of functions of two variables is also obtained under the above limit process instead of the restricted limit process.

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