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GENERALIZED HARMONIC ANALYSIS OF FUNCTIONS OF TWO VARIABLES

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ABSTRACT

On the basis of the Wiener formula of functions of two variables under a restricted limit process, a new approach to the generalized harmonic analysis of functions of two variables is shown.

§ 1. Introduction

Wiener [10] established the generalized harmonic analysis for the analysis of paths of the Brownian motion. Wiener and A. C. Berry also proved the case of functions of two variables (*Wiener* [10]). As is well-known, it is based on the so-called Wiener formula. They use the circular mean concerning the double limit process. It seems to be something restricted.

Now, it is worthwhile to consider a possibility of approaching this problem by a more relaxed limit process. Recently, Anzai, Koizumi and Matsuoka [1] proved the two-dimensional Wiener formula under a restricted rectangular mean concerning the double limit process. In this paper, we shall establish the generalized harmonic analysis of functions of two variables under this restricted limit process.

The proofs can be done along the similar lines as in Wiener [11].

§ 2. Definitions and theorems

First, we introduce the following class of functions.

DEFINITION 1. By $W(R^2)$, we denote the class of functions such that $f(x_1, x_2) \in L^2_{loc}(R^2)$ and

$$\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s, t)|^2 ds dt \tag{2.1}$$

is bounded in $S, T>0$.

Then we shall prove

THEOREM 1. If $f(x_1, x_2)$ belongs to the class $W(R^2)$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(s,t)|^2}{(1+s^2)(1+t^2)}\,dsdt<\infty\,. \tag{2.2}$$

Next, we introduce the double generalized Fourier transform due to Wiener [10, 11]. This is defined by

$$\begin{aligned} s(u,v;f)=&\underset{A\to\infty}{\mathrm{l.i.m.}}\,\frac{1}{2\pi}\left[\int_1^A+\int_{-A}^{-1}\right]\left[\int_1^A+\int_{-A}^{-1}\right]f(s,t)\frac{e^{-ius}}{-is}\frac{e^{-ivt}}{-it}\,dsdt\\ &+\underset{A\to\infty}{\mathrm{l.i.m.}}\,\frac{1}{2\pi}\left[\int_1^A+\int_{-A}^{-1}\right]\int_{-1}^{1}f(s,t)\frac{e^{-ius}-1}{-is}\frac{e^{-ivt}}{-it}\,dsdt\\ &+\underset{A\to\infty}{\mathrm{l.i.m.}}\,\frac{1}{2\pi}\int_{-1}^{1}\left[\int_1^A+\int_{-A}^{-1}\right]f(s,t)\frac{e^{-ius}}{-is}\frac{e^{-ivt}-1}{-it}\,dsdt\\ &+\frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1}f(s,t)\frac{e^{-ius}-1}{-is}\frac{e^{-ivt}-1}{-it}\,dsdt\,. \end{aligned} \tag{2.3}$$

If $f(x_1, x_2)$ belongs to the class $W(R^2)$, then by Theorem 1 and the Plancherel theorem, the double generalized Fourier transform $s(u, v; f)$ is defined and we have

$$\begin{aligned} \Delta_{\varepsilon,\eta}s(u,v;f)&=s(u+\varepsilon,v+\eta;f)-s(u-\varepsilon,v+\eta;f)-s(u+\varepsilon,v-\eta;f)+s(u-\varepsilon,v-\eta;f)\\ &=\underset{A\to\infty}{\mathrm{l.i.m.}}\,\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}f(s,t)\,\frac{2\sin\varepsilon s}{s}\,\frac{2\sin\eta t}{t}\,e^{-i(us+vt)}dsdt \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv\\ &\quad=\frac{1}{\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(s,t)|^2\,\frac{\sin^2\varepsilon s}{s^2}\,\frac{\sin^2\eta t}{t^2}\,dsdt\,. \end{aligned} \tag{2.5}$$

On the other hand, Anzai, Koizumi and Matsuoka [1] proved the following two-dimensional Wiener formula in the first quadrant R_+^2 of the plane.

THEOREM 2. Suppose that $f(x_1, x_2)\geqq 0$ in $(x_1, x_2)\in R_+^2$, $f(x_1, x_2)\in L^1_{\mathrm{loc}}(R_+^2)$ and

$$\frac{1}{ST}\int_0^T\int_0^S f(s,t)dsdt \tag{2.6}$$

is bounded in $S, T>0$. Then the limit relations

$$\lim_{T\to\infty}\frac{1}{C_1C_2T^2}\int_0^{C_2T}\int_0^{C_1T}f(x_1,x_2)dx_1dx_2=A\quad(\forall C_1, C_2>0) \tag{2.7}$$

and

$$\lim_{\varepsilon\to 0}\frac{4C_1C_2}{\pi^2\varepsilon^2}\int_0^{\infty}\int_0^{\infty}f(x_1,x_2)\frac{\sin^2\varepsilon C_1^{-1}x_1}{x_1^2}\frac{\sin^2\varepsilon C_2^{-1}x_2}{x_2^2}\,dx_1dx_2=A\quad(\forall C_1, C_2>0) \tag{2.8}$$

are equivalent in the sense that if either of the limits (2.7) or (2.8) exists, then the other limit exists and assumes the same value.

By the way, if we put S, T, ε, η and C instead of C_1T, C_2T, εC_1^{-1}, εC_2^{-1} and C_1/C_2 respectively, then $S=CT$ and $\eta=C\varepsilon$ hold. Hence, (2.7) and (2.8) of Theorem 2 are equivalent to the statements

$$\text{“} \lim_{S,T\to\infty}\frac{1}{ST}\int_0^T\int_0^S f(s,t)dsdt$$

exists and has the same limit for every positive constant C whenever S and T tend to infinity in such a way that $S=CT$ ” and

$$\text{“} \lim_{\varepsilon,\eta\to 0}\frac{4}{\pi^2\varepsilon\eta}\int_0^\infty\int_0^\infty f(s,t)\frac{\sin^2\varepsilon s}{s^2}\frac{\sin^2\eta t}{t^2}dsdt$$

exists and has the same limit for every positive constant C whenever ε and η tend to zero in such a way that $\eta=C\varepsilon$ ” respectively.

For the sake of simplicity, we shall from now on use the notations $\mathcal{R}_1$ and $\mathcal{R}_2$ instead of denoting the above limit processes respectively. And we shall refer to these limits as the restricted limits. Therefore, using the notations $\mathcal{R}_1$ and $\mathcal{R}_2$, Theorem 2 is rewritten as follows:

THEOREM 2′. If $f(x_1, x_2)$ satisfies the hypotheses of Theorem 2, and either of the limits

$$\mathcal{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{ST}\int_0^T\int_0^S f(s,t)dsdt \tag{2.9}$$

or

$$\mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{4}{\pi^2\varepsilon\eta}\int_0^\infty\int_0^\infty f(s,t)\frac{\sin^2\varepsilon s}{s^2}\frac{\sin^2\eta t}{t^2}dsdt \tag{2.10}$$

exists, then the other limit exists and assumes the same value.

From this theorem and (2.5), we get

THEOREM 3. If $f(x_1, x_2)$ belongs to the class $W(R^2)$, then we have

$$\begin{aligned}\mathcal{R}_1\text{-}\lim_{S,T\to\infty}&\frac{1}{4ST}\int_{-T}^T\int_{-S}^S |f(s,t)|^2dsdt\\ &=\mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^\infty\int_{-\infty}^\infty|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv,\end{aligned} \tag{2.11}$$

in the sense that if either side of (2.11) exists, the other side exists and assumes the same value.

Now, we introduce the following two classes of functions which are due to Wiener [10, 11].

DEFINITION 2. By $S(R^2)$, we denote the class of functions such that $f(x_1, x_2) \in W(R^2)$ and

$$\phi(x_1, x_2; f) = \mathcal{R}_1\text{-}\lim_{S,T\to\infty} \frac{1}{4ST} \int_{-T}^{T} \int_{-S}^{S} f(x_1+s, x_2+t)\overline{f(s, t)}\,ds\,dt \tag{2.12}$$

exists for all $(x_1, x_2) \in R^2$.

DEFINITION 3. By $S'(R^2)$, we denote the class of functions such that $f(x_1, x_2) \in S(R^2)$ and $\phi(x_1, x_2; f)$ is continuous on R^2.

$S'(R^2)$ is a proper subclass of $S(R^2)$. In other words, there is a function $\phi(x_1, x_2; f)$ defined by (2.12) which is not continuous on R^2. For example, take $f(x_1, x_2) = \exp\{i(x_1^2 + x_2^2)\}$. Then

$$\begin{aligned}\phi(x_1, x_2; f) &= \mathcal{R}_1\text{-}\lim_{S,T\to\infty} \frac{1}{4ST} \int_{-T}^{T} \int_{-S}^{S} e^{i((x_1+s)^2+(x_2+t)^2)} e^{-i(s^2+t^2)}\,ds\,dt \\ &= \begin{cases} 1 & ((x_1, x_2) = (0, 0)) \\ 0 & (\text{elsewhere}). \end{cases}\end{aligned}$$

Here, we shall consider the properties of functions of $S(R^2)$ and $S'(R^2)$.

THEOREM 4. If $f(x_1, x_2)$ belongs to the class $S(R^2)$, then

$$|\phi(x_1, x_2; f)| \leqq \phi(0, 0; f) \tag{2.13}$$

for all $(x_1, x_2) \in R^2$.

THEOREM 5. Suppose that $f(x_1, x_2)$ belongs to the class $S(R^2)$ and $\phi(x_1, x_2; f)$ is continuous at $(x_1, x_2) = (0, 0)$. Then $\phi(x_1, x_2; f)$ is continuous at every point of R^2 and $f(x_1, x_2)$ belongs to the class $S'(R^2)$.

THEOREM 6. If $f(x_1, x_2)$ belongs to the class $S(R^2)$, then

$$\phi(x_1, x_2; f) = \mathcal{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(ux_1+vx_2)} |\Delta_{\varepsilon,\eta} s(u, v; f)|^2\,du\,dv. \tag{2.14}$$

THEOREM 7. If $f(x_1, x_2)$ belongs to the class $S(R^2)$, then it will belong to the class $S'(R^2)$ when and only when both

$$\lim_{A\to\infty} \mathcal{R}_2\text{-}\overline{\lim_{\varepsilon,\eta\to 0}} \frac{1}{16\pi^2\varepsilon\eta} \int_{-\infty}^{\infty} \left[\int_{A}^{\infty} + \int_{-\infty}^{-A}\right] |\Delta_{\varepsilon,\eta} s(u, v; f)|^2\,du\,dv = 0 \tag{2.15}$$

and

$$\lim_{A\to\infty} \mathcal{R}_2\text{-}\overline{\lim_{\varepsilon,\eta\to 0}} \frac{1}{16\pi^2\varepsilon\eta} \left[\int_{A}^{\infty} + \int_{-\infty}^{-A}\right] \int_{-\infty}^{\infty} |\Delta_{\varepsilon,\eta} s(u, v; f)|^2\,du\,dv = 0 \tag{2.16}$$

are true.

Finally, we shall show Theorem 8 below concerning the spectral representation of the correlation function $\phi(x_1, x_2; f)$ of $f(x_1, x_2)$. For this, we first observe the following two well-known theorems for positive definite functions. Now, let $\mathscr{B}_2$ be the Borel field in R^2.

Theorem (Bochner's representation theorem).* If $\phi(x_1, x_2)$ is a positive definite function, then there exists a measure μ on $(R^2, \mathscr{B}_2)$ such that $\mu(R^2)=\phi(0, 0)$, and

$$\phi(x_1, x_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\xi_1 x_1+\xi_2 x_2)}d\mu(\xi_1, \xi_2) \tag{2.17}$$

holds.

Theorem (Lévy's inversion formula). The measure μ of (2.17) of Bochner's representation theorem is represented, by $\phi(x_1, x_2)$, as follows: If $I=(\alpha_1, \alpha_2]\times(\beta_1, \beta_2]$ is a finite continuity interval of μ, then we have

$$\mu(I)=\lim_{A\to\infty}\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}\frac{e^{-i\alpha_2 x_1}-e^{-i\alpha_1 x_1}}{-ix_1}\frac{e^{-i\beta_2 x_2}-e^{-i\beta_1 x_2}}{-ix_2}\phi(x_1, x_2)dx_1dx_2\,. \tag{2.18}$$

Thus, the positive definite function $\phi(x_1, x_2)$ determines uniquely the measure μ.

We second observe the following definition.

Definition 4. A function $A(u, v)$ is called monotone increasing if

$$A(u+h, v+k)-A(u, v+k)-A(u+h, v)+A(u, v)\geqq 0 \tag{2.19}$$

for all $h, k\geqq 0$.

Then we can state

Theorem 8. If $f(x_1, x_2)$ belongs to the class $S'(R^2)$, then $\phi(x_1, x_2; f)$ is positive definite. Thus, there exists a monotone increasing function $A(u, v)$ such that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|dA(u, v)|=\phi(0, 0; f)$, and

$$\phi(x_1, x_2; f)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(ux_1+vx_2)}dA(u, v) \tag{2.20}$$

holds. Moreover, if $(\alpha_1, \alpha_2]\times(\beta_1, \beta_2]$ is a finite continuity interval of the interval function generated by $A(u, v)$, then we have

$$A(\alpha_2, \beta_2)-A(\alpha_1, \beta_2)-A(\alpha_2, \beta_1)+A(\alpha_1, \beta_1)$$
$$=\lim_{A\to\infty}\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}\frac{e^{-i\alpha_2 x_1}-e^{-i\alpha_1 x_1}}{-ix_1}\frac{e^{-i\beta_2 x_2}-e^{-i\beta_1 x_2}}{-ix_2}\phi(x_1, x_2; f)dx_1dx_2\,. \tag{2.21}$$

* Here, by positive definiteness, we mean the function under consideration to be continuous.

§ 3. Proofs of theorems

First, we prove Theorem 1.

Proof of THEOREM 1. If we integrate by parts for each variable, then

$$\int_{-B}^{B}\int_{-A}^{A}\frac{|f(s,t)|^2}{(1+s^2)(1+t^2)}dsdt$$
$$=\frac{1}{(1+A^2)(1+B^2)}\int_{-B}^{B}\int_{-A}^{A}|f(\xi_1,\xi_2)|^2d\xi_1d\xi_2$$
$$+\frac{1}{1+A^2}\int_0^B\frac{4t^2}{(1+t^2)^2}dt\frac{1}{2t}\int_{-t}^{t}\int_{-A}^{A}|f(\xi_1,\xi_2)|^2d\xi_1d\xi_2$$
$$+\frac{1}{1+B^2}\int_0^A\frac{4s^2}{(1+s^2)^2}ds\frac{1}{2s}\int_{-B}^{B}\int_{-s}^{s}|f(\xi_1,\xi_2)|^2d\xi_1d\xi_2$$
$$+\int_0^B\int_0^A\frac{4s^2}{(1+s^2)^2}\frac{4t^2}{(1+t^2)^2}dsdt\frac{1}{4st}\int_{-t}^{t}\int_{-s}^{s}|f(\xi_1,\xi_2)|^2d\xi_1d\xi_2$$
$$\leqq\left\{\frac{4AB}{(1+A^2)(1+B^2)}+\frac{8A}{1+A^2}\tan^{-1}B+\frac{8B}{1+B^2}\tan^{-1}A+16\tan^{-1}A\tan^{-1}B\right\}$$
$$\cdot\sup_{0<S,T<\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt$$
$$\leqq\{1+4\pi+4\pi^2\}\cdot\sup_{0<S,T<\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt<\infty.$$

Thus, (2.2) is established.

Next, in order to prove Theorems 4, 5 and 6, we show several lemmas which are due to Wiener [11].

LEMMA 1. If $\phi(x_1, x_2; f)$ defined by (2.12) exists at $(x_1, x_2)=(0,0)$, then for any real numbers a, b,

$$\mathcal{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s+a,t+b)|^2dsdt$$
$$=\mathcal{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt. \tag{3.1}$$

Proof. Let us notice that if $S>|a|$, $T>|b|$, then

$$\left(1-\frac{|a|}{S}\right)\left(1-\frac{|b|}{T}\right)\frac{1}{4(S-|a|)(T-|b|)}\int_{-T+|b|}^{T-|b|}\int_{-S+|a|}^{S-|a|}|f(s,t)|^2dsdt$$
$$\leqq\frac{1}{4ST}\int_{-T+b}^{T+b}\int_{-S+a}^{S+a}|f(s,t)|^2dsdt=\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s+a,t+b)|^2dsdt$$
$$\leqq\left(1+\frac{|a|}{S}\right)\left(1+\frac{|b|}{T}\right)\frac{1}{4(S+|a|)(T+|b|)}\int_{-T-|b|}^{T+|b|}\int_{-S-|a|}^{S+|a|}|f(s,t)|^2dsdt.$$

Thus, since both the first and the last of these expressions tend to $\phi(0,0;f)$ as

$S, T\to\infty\,(\mathscr{R}_1)$, we have (3.1).

Now, we have

$$f(x_1+s,\, x_2+t)\overline{f(s,\,t)}=\frac{1}{4}\{|f(x_1+s,\, x_2+t)+f(s,\,t)|^2-|f(x_1+s,\, x_2+t)-f(s,\,t)|^2 \\ +i|f(x_1+s,\, x_2+t)+if(s,\,t)|^2-i|f(x_1+s,\, x_2+t)-if(s,\,t)|^2\}\,. \quad (3.2)$$

Using this, we can show

LEMMA 2. If $f(x_1, x_2)$ belongs to the class $S(R^2)$, then for any real numbers a, b,

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}f(x_1+s+a,\, x_2+t+b)\overline{f(s+a,\, t+b)}dsdt \\ =\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}f(x_1+s,\, x_2+t)\overline{f(s,\, t)}dsdt\,. \quad (3.3)$$

Proof. Since $f(x_1, x_2)$ belongs to the class $S(R^2)$, by (3.2) and Lemma 1,

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}f(x_1+s+a,\, x_2+t+b)\overline{f(s+a,\, t+b)}dsdt \\ =\frac{1}{4}\,\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T+b}^{T+b}\int_{-S+a}^{S+a}\{|f(x_1+s,\, x_2+t)+f(s,\,t)|^2 \\ -|f(x_1+s,\, x_2+t)-f(s,\,t)|^2+i|f(x_1+s,\, x_2+t)+if(s,\,t)|^2 \\ -i|f(x_1+s,\, x_2+t)-if(s,\,t)|^2\}dsdt \\ =\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}f(x_1+s,\, x_2+t)\overline{f(s,\, t)}dsdt\,.$$

Thus, the lemma is established.

Furthermore, we have for any real or complex number w such as $|w|=1$,

$$|f(x_1+s,\, x_2+t)+wf(s,\,t)|^2=|f(x_1+s,\, x_2+t)|^2+|f(s,\,t)|^2 \\ +\bar{w}f(x_1+s,\, x_2+t)\overline{f(s,\,t)}+w\overline{f(x_1+s,\, x_2+t)}f(s,\,t)\,, \quad (3.4)$$

from which we have

LEMMA 3. If $f(x_1, x_2)$ belongs to the class $S(R^2)$, then for any real or complex number w such as $|w|=1$,

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(x_1+s,\, x_2+t)+wf(s,\,t)|^2dsdt \quad (3.5)$$

exists for all $(x_1, x_2)\in R^2$.

On the other hand, we can obtain

Lemma 4. If $f(x_1, x_2)$ belongs to the class $W(R^2)$, then we have for any $x_1 \in (-\infty, \infty)$,

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Delta_{\varepsilon,\eta}s_{x_1}(u, v; f) - e^{iux_1}\Delta_{\varepsilon,\eta}s(u, v; f)|^2 du dv = 0 \tag{3.6}$$

and for any $x_2 \in (-\infty, \infty)$,

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Delta_{\varepsilon,\eta}s_{x_2}(u, v; f) - e^{ivx_2}\Delta_{\varepsilon,\eta}s(u, v; f)|^2 du dv = 0\,, \tag{3.7}$$

where

$$\Delta_{\varepsilon,\eta}s_{x_1}(u, v; f) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A} f(x_1+s, t)\frac{2\sin\varepsilon s}{s}\frac{2\sin\eta t}{t}e^{-i(us+vt)}dsdt \tag{3.8}$$

and

$$\Delta_{\varepsilon,\eta}s_{x_2}(u, v; f) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A} f(s, x_2+t)\frac{2\sin\varepsilon s}{s}\,\frac{2\sin\eta t}{t}e^{-i(us+vt)}dsdt\,, \tag{3.9}$$

respectively.

Before proving the lemma, we observe the following: If $f(x_1, x_2)$ belongs to the class $W(R^2)$, then in a way similar to the proof of Theorem 1,

$$\frac{1}{2T}\int_{-T}^{T}dt\int_{-A}^{A}\frac{|f(x_1, t)|^2}{1+x_1^2}dx_1 \leqq (1+2\pi)\cdot\sup_{0<S,T<\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s, t)|^2 dsdt < \infty\,.$$

It follows, therefore, that

$$\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}dt\int_{-\infty}^{\infty}\frac{|f(x_1, t)|^2}{1+x_1^2}dx_1 < \infty\,. \tag{3.10}$$

Similarly,

$$\overline{\lim_{S\to\infty}}\frac{1}{2S}\int_{-S}^{S}ds\int_{-\infty}^{\infty}\frac{|f(s, x_2)|^2}{1+x_2^2}dx_2 < \infty\,. \tag{3.11}$$

Proof. Let us notice that

$$\left|\frac{2\sin\varepsilon(\xi-x)}{\xi-x} - \frac{2\sin\varepsilon\xi}{\xi}\right| \leqq \begin{cases} \dfrac{16\varepsilon|x|}{|\xi|+|x|} & (|\xi|>2|x|) \\ 4\varepsilon & (|\xi|\leqq 2|x|)\,. \end{cases} \tag{3.12}$$

As for detailed calculations, refer to Wiener [11].

Now, since

$$\begin{aligned}&\Delta_{\varepsilon,\eta}s_{x_1}(u, v; f) - e^{iux_1}\Delta_{\varepsilon,\eta}s(u, v; f) \\ &= \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A} f(s, t)\left[\frac{2\sin\varepsilon(s-x_1)}{s-x_1} - \frac{2\sin\varepsilon s}{s}\right]\frac{2\sin\eta t}{t}e^{-i\{u(s-x_1)+vt\}}dsdt\,,\end{aligned}$$

it follows from the Plancherel theorem and (3.12) that

$$\begin{aligned}&\overline{\lim_{\varepsilon\to0}}\frac{1}{16\pi^2C\varepsilon^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,C\varepsilon}s_{x_1}(u,v;f)-e^{iux_1}\Delta_{\varepsilon,C\varepsilon}s(u,v;f)|^2dudv\\&\quad\leq\overline{\lim_{\varepsilon\to0}}\frac{(8|x_1|)^2}{\pi}\varepsilon\cdot\frac{1}{\pi C\varepsilon}\int_{-\infty}^{\infty}\frac{\sin^2C\varepsilon t}{t^2}dt\int_{|s|>2|x_1|}\frac{|f(s,t)|^2}{(|s|+|x_1|)^2}ds\\&\qquad+\overline{\lim_{\varepsilon\to0}}\frac{4}{\pi}\varepsilon\cdot\frac{1}{\pi C\varepsilon}\int_{-\infty}^{\infty}\frac{\sin^2C\varepsilon t}{t^2}dt\int_{|s|\leq2|x_1|}|f(s,t)|^2ds\,.\end{aligned}$$

Consequently, using the one-sided Wiener formula of Koizumi [6], the last expression does not exceed, for every positive constant C,

$$\begin{aligned}&\overline{\lim_{\varepsilon\to0}}\,O(\varepsilon)\cdot\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}dt\int_{|s|>2|x_1|}\frac{|f(s,t)|^2}{(|s|+|x_1|)^2}ds\\&\quad+\overline{\lim_{\varepsilon\to0}}\,O(\varepsilon)\cdot\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}dt\int_{|s|\leq2|x_1|}|f(s,t)|^2ds\,,\end{aligned}$$

which turns out to be 0 by (3.10), and (3.6) is proved. Similarly, (3.7) is proved by (3.11).

Moreover, we can easily obtain

LEMMA 5. If $f(x_1,x_2)$ belongs to the class $W(R^2)$, then we have for any $(x_1,x_2)\in R^2$,

$$\begin{aligned}&\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s_{x_1,x_2}(u,v;f)\\&\qquad-e^{i(ux_1+vx_2)}\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv=0\,,\end{aligned}\tag{3.13}$$

where

$$\begin{aligned}&\Delta_{\varepsilon,\eta}s_{x_1,x_2}(u,v;f)\\&\quad=\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}f(x_1+s,x_2+t)\frac{2\sin\varepsilon s}{s}\frac{2\sin\eta t}{t}e^{-i(us+vt)}dsdt\,.\end{aligned}\tag{3.14}$$

Proof. First, we have

$$\begin{aligned}&\Delta_{\varepsilon,\eta}s_{x_1,x_2}(u,v;f)-e^{i(ux_1+vx_2)}\Delta_{\varepsilon,\eta}s(u,v;f)\\&\quad=\{\Delta_{\varepsilon,\eta}s_{x_1,x_2}(u,v;f)-e^{ivx_2}\Delta_{\varepsilon,\eta}s_{x_1}(u,v;f)\\&\qquad-e^{iux_1}\Delta_{\varepsilon,\eta}s_{x_2}(u,v;f)+e^{i(ux_1+vx_2)}\Delta_{\varepsilon,\eta}s(u,v;f)\}\\&\qquad+\{e^{ivx_2}\Delta_{\varepsilon,\eta}s_{x_1}(u,v;f)-e^{i(ux_1+vx_2)}\Delta_{\varepsilon,\eta}s(u,v;f)\}\\&\qquad+\{e^{iux_1}\Delta_{\varepsilon,\eta}s_{x_2}(u,v;f)-e^{i(ux_1+vx_2)}\Delta_{\varepsilon,\eta}s(u,v;f)\}\\&\quad=\Delta_1(u,v)+\Delta_2(u,v)+\Delta_3(u,v)\,,\quad\text{say.}\end{aligned}$$

Here $\Delta_{\varepsilon,\eta}s_{x_1}(u,v;f)$ and $\Delta_{\varepsilon,\eta}s_{x_2}(u,v;f)$ are defined respectively by (3.8) and (3.9) of Lemma 4. Now, since

$$\Delta_1(u,v)=\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}f(s,t)\left[\frac{2\sin\varepsilon(s-x_1)}{s-x_1}-\frac{2\sin\varepsilon s}{s}\right]$$
$$\cdot\left[\frac{2\sin\eta(t-x_2)}{t-x_2}-\frac{2\sin\eta t}{t}\right]e^{-i\{u(s-x_1)+v(t-x_2)\}}dsdt\,,$$

it follows from the Plancherel theorem and (3.12) that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_1(u,v)|^2dudv$$
$$\leqq(16\varepsilon|x_1|)^2(16\eta|x_2|)^2\int_{|t|>2|x_2|}\int_{|s|>2|x_1|}\frac{|f(s,t)|^2}{(|s|+|x_1|)^2(|t|+|x_2|)^2}dsdt$$
$$+(16\varepsilon|x_1|)^2(4\eta)^2\int_{|t|\leqq 2|x_2|}\int_{|s|>2|x_1|}\frac{|f(s,t)|^2}{(|s|+|x_1|)^2}dsdt$$
$$+(4\varepsilon)^2(16\eta|x_2|)^2\int_{|t|>2|x_2|}\int_{|s|\leqq 2|x_1|}\frac{|f(s,t)|^2}{(|t|+|x_2|)^2}dsdt$$
$$+(4\varepsilon)^2(4\eta)^2\int_{|t|\leqq 2|x_2|}\int_{|s|\leqq 2|x_1|}|f(s,t)|^2dsdt\,.$$

Consequently, by Theorem 1,

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_1(u,v)|^2dudv=0\,.$$

Also, by Lemma 4,

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_2(u,v)|^2dudv=0$$

and

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_3(u,v)|^2dudv=0\,,$$

respectively. Thus, (3.13) is proved.

Therefore, we have

LEMMA 6. If $f(x_1, x_2)$ belongs to the class $S(R^2)$, then for any real or complex number w such as $|w|=1$,

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(x_1+s,x_2+t)+wf(s,t)|^2dsdt$$
$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{2+we^{-i(ux_1+vx_2)}+\bar{w}e^{i(ux_1+vx_2)}\}$$
$$\cdot|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv\,. \qquad (3.15)$$

Proof. It follows from Lemma 3, Theorem 3 and Minkowski's inequality that

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\left\{\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(x_1+s,\,x_2+t)+wf(s,t)|^2dsdt\right\}^{1/2}$$
$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\left\{\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s_{x_1,x_2}(u,v\,;f)\right.$$
$$\left.+w\Delta_{\varepsilon,\eta}s(u,v\,;f)|^2dudv\right\}^{1/2}$$
$$\leqq\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\left\{\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s_{x_1,x_2}(u,v\,;f)\right.$$
$$\left.-e^{i(ux_1+vx_2)}\Delta_{\varepsilon,\eta}s(u,v\,;f)|^2dudv\right\}^{1/2}$$
$$+\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\left[\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{2+we^{-i(ux_1+vx_2)}+\overline{w}e^{i(ux_1+vx_2)}\}\right.$$
$$\left.\cdot|\Delta_{\varepsilon,\eta}s(u,v\,;f)|^2dudv\right]^{1/2}.$$

Thus, by Lemma 5, (3.15) is proved.

Proof of THEOREM 4. By the Schwarz inequality and Lemma 1, we have

$$|\phi(x_1,x_2\,;f)|\leqq\mathscr{R}_1\text{-}\overline{\lim_{S,T\to\infty}}\left\{\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(x_1+s,\,x_2+t)|^2dsdt\right\}^{1/2}$$
$$\cdot\left\{\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt\right\}^{1/2}$$
$$=\phi(0,0\,;f)\,.$$

Thus, (2.13) is proved.

Proof of THEOREM 5. By the Schwarz inequality, we have

$$|\phi(x_1+\varepsilon,\,x_2+\eta\,;f)-\phi(x_1,x_2\,;f)|$$
$$\leqq\mathscr{R}_1\text{-}\overline{\lim_{S,T\to\infty}}\left\{\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(x_1+\varepsilon+s,\,x_2+\eta+t)-f(x_1+s,\,x_2+t)|^2dsdt\right\}^{1/2}$$
$$\cdot\left\{\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt\right\}^{1/2}.$$

If we now appeal to Lemmas 1, 3 and 4, then

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(x_1+\varepsilon+s,\,x_2+\eta+t)-f(x_1+s,\,x_2+t)|^2dsdt$$
$$=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(x_1+\varepsilon+s,\,x_2+\eta+t)|^2dsdt$$
$$+\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(x_1+s,\,x_2+t)|^2dsdt$$
$$-\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}f(x_1+\varepsilon+s,\,x_2+\eta+t)\overline{f(x_1+s,\,x_2+t)}dsdt$$

$$-\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}f(x_1+s,\,x_2+t)\overline{f(x_1+\varepsilon+s,\,x_2+\eta+t)}dsdt$$
$$=2\phi(0,0;f)-\phi(\varepsilon,\eta;f)-\phi(-\varepsilon,-\eta;f)\,.$$

Thus, we have

$$|\phi(x_1+\varepsilon,\,x_2+\eta;f)-\phi(x_1,x_2;f)|$$
$$\leqq[\phi(0,0;f)\{2\phi(0,0;f)-\phi(\varepsilon,\eta;f)-\phi(-\varepsilon,-\eta;f)\}]^{1/2}\,, \tag{3.16}$$

from which the theorem follows immediately.

Proof of THEOREM 6. If we take w in (3.15) of Lemma 6 successively to equal ± 1, $\pm i$, and combine four expressions, then we obtain (2.14) by (3.2).

Finally, we prove Theorems 7 and 8.

Proof of THEOREM 7. Let $f(x_1, x_2)$ belong to the class $S(R^2)$. Now, let us define

$$\phi_{\varepsilon,\eta}(x_1,x_2;f)=\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(ux_1+vx_2)}|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv\,. \tag{3.17}$$

Then, it follows at once that

$$|\phi_{\varepsilon,\eta}(x_1,x_2;f)|\leqq\phi_{\varepsilon,\eta}(0,0;f)\,.$$

Inasmuch as $\phi_{\varepsilon,\eta}(0,0;f)$ tends to $\phi(0,0;f)$ as $\varepsilon,\eta\to 0\,(\mathscr{R}_2)$, $\phi_{\varepsilon,\eta}(x_1,x_2;f)$ is bounded for all $(x_1,x_2)\in R^2$ and small values of ε,η. It, therefore, tends boundedly to $\phi(x_1,x_2;f)$ as $\varepsilon,\eta\to 0\,(\mathscr{R}_2)$. Thus, by the bounded convergence theorem and Fubini's theorem, we get

$$\frac{1}{\lambda}\int_{-\lambda}^{\lambda}\left(1-\frac{|x_1|}{\lambda}\right)\phi(x_1,0;f)dx_1$$
$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{\lambda}\int_{-\lambda}^{\lambda}\left(1-\frac{|x_1|}{\lambda}\right)\phi_{\varepsilon,\eta}(x_1,0;f)dx_1$$
$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{4\sin^2\dfrac{u\lambda}{2}}{u^2\lambda^2}|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv\,.$$

Hence, if $f(x_1,x_2)$ belongs to the class $S'(R^2)$, then we have

$$\phi(0,0;f)=\lim_{\lambda\to 0}\frac{1}{\lambda}\int_{-\lambda}^{\lambda}\left(1-\frac{|x_1|}{\lambda}\right)\phi(x_1,0;f)dx_1\,,$$

and

$$\lim_{\lambda\to 0}\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[1-\frac{4\sin^2\dfrac{u\lambda}{2}}{u^2\lambda^2}\right]|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv=0\,. \tag{3.18}$$

Since, when $|u\lambda|>\pi$,

$$1-\frac{4\sin^2\dfrac{u\lambda}{2}}{u^2\lambda^2}>1-\frac{4}{\pi^2},$$

it follows from the positiveness of the integrand in (3.18) that

$$\lim_{\lambda\to 0}\mathcal{R}_2\text{-}\overline{\lim_{\varepsilon,\eta\to 0}}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\left[\int_{\pi/\lambda}^{\infty}+\int_{-\infty}^{-\pi/\lambda}\right]|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv=0$$

or simply

$$\lim_{A\to\infty}\mathcal{R}_2\text{-}\overline{\lim_{\varepsilon,\eta\to 0}}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\left[\int_{A}^{\infty}+\int_{-\infty}^{-A}\right]|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv=0,$$

and we obtain (2.15). Similarly, we obtain (2.16).

Again, let $f(x_1, x_2)$ belong to the class $S(R^2)$ and let (2.15) and (2.16) hold. Then, by (2.14) of Theorem 6,

$$\begin{aligned}
&|\phi(x_1,x_2;f)-\phi(0,0;f)|\\
&\leqq 2\,\mathcal{R}_2\text{-}\overline{\lim_{\varepsilon,\eta\to 0}}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\left[\int_{A}^{\infty}+\int_{-\infty}^{-A}\right]|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv\\
&\quad+2\,\mathcal{R}_2\text{-}\overline{\lim_{\varepsilon,\eta\to 0}}\frac{1}{16\pi^2\varepsilon\eta}\left[\int_{A}^{\infty}+\int_{-\infty}^{-A}\right]\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv\\
&\quad+\mathcal{R}_2\text{-}\overline{\lim_{\varepsilon,\eta\to 0}}\frac{1}{16\pi^2\varepsilon\eta}\int_{-A}^{A}\int_{-A}^{A}|e^{i(ux_1+vx_2)}-1||\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv\\
&=I_1+I_2+I_3,\quad \text{say.}
\end{aligned}$$

Now, let us choose A, as is possible by (2.15) and (2.16), so large that I_1 and I_2 do not exceed $\delta/3$. Let us then choose $|x_1|$ and $|x_2|$ so small that over $(-A, A)\times(-A, A)$

$$|e^{i(ux_1+vx_2)}-1|\leqq\frac{\delta}{3\phi(0,0;f)},$$

and, therefore, I_3 does not exceed $\delta/3$. Thus, since we have

$$|\phi(x_1,x_2;f)-\phi(0,0;f)|\leqq\delta,$$

it follows from Theorem 5 that $f(x_1, x_2)$ belongs to the class $S'(R^2)$. This completes the proof of Theorem 7.

Proof of THEOREM 8. From Theorem 4 and Definition 3, it follows immediately that $\phi(x_1, x_2; f)$ is bounded and continuous on R^2. Also,

$$\overline{\phi(-x_1,-x_2;f)}=\phi(x_1,x_2;f)$$

for all $(x_1, x_2)\in R^2$. Furthermore, for any finite sequence $\{(\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)\}$ of points in R^2 and any finite sequence $\{z_1, \cdots, z_n\}$ of complex numbers,

$$\sum_{\mu,\nu=1}^{n} \phi(\alpha_\mu-\alpha_\nu, \beta_\mu-\beta_\nu)z_\mu\bar{z}_\nu$$
$$=\mathcal{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\sum_{\mu,\nu=1}^{n}\int_{-T}^{T}\int_{-S}^{S} f(\alpha_\mu-\alpha_\nu+s, \beta_\mu-\beta_\nu+t)\overline{f(s,t)}dsdtz_\mu\bar{z}_\nu$$
$$=\mathcal{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}\left|\sum_{\mu=1}^{n} f(\alpha_\mu+s, \beta_\mu+t)z_\mu\right|^2 dsdt\geqq 0\,.$$

Consequently, $\phi(x_1, x_2; f)$ is positive definite. Thus, by Bochner's representation theorem and Lévy's inversion formula, there exists a measure μ on $(R^2, \mathcal{B}_2)$ such that $\mu(R^2)=\phi(0, 0; f)$, and (2.17) and (2.18) hold with $\phi(x_1, x_2; f)$ instead of $\phi(x_1, x_2)$.

Now, let us put

$$\Lambda(u, v)=\int_{-\infty}^{v}\int_{-\infty}^{u} d\mu(\xi_1, \xi_2)\,. \tag{3.19}$$

Then, $\Lambda(u, v)$ is monotone increasing and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|d\Lambda(u, v)|=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\mu(\xi_1, \xi_2)=\mu(R^2)=\phi(0, 0; f)\,.$$

Moreover, by (2.17) and (2.18) with $\phi(x_1, x_2; f)$ instead of $\phi(x_1, x_2)$, we have (2.20) and (2.21). This completes the proof.

Remark. The generalized harmonic analysis we established in this paper was based on the two-dimensional Wiener formula (Theorem 2 or 2′). Its limit process, therefore, also depended on the limit process of the above two-dimensional Wiener formula. On the other hand, the two-dimensional Wiener formula is also proved under the unrestricted rectangular mean concerning the double limit process. For details, refer to Pitt [8] and Rudin [9]. Thus, if we base the argument on this sort of two-dimensional Wiener formula, then the generalized harmonic analysis also holds under the above limit process instead of the restricted limit process.

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