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# ON A CONSTRUCTION OF A SOLUTION FOR $\partial u/\partial t=\phi(u'')-uu'$ WITH INITIAL AND BOUNDARY CONDITIONS

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## 0. Introduction

This paper is concerned with a construction of a continuous solution $u(t, x)$ for the scalar equation

$$\frac{\partial u}{\partial t}=\varphi(u'')-uu'\left('=\frac{\partial}{\partial x}\right) \text{ for } (t, x)\in[0, T]\times[0, 1] \tag{1}$$

with initial and boundary conditions

$$u(0, x)=u_0(x) \quad \text{for } 0\leqq x\leqq 1, \tag{2}$$
$$u(t, 0)=0,\ u(t, 1)=0 \quad \text{for } 0\leqq t\leqq T.$$

Now we assume that the function $\varphi$ in (1) satisfies the following conditions:

$$\varphi\in C^5(-\infty, \infty),\ \varphi(0)=0,\ \varphi''(0)=0, \tag{3}$$
$$0<\lambda\leqq\varphi'(u)\leqq\mu<\infty \quad \text{for } -\infty<u<\infty.$$

The compatibility conditions for initial and boundary conditions are the following

$$u_0\in C^5[0, 1], \tag{4}$$
$$u_0(0)=u_0(1)=0,\ u_0''(0)=u_0''(1)=0.$$

Under the assumptions stated above we have the following theorem.

**Theorem.** *There exists a continuous 'solution' satisfying* (1), (2) *in the domain* $[0, T]\times[0, 1]$, *where $T$ is an arbitrary positive number.*

The meaning of 'solution' will be explained later.

This paper is stimulated by M. Hukuhara's papers [2], [3] and is treated by Rothe's method. Following S. N. Bernstein's paper [1] and O. A. Oleinik and T. D. Venttsel's paper [5] we find estimates for the derivatives of approximate solutions and we further estimate derivatives of higher order up to the boundary. To construct an approximate solution we use M. Nagumo's existence theorem of solutions for a second order differential equation with boundary conditions. Here we shall quote the following theorem ([4]):

Nagumo's Theorem. Consider a second order differential equation

$$y''=g(x, y, y'), \tag{5}$$

where $g(x, y, y')$ is a continuous function defined on a closed domain

$$0\leqq x\leqq 1,\ \alpha(x)\leqq y\leqq \omega(x),\ -\infty<y'<\infty.$$

$\alpha(x)$ and $\omega(x)$ is a minorant and majorant function of class $C^2[0, 1]$ respectively satisfying

$$\alpha(0)=\omega(0)=0,\quad \alpha(1)\leqq 0\leqq \omega(1),$$

$$\alpha''(x)\geqq g(x, \alpha(x), \alpha'(x)),$$

$$\omega''(x)\leqq g(x, \omega(x), \omega'(x)).$$

If $g(x, y, y')$ satisfies the following inequality

$$|g(x, y, y')|\leqq G(1+(y')^2)\ (G>0),$$

then there exists a solution $y(x)$ of (5) satisfying

$$\alpha(x)\leqq y(x)\leqq \omega(x)\quad \text{for } 0\leqq x\leqq 1,$$

$$y(0)=0,\ y(1)=0.$$

## 1. Explanation of the method

Let $X=C[0, 1]$ be the Banach space of real-valued functions $u(x)$ continuous on $[0, 1]$ with the norm

$$|u|=\sup\{|u(x)|\,;\ 0\leqq x\leqq 1\}.$$

Let $N$ be a positive integer sufficiently large and put

$$h=T/N(\ll 1),\ t_n=nh\ (n=0, 1, \cdots, N).$$

We define $\{u_n\}\subset X(n=1, 2, \cdots, N)$ inductively by:

$$v_n\equiv\frac{u_n-u_{n-1}}{h}=\varphi(u_n'')-u_nu_n', \tag{6}$$

$$u_n(0)=u_n(1)=0,$$

and define a Cauchy polygon in $X$ by

$$P_N(t)=u_{n-1}+\frac{t-t_{n-1}}{h}(u_n-u_{n-1})\ \text{for } t_{n-1}\leq t\leq t_n.$$

We have by (6) that

$$\frac{d^+P_N(t)}{dt}=\varphi(P_N(t_n)'')-P_N(t_n)P_N(t_n)' \quad \text{for } t_{n-1}\leqq t<t_n,$$

where $d^+/dt$ denotes right-hand derivative with respect to the topology of $X$. Here we define operators $A$ and $B$ in $X$ by

$$A:\quad \mathcal{D}(A)=\{u=u(x)\in C^2[0,1];\ u(0)=u(1)=0\}\to X,$$
$$(Au)(x)=u''(x),$$
$$B:\quad \mathcal{D}(B)=\{u=u(x)\in C^1[0,1];\ u(0)=u(1)=0\}\to X,$$
$$(Bu)(x)=u'(x),$$

which can be verified to be closed in $X$.

If we can select a subsequence $\{P_N(t)\}\subset X$ such that

$$P_N(t)\to P(t),\ \frac{d^+}{dt}P_N(t)\to Q(t),\ AP_N(t)\to R(t),\ BP_N(t)\to S(t)$$

uniformly on $[0, T]$ and $Q(t)$ is continuous on $[0, T]$, we have by noticing the closedness of $A$ and $B$ that

$$P(t)\in\mathcal{D}(A),\ \frac{d}{dt}P(t)=Q(t),\ AP(t)=R(t),\ BP(t)=S(t)$$

and this $P(t)$ satisfies

(7) $$\frac{d}{dt}P(t)=\varphi(AP(t))-P(t)BP(t),$$
$$P(t)|_{t=0}=u_0.$$

Under a solution of (1) and (2) we understand a function $P(t)$ satisfying (7).

## 2. Construction of $\{u_n\}$

In order to construct $u_n$ satisfying (6) we shall consider the following equation with 0-Dirichlet condition

(8) $$\frac{u-\bar{u}}{h}=\varphi(u'')-uu',$$
$$u(0)=u(1)=0,$$

where $\bar{u}$ is a function of class $C^2[0,1]$ satisfying

$$\bar{u}(0)=\bar{u}(1)=0,\ \bar{u}''(0)=\bar{u}''(1)=0.$$

Here we shall construct $\omega(x)$ of the type

$$\omega(x)=\begin{cases} ax-bx^2 & 0\leqq x\leqq\dfrac{a}{2b}, \\ \dfrac{a^2}{4b} & \dfrac{a}{2b}\leqq x\leqq 1\ (a, b>0, a\leqq 2b) \end{cases}$$

and define $\alpha(x)$ by

$$\alpha(x)=-\omega(x).$$

Such $\alpha(x)$ and $\omega(x)$ can be verified to be minorant and majorant function for (8), if we take $a, b$ suitably large. Hence, by using Nagumo's Theorem we can construct $u_n$ satisfying (6). ($\alpha(x), \omega(x)$ don't belong to $C^2[0,1]$ but Nagumo's Theorem is known to be applicable if $\alpha(x)$ and $\omega(x)$ are such functions.)

### 3. Estimates of $\{u_n\}$, $\{v_n\}$ and their derivatives

Here we shall prove the following estimates: there exists positive numbers $U_k$, $V_k$ independent of $N$ such that

$$|u_n^{(k)}|\leqq U_k \quad (k=0,1,2,3,4), \tag{9}$$

$$|v_n^{(k)}|\leqq V_k \quad (k=1,2,3). \tag{10}$$

From the relation (6) we note that

$$u_n=0, \quad u_n''=0, \quad u_n^{(4)}=0 \tag{11}$$

$$v_n=0, \tag{12}$$

at the boundary (i.e., $x=0,1$).

3.1. Estimate of $\{u_n\}$

By means of maximum principle for (6) we have

$$|u_n|\leqq|u_{n-1}|\leqq\cdots\leqq|u_0|.$$

3.2. Estimate of $\{u_n'\}$ at the boundary

We make the substitution $u_n=\log(1+p_n)$. Then we have from (6) that

$$\mu\frac{p_n''}{1+p_n}-\frac{1}{h}[\log(1+p_n)-\log(1+p_{n-1})] \tag{13}$$

$$\geqq\mu\frac{(p_n')^2}{(1+p_n)^2}+u_n\frac{p_n'}{(1+p_n)}\geqq-\frac{1}{\mu}u_n^2\geqq-\frac{1}{\mu}U_0^2.$$

We consider a function $q_n=p_n+ke^{-x}(k>0)$ and put

$$Lq_n=\mu\frac{q_n''}{1+p_n}-\frac{1}{h}[\log(1+q_n)-\log(1+q_{n-1})]. \tag{14}$$

From (13) and (14) we have

$$Lq_n\geqq\frac{k\mu}{e(1+p_n)}-\frac{U_0^2}{\mu}$$

$$+\frac{p_n-p_{n-1}}{h}\int_0^1\left[\frac{1}{1+p_{n-1}+s(p_n-p_{n-1})}-\frac{1}{1+p_{n-1}+ke^{-x}+s(p_n-p_{n-1})}\right]ds.$$

At the point $x$ where $p_n(x)\geqq p_{n-1}(x)$ we have

$$Lq_n\geqq\frac{k\mu}{e^{U_0+1}}-\frac{U_0^2}{\mu}>0$$

if we take $k$ sufficiently large such that

$$k>eU_0^2e^{U_0}\mu^{-2}.$$

This implies that $\{q_n\}$ $(n=0, 1, \cdots, N)$ can have a maximum value at $x=0$ if $k$ is chosen sufficiently large. Consequently, we have

$$\left.\frac{\partial p_n}{\partial x}\right|_{x=0}\leqq k \ (n=0, 1, \cdots, N).$$

By considering the function $p_n-ke^{-x}$ we can similarly verify that

$$\left.\frac{\partial p_n}{\partial x}\right|_{x=0}\geqq -k \ (n=0, 1, \cdots, N).$$

We can further similarly have that

$$\left|\left.\frac{\partial p_n}{\partial x}\right|_{x=1}\right|\leqq k \ (n=0, 1, \cdots, N).$$

Hence, we have an equi-boundedness of $\{u_n'\}$ at the boundary.

3.3. Estimate of $\{u_n'\}$ in the interior

In order to estimate $\{u_n'\}$ we consider the following transformation

$$u=\psi(v),\ \psi'(v)=e^{-kv^2}(k>0).$$

We then have from (8) that

$$\frac{v-\bar{v}}{h}\int_0^1\psi'(\bar{v}+s(v-\bar{v}))ds=\varphi(\psi'v_{xx}+\psi''v_x^2)-\psi\psi'v_x, \tag{15}$$

where $\bar{u}=\psi(\bar{v})$. Differentiating this equation with respect to $x$ we have

$$\begin{aligned}&\frac{v_x-\bar{v}_x}{h}\int_0^1[\psi'(\bar{v}+s(v-\bar{v}))+s(v-\bar{v})\psi''(\bar{v}+s(v-\bar{v}))]ds\\&\quad+\frac{v-\bar{v}}{h}\,\bar{v}_x\int_0^1\psi''(\bar{v}+s(v-\bar{v}))ds\\&=(\psi'''v_x^3+3\psi''v_xv_{xx}+\psi'v_{xxx})\varphi'-(\psi')^2v_x^2-\psi\psi''v_x^2-\psi\psi'v_{xx},\end{aligned}$$

where the argument of $\varphi'$ is $\psi'v_{xx}+\psi''v_x^2$. For $k$ sufficiently small we can have positive $e_0, e_\infty$ such that

$$e_0\leqq E\equiv\int_0^1[\psi'(\bar{v}+s(v-\bar{v}))+s(v-\bar{v})\psi''(\bar{v}+s(v-\bar{v}))]ds\leqq e_\infty$$

for every $v, \bar{v}(|v|, |\bar{v}|\leqq|v_0|, u_0=\psi(v_0))$. Multiplying $v_x$ to both sides of the above equation, we have

$$E\frac{v_x-\bar v_x}{h}v_x+\frac{v-\bar v}{h}v_x\bar v_x\int_0^1\psi''(\bar v+s(v-\bar v))ds$$
$$=(\psi'''v_x^4+3\psi''v_x^2v_{xx}+\psi'v_xv_{xxx})\varphi'$$
$$-(\psi')^2v_x^2-\psi\psi''v_x^3-\psi\psi'v_xv_{xx}.$$

At the point $x_0\in(0,1)$ where the maximum of $|v_x|$ is attained, we have from (15) that

$$\frac{v-\bar v}{h}F=\varphi(\psi''v_x^2)-\psi\psi'v_x,$$

where

$$f_0\leqq F\equiv\int_0^1\psi'(\bar v(s_0)+s(v(x_0)-\bar v(x_0)))ds\leqq f_\infty$$

for some positive $f_0, f_\infty$ independent of $x_0$ and $v, \bar v(|v|, |\bar v|\leqq v_0|)$. Hence, we have at $x_0$ that

$$E\frac{v_x-\bar v_x}{h}v_x+\frac{\varphi(\psi''v_x^2)-\psi\psi'v_x}{F}v_x\bar v_x\int_0^1\psi''(\bar v+s(v-\bar v))ds$$
$$=\varphi'\psi'''v_x^4+\varphi'\psi'v_xv_{xxx}-[(\psi')^2+\psi\psi'']v_x^3.$$

By applying the same procedure for (6) and taking $k$ sufficiently small we can show that $\{|u_{nx}|\}$ is equi-bounded.

3.4. Estimate of $\{v_n\}$

From (6) we have

$$hv_n''\int_0^1\varphi'(u_{n-1}''+s(u_n''-u_{n-1}''))ds \tag{16}$$
$$=v_n-v_{n-1}+hu_{n-1}'v_n+hu_nv_n'.$$

By means of maximum principle for (16) we have the equi-boundedness of $\{|v_n|\}$ and hence of $\{|u_n''|\}$.

3.5. Estimate of $\{v_n'\}$ at the boundary

We make the substitution $v_n=\log(1+p_n)$. Then we have from (16) that

$$E_n\frac{p_n''}{1+p_n}-\frac{1}{h}[\log(1+p_n)-\log(1+p_{n-1})] \tag{17}$$
$$=E_n\frac{(p_n')^2}{(1+p_n)^2}+u_n\frac{p_n'}{1+p_n}+u_{n-1}'v_n$$
$$\geqq -W$$

for some positive $W$ independent of $n, N$, where

$$E_n\equiv\int_0^1\varphi'(u_{n-1}''+s(u_n''-u_{n-1}''))ds.$$

We consider a function $q_n=p_n+ke^{-x}$ and put

$$(18)\qquad Lq_n=E_n\frac{q_n''}{1+p_n}-\frac{1}{h}[\log(1+q_n)-\log(1+q_{n-1})].$$

From (17) and (18) we have

$$Lq_n\geqq -W+k\lambda e^{-V_0-1}>0,$$

if we take $k$ sufficiently large such that

$$k>\lambda^{-1}e^{1+V_0}W.$$

This implies that $\{q_n\}$ $(n=0,1,\cdots,N)$ can have a maximum value at $x=0$ and hence we have

$$\left.\frac{dp_n}{dx}\right|_{x=0}\leqq k\quad(n=0,1,\cdots,N)$$

By using the same method for establishing the estimate $\{u_n'\}$ at the boundary, we have the equi-boundedness of $\{v_n'\}$ at the boundary.

3.6. Estimate of $\{v_n'\}$ in the interior

Differetiating both sides of (16) we have

$$(19)\qquad \frac{v_n'-v_{n-1}'}{h}=v_n'''\int_0^1\varphi'(u_{n-1}''+s(u_n''-u_{n-1}''))ds$$

$$+v_n''\int_0^1\varphi''(u_{n-1}''+s(u_n''-u_{n-1}''))\ [u_{n-1}'''+s(u_n'''-u_{n-1}''')]ds$$

$$-(u_{n-1}'v_n'+u_{n-1}''v_n+u_n'v_n'+u_nv_n'').$$

By means of maximum principle for (19) we have

$$|v_n'|\frac{|v_n'|-|v_{n-1}'|}{h}\leqq U_1|v_n'|^2+U_2V_0|v_n'|+U_1|v_n'|^2,$$

which implies the equi-boundedness of $\{|v_n'|\}$ and hence of $\{|u_n'''|\}$.

3.7. Estimate of $\{u_n^{(4)}\}$

Differentiating both sides of (6) three times we have

$$\frac{u_n'''-u_{n-1}'''}{h}=\varphi'u_n^{(5)}+3\varphi''u_n'''u_n^{(4)}+\varphi'''(u_n''')^3$$

$$-u_nu_n^{(4)}-4u_n'u_n'''-3(u_n'')^2,$$

where the argument of $\varphi$ is $u_n''$. Put $p_n=u_n'''$ and hence we have

$$(20)\qquad \frac{p_n-p_{n-1}}{h}=\varphi'p_n''+3\varphi''p_np_n'+\varphi'''p_n^3$$

$$-u_np_n'-4u_n'p_n-3(u_n'')^2.$$

Here we consider the following transformation

$$p_n=\psi(q_n),\ \psi'(q)=e^{-kq^2}(k>0).$$

We then have from (20) that

$$\frac{q_n-q_{n-1}}{h}\int_0^1\psi'(q_{n-1}+s(q_n-q_{n-1}))ds=(\psi'q_n''+\psi''(q_n')^2)\varphi'$$
$$+3\varphi''\psi'p_nq_n'+\varphi'''p_n^3-\psi'u_nq_n'-4u_n'p_n-3(u_n'')^2.$$

Differentiating this equation with respect to $x$ we have

$$\frac{q_n'-q_{n-1}'}{h}\int_0^1[\psi'(q_{n-1}+s(q_n-q_{n-1}))+s(q_n-q_{n-1})\psi''(q_{n-1}+s(q_n-q_{n-1}))]ds$$
$$+\frac{q_n-q_{n-1}}{h}q_{n-1}'\int_0^1\psi''(q_{n-1}+s(q_n-q_{n-1}))ds$$
$$=(\psi'q_n'''+3\psi''q_n'q_n''+\psi'''(q_n')^3)\varphi'+(\psi'q_n''+\psi''(q_n')^2)\varphi''p_n$$
$$+3\varphi''\psi'p_nq_n''+3\varphi''(\psi')^2(q_n')^2+3\varphi''\psi''p_n(q_n')^2+3\varphi'''\psi'(p_n)^2q_n'$$
$$+3\varphi'''\psi'p_n^2q_n'+\varphi^{(4)}p_n^4-\psi'u_nq_n''-\psi''u_n(q_n')^2-5\psi'u_n'q_n'-10u_n''p_n.$$

Multiplying $q_n'$ to both sides of the above equation, we have

$$(21)\qquad E\frac{q_n'-q_{n-1}'}{h}q_n'+\frac{q_n-q_{n-1}}{h}q_n'q_{n-1}'\int_0^1\psi''(q_{n-1}+s(q_n-q_{n-1}))ds$$
$$=(\psi'q_n'q_n'''+3\psi''(q_n')^2q_n''+\psi'''(q_n')^4)\varphi'+(\psi'q_n'q_n''+\psi''(q_n')^3)\varphi''p_n$$
$$+3\varphi''\psi'p_nq_n'q_n''+3\varphi''(\psi')^2(q_n')^3+3\varphi''\psi''p_n(q_n')^3+3\varphi'''\psi'p_n^2(q_n')^2$$
$$+3\varphi'''\psi'p_n^2(q_n')^2+\varphi^{(4)}p_n^4q_n'-\psi'u_nq_n'q_n''-\psi''u_n(q_n')^3-5\psi'u_n'(q_n')^2-10u_n''p_nq_n',$$

where, by taking $k$ sufficiently small,

$$e_0\leqq E\equiv\int_0^1[\psi'(q_{n-1}+s(q_n-q_{n-1}))+s(q_n-q_{n-1})\psi''(q_{n-1}+s(q_n-q_{n-1}))]ds\leqq e_\infty$$

for some positive $e_0, e_\infty$ independent of $x\in[0,1]$, $n$ and $N$. At the point $x_0\in(0,1)$ such that $|q_n'|=|q_n'(x_0)|$ we have

$$(22)\qquad \frac{q_n-q_{n-1}}{h}F=\psi''(q_n')^2\varphi'+3\varphi''\psi'p_nq_n'+\varphi'''p_n^3$$
$$-\psi'u_nq_n'-4u_n'p_n-3(u_n'')^2,$$

where

$$f_0\leqq F\equiv\int_0^1\psi'(q_{n-1}+s(q_n-q_{n-1}))ds\leqq f_\infty$$

for some positive $f_0, f_\infty$ independent of $x_0$, $n$ and $N$. From (21) and (22) we have at $x_0$ that

$$E\frac{q_n'-q_{n-1}'}{h}q_n'+\frac{q_n'q_{n-1}'}{F}\int_0^1\psi''(q_{n-1}+s(q_n-q_{n-1}))ds$$

$$\times(\varphi'\psi''(q_n')^2+3\varphi''\psi'p_nq_n'+\varphi'''p_n^3-\psi'u_nq_n'-4u_n'p_n-3(u_n'')^2)$$

$$=(\psi'q_n'q_n'''+\psi'''(q_n')^4)\varphi'+\varphi''\psi''p_n(q_n')^3$$

$$+3\varphi''(\psi')^2(q_n')^3+3\varphi''\psi''p_n(q_n')^3+3\varphi'''\psi'p_n^2(q_n')^2$$

$$+3\varphi'''\psi'p_n^2(q_n')^2+\varphi^{(4)}p_n^4q_n'-\psi''u_n(q_n')^3-5\psi'u_n'(q_n')^2-10u_n''p_nq_n'.$$

By taking $k$ sufficiently small we can show the equi-boundedness of $\{|u_n^{(4)}|\}$ and hence of $\{|v_n''|\}$.

3.8. Estimate of $\{u_n^{(5)}\}$ at the boundary

Differentiating both sides of (6) four times we have

$$\frac{u_n^{(4)}-u_{n-1}^{(4)}}{h}=\varphi'u_n^{(6)}+4\varphi''u_n'''u_n^{(5)}+3\varphi''(u_n^{(4)})^2$$

$$+6\varphi'''(u_n''')^2u_n^{(4)}+\varphi^{(4)}(u_n''')^4$$

$$-u_nu_n^{(5)}-5u_n'u_n^{(4)}-10u_n''u_n''',$$

where the argument of $\varphi$ is $u_n''$. Put $p_n=u_n^{(4)}$ and hence we have

$$\frac{p_n-p_{n-1}}{h}=\varphi'p_n''+4\varphi''u_n'''p_n'+3\varphi''p_n^2 \tag{23}$$

$$+6\varphi'''(u_n''')^2p_n+\varphi^{(4)}(u_n''')^4$$

$$-u_np_n'-5u_n'p_n-10u''u_n'''.$$

In (23) we make the substitution $p_n=\log(1+q_n)$. Then we have

$$\varphi'\frac{q_n''}{1+q_n}-\frac{1}{h}[\log(1+q_n)-\log(1+q_{n-1})]$$

$$=\varphi'\frac{(q_n')^2}{(1+q_n)^2}-4\varphi''u_n'''\frac{q_n'}{1+q_n}-3\varphi''p_n^2$$

$$-\varphi'''(u_n''')^2p_n-\varphi^{(4)}(u_n''')^4$$

$$+u_n\frac{q_n'}{1+q_n}-5u_n'p_n-10u_n''u_n'''$$

$$\geqq -W,$$

where $W$ is a positive number independent of $x, n$ and $N$. By using the same method for establishing the estimate of $\{u_n'\}$ at the boundary, we have the equi-boundedness of $\{u_n^{(5)}\}$ at the boundary.

3.9. Estimate of $\{u_n^{(5)}\}$ in the interior

By the following transformation

$$p_n=\psi(q_n),\ \psi'(q)=e^{-kq^2}(k>0)$$

we have from (23) that

$$\frac{q_n-q_{n-1}}{h}\int_0^1 \phi'(q_{n-1}+s(q_n-q_{n-1}))ds$$
$$-(\phi' q_n''+\phi''(q_n')^2)\varphi'+4\varphi''\phi' u_n''' q_n'+3\varphi'' p_n^2$$
$$+6\varphi'''(u_n''')^2 p_n+\varphi^{(4)}(u_n''')^4-\phi' u_n q_n'-5u_n' p_n-10u_n'' u_n'''.$$

Differentiating this equation with respect to $x$ we have

$$\frac{q_n'-q_{n-1}'}{h}\int_0^1 [\phi'(q_{n-1}+s(q_n-q_{n-1}))+s(q_n-q_{n-1})\phi''(q_{n-1}+s(q_n-q_{n-1}))]ds \tag{24}$$
$$+\frac{q_n-q_{n-1}}{h}\, q_{n-1}'\int_0^1 \phi''(q_{n-1}+s(q_n-q_{n-1}))ds$$
$$=(\phi' q_n'''+3\phi'' q_n' q_n''+\phi'''(q_n')^3)\varphi'+5\varphi''\phi' u_n''' q_n''+5\varphi''\phi'' u_n'''(q_n')^2$$
$$+10\varphi''\phi' p_n q_n'+10\varphi'''\phi'(u_n''')^2 q_n'+15\varphi''' u_n''' p_n^2$$
$$+4\varphi^{(4)} u_n''' p_n+\varphi^{(5)}(u_n''')^5$$
$$-\phi' u_n q_n''-\phi'' u_n(q_n')^2-6\phi' u_n' q_n'-15u_n'' p_n-10(u_n''')^3.$$

At the point $x_0\in(0,1)$ such that $|q_n'|=|q_n'(x_0)|$ we have

$$\frac{q_n-q_{n-1}}{h}F=\varphi'\phi''(q_n')^2+4\varphi''\phi' u_n''' q_n' \tag{25}$$
$$+3\varphi'' p_n^2+6\varphi'''(u_n''')^2 p_n+\varphi^{(4)}(u_n''')^4$$
$$-\phi' u_n q_n'-5u_n' p_n-10u_n'' u_n'''.$$

From (24) and (25) we have at $x_0$ that

$$Eq_n'\frac{q_n'-q_{n-1}'}{h}+\frac{1}{F}\int_0^1 \phi''(q_{n-1}+s(q_n-q_{n-1}))ds$$
$$\times q_n' q_{n-1}'(\varphi'\phi''(q_n')^2+4\varphi''\phi' u_n''' q_n'+3\varphi'' p_n^2+6\varphi'''(u_n''')^2 p_n$$
$$+\varphi^{(4)}(u_n''')^4-\phi' u_n q_n'-5u_n' p_n-10u_n'' u_n''')$$
$$=(\phi' q_n' q_n'''+\phi'''(q_n')^4)\varphi'+5\varphi''\phi'' u_n'''(q_n')^3+10\varphi''\phi' p_n(q_n')^2$$
$$+10\varphi'''\phi'(u_n''')^2(q_n')^2+15\varphi''' u_n''' p_n^2 q_n'+6\varphi^{(4)}(u_n''')^3 p_n q_n'$$
$$+4\varphi^{(4)} u_n''' p_n q_n'+\varphi^{(5)}(u_n''')^5 q_n'-\phi'' u_n(q_n')^3$$
$$-6\phi' u_n'(q_n')^2-15u_n'' p_n q_n'-10(u_n''')^3 q_n',$$

where $E$ and $F$ are similar functions as defined in 3.3. By taking $k$ sufficiently small we can show the equi-boundedness of $\{|u_n^{(5)}|\}$ and hence of $\{|v_n^{(3)}|\}$.

## 4. Estimate of $\{|u_n''-u_{n-1}''|\}$

From the construction (6) of $u_n(n\geqq 2)$ we have

$$h(u_n''-u_{n-1}'')\int_0^1\varphi'(u_{n-1}''+s(u_n''-u_{n-1}''))ds \tag{24}$$
$$=(u_n-u_{n-1})-(u_{n-1}-u_{n-2})+h^2u_{n-1}v_n'+h^2u_{n-1}'v_n+h^3v_nv_n'.$$

Differentiating both sides of (24) twice we have

$$\begin{aligned} &h(u_n''''-u_{n-1}'''')\int_0^1\varphi'(u_{n-1}''+s(u_n''-u_{n-1}''))ds \\ &+2h(u_n'''-u_{n-1}''')\int_0^1\varphi'(u_{n-1}''+s(u_n''-u_{n-1}''))[u_{n-1}'''+s(u_n'''-u_{n-1}''')]ds \\ &+h(u_n''-u_{n-1}'')\int_0^1\{\varphi'''(u_{n-1}''+s(u_n''-u_{n-1}''))[u_{n-1}'''+s(u_n'''-u_{n-1}''')]^2 \\ &+\varphi''(u_{n-1}''+s(u_n''-u_{n-1}''))[u_{n-1}''''+s(u_n''''-u_{n-1}'''')]\}ds \\ &=(u_n''-u_{n-1}'')-(u_{n-1}''-u_{n-2}'') \\ &+h^2u_{n-1}v_n'''+3h^2u_{n-1}'v_n''+3h^2u_{n-1}''v_n'+h^2u_n'''v_n+3h^3v_n'v_n''+h^3v_nv_n'''. \end{aligned} \tag{25}$$

By means of maximum principle for (25) we have

$$|u_n''-u_{n-1}''|\leqq|u_{n-1}''-u_{n-2}''|+hK_1|u_n''-u_{n-1}''|+h^2K_2, \tag{26}$$

where

$$K_1=\sup_{|u|\leq U_2}|\varphi'''(u)|+\sup_{|u|\leq U_2}|\varphi''(u)|U_4,$$

$$K_2=U_0V_3+3U_1V_2+3U_2V_1+U_3V_0+3V_1V_2+V_0V_3.$$

On the other hand, from the relation

$$u_1-u_0=h\varphi(u_1'')-hu_1u_1'$$

we have

$$u_1''-u_0''=h\varphi'(u_1'')u_1''''+h\varphi''(u_1'')(u_1''')^2-3hu_1'u_1''-hu_1u_1''' \tag{27}$$

and hence we have by using maximum principle for (27) that

$$|u_1''-u_0''|\leqq hK_3, \tag{28}$$

where

$$K_3=\sup_{|u|\leq U_2}|\varphi''(u)|U_3^2+3U_1U_2+U_0U_3+\mu U_4.$$

Hence, we have from (26) and (28) that

$$|u_n''-u_{n-1}''|\leqq h\frac{K_3+K_2K_1^{-1}\{1-(1-hK_1)^{n-1}\}}{(1-hK_1)^{n-1}}\leqq hK \tag{29}$$

for some constant $K$ independent of $N$ and $n$.

## 5. Completion of the proof

The equi-boundedness (9) of $\{|u_n'|\}$ implies that $\{P_N(t)\}$ $(0\leqq t\leqq T)$ belongs to a compact set in $X$ and the equi-boundedness of $\{|u_n|\}$, $\{|u_n'|\}$, $\{|u_n''|\}$ implies the equi-continuity of $\{P_N(t)\}$. Hence, $\{P_N(t)\}$ can be assumed to be a normal family of functions with values in $X$ converging to a continuous function $P(t)\in X$ uniformly on $[0, T]$.

The equi-boundedness (9) of $\{|u_n'''|\}$ similarly implies that $\{d^+P_N(t)/dt\}$ $(0\leqq t\leqq T)$ belongs to a compact set in $X$. From the estimates (9), (10) and (29) we have

$$|\varphi(u_n'')-u_nu_n'-\varphi(u_{n-1}'')+u_{n-1}u_{n-1}'|$$

$$\leqq h(\mu K+U_1V_0+V_1U_0+V_0V_1) \quad (n\geqq 1),$$

which shows that $\{d^+P_N(t)/dt\}$ is equi-continuous. Hence, $\{d^+P_N(t)/dt\}$ can be assumed to be a normal family of functions with values in $X$ converging to a continuous function uniformly on $[0, T]$.

Noticing the closedness of $A$ and $B$, we have by letting $N\to\infty$ in (6) that

$$\frac{d}{dt}P(t)=\varphi(AP(t))-P(t)BP(t) \quad 0\leqq t\leqq T,$$

$$P(t)\in\mathcal{D}(A),$$

$$P(t)|_{t=0}=u_0.$$

Hence, a solution for (1) and (2) was constructed.

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