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SIMULTANEOUS ESTIMATION OF VARIANCES OF NORMAL DISTRIBUTIONS WITH KNOWN MEANS UNDER SQUARED ERROR LOSS

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ABSTRACT

Simultaneous estimation of variances of normal distributions with known means is considered under squared error loss. We show inadmissibility of the usual estimator (best multiple of sum of squared deviations from mean) by giving a class of improved estimators which expand the usual one towards infinity. The improvement is possible if the number of populations is greater than one.

1. Introduction

A familiar problem in simultaneous estimation is that of estimating the mean vector of a multivariate normal distribution. Stein (1955) showed that the usual estimator, the sample mean, is inadmissible under squared error loss if the dimension is larger than two. An explicit estimator which improves upon the usual one was given by James and Stein (1960). Since then, various kinds of extensions have been made by many authors.

Recently attention is also brought to the problem of improving upon the usual estimator under squared error loss in multiparameter estimation of nonnormal distributions. Clevenson and Zidek (1975) showed that in simultaneous estimation of Poisson means the usual estimator is dominated by a proposed one if the number of populations is larger than one. For some class of distributions in the exponential family Hudson (1978) has also given improved estimators.

Here we deal with the problem of simultaneous estimation of variances of normal distributions with known means. If $(z_1, \ldots, z_n)$ is a random sample from a normal population with known mean μ_0 and unknown variance θ, then $x=\sum_{i=1}^{n}(z_i-\mu_0)^2$ is distributed as θy, where y is a chi square variable with n degrees of freedom. A natural estimator of θ is $x/(n+2)$. This estimator is biased, but have minimum mean square error in the class of estimators of the form cx, where

c is a constant. The admissibility of this estimator under squared error loss was shown by Hodges and Lehmann (1951).

We combine p such estimation problems by defining the loss by sum of squared errors. In section 2 we show that if $p \geqq 2$, the natural estimator is inadmissible by giving a class of improved estimators.

It should be remarked that in personal communication J. O. Berger has informed the author that he has solved the same problem in more general framework.

2. A class of improved estimators

Let x_i be independently distributed as $\theta_i y_i$, where y_i is a chi square variable with n_i degrees of freedom, $i=1, \cdots, p$. We assume that $n_i \geqq 5$, $i=1, \cdots, p$, and that $p \geqq 2$. Suppose we want to estimate $\theta_1, \cdots, \theta_p$ with the loss

$$\sum_{i=1}^{p} (\hat{\theta}_i - \theta_i)^2.$$

We consider estimators of the form

$$\phi_i = \left\{1 + a \frac{f(x_1^2/b_1, \cdots, x_p^2/b_p)}{x_i^2/c_i}\right\} \frac{x_i}{n_i+2}, \qquad i=1, \cdots, p,$$

where $b_i = (n_i+2)^2 (n_i-2)^2$ and $c_i = (n_i+2)^2 (n_i-2)$, $i=1, \cdots, p$. We show that for appropriate choice of $f = f(x_1^2/b_1, \cdots, x_p^2/b_p)$ and a, $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_p)$ has uniformly smaller risk than $\boldsymbol{\phi}^0 = (x_1/(n_1+2), \cdots, x_p/(n_p+2))$.

We first state the following

Lemma. Let y be a chi square variable with n degrees of freedom, g be an absolutely continuous function, and g' denote the derivative of g. Then

$$\mathrm{E}\{g(y)/y\} = \mathrm{E}\{g(y)\}/(n-2) - 2\ \mathrm{E}\{g'(y)\}/(n-2),$$

provided that both expected values exist and are finite.

We can prove this lemma by using an integration by parts. This lemma was noted in Efron and Morris (1976) and was also used in Berger ane Bock (1976a, b).

Theorem. If f is

(i) $p^{-1} \min_k (x_k^2/b_k)$ or (ii) $1/(\sum_k b_k/x_k^2)$,

and if $0 < a < 8(p-e)$, then the estimator $\boldsymbol{\phi}$ has uniformly smaller risk than $\boldsymbol{\phi}^0$, where $e = \mathrm{E}\,[\max_k\{(n_k+2)/y_k\}]$.

Proof. First we evaluate the difference between the risks of two estimators ϕ_i and $x_i/(n_i+2)$ as an estimator of θ_i when the loss is squared error. To do this, we see that

$$\{x_i/(n_i+2) - \theta_i\}^2 - (\phi_i - \theta_i)^2$$

$$= -2a \frac{c_i}{x_i} \frac{f}{n_i+2}\left(\frac{x_i}{n_i+2} - \theta_i\right) - a^2 \frac{c_i^2}{x_i^2} \frac{f^2}{(n_i+2)^2}$$

$$=2a\left\{\frac{c_i}{y_i}\frac{f}{n_i+2}-\frac{c_i f}{(n_i+2)^2}-\frac{a}{2}\frac{f^2}{x_i^2/b_i}\right\}$$

$$=2aK_i, \text{ say.}$$

By using Lemma, we have

$$E\left\{\frac{c_i}{y_i}\frac{f}{n_i+2}\right\}=\frac{c_i}{n_i+2}\left\{\frac{E(f)}{n_i-2}-\frac{2}{n_i-2}E\left(\frac{\partial f}{\partial y_i}\right)\right\}.$$

Therefore we have

$$E(K_i)=\frac{4c_i\ E(f)}{(n_i+2)^2(n_i-2)}-\frac{2c_i}{(n_i+2)(n_i-2)}E\left(\frac{\partial f}{\partial y_i}\right)-\frac{a}{2}E\left(\frac{f^2}{x_i^2/b_i}\right)$$

$$=4\ E(f)-2(n_i+2)\ E\left(\frac{\partial f}{\partial y_i}\right)-\frac{a}{2}E\left(\frac{f^2}{x_i^2/b_i}\right).$$

(i) If $f=p^{-1}\min_k (x_k^2/b_k)$,

$$\frac{\partial f}{\partial y_i}=2p^{-1}\theta_i^2 y_i/b_i, \quad \text{if } \min_k (x_k^2/b_k)=x_i^2/b_i$$

$$=0, \qquad \text{if } \min_k (x_k^2/b_k)<x_i^2/b_i.$$

Let Ω_i denote the set such that $\min_k (x_k^2/b_k)=x_i^2/b_i$ and let $I_{\Omega i}$ be its indicator function. Then we have

$$\frac{\partial f}{\partial y_i}=2f\frac{I_{\Omega i}}{y_i}\ .$$

Therefore we have

$$E(\sum_{i=1}^p K_i)=4pE(f)-4E\{f\sum_{i=1}^p (n_i+2)I_{\Omega i}/y_i\}-a/2\cdot E\{\sum_{i=1}^p f^2/(x_i^2/b_i)\}$$

$$\geqq 4p\ E(f)-4E[f\ \max_k\{n_k+2)/y_k\}]-a/2\cdot E(f).$$

Since f is a nondecreasing function of y_k and $\max_k \{(n_k+2)/y_k\}$ is a nonincreasing function of y_k, we have

$$E(\sum_{i=1}^p K_i)\geqq 4\ E(f)(p-e-a/8).$$

(ii) If $f=1/(\sum b_k/x_k^2)$, we have $\partial f/\partial y_i=2f^2 b_i/(x_i^2 y_i)$. By similar arguments we can show that $E(\sum_{i=1}^p K_i)\geqq 4\ E(f)(p-e-a/8)$.
This completes the proof.

Remark 1. To apply Theorem, it is necessary to calculate e (Improvement is obtainable only if $e < p$.). Similar problem is discussed in Berger and Bock (1976b). If min $n_k \to \infty$, then $e \to 1$. And as is shown in Berger and Bock (1976b), if the n_i are even, an exact formula for e can be given as follows

$$\sum_{i=1}^{p} \frac{(m_i+1)^{m_i}}{(m_i-1)!} \sum \left\{ \left[\prod_{l \neq i} \frac{(m_l+1)^{m_l - h(l)}}{(m_l - h(l))!} \right] \frac{(m-H(i)-2)!}{M^{m-H(i)-1}} \right\}$$

where $m_i = n_i/2$, $m = \sum_{i=1}^{p} m_i$, $H(i) = \sum_{l \neq i} h(l)$, $M = \sum_k (n_k+2)/2$, and the inner summation is over all combinations $(h(1), h(2), \cdots, h(i-1), h(i+1), \cdots\cdots, h(p))$, where the $h(l)$'s are integers between 1 and m_i inclusive. If $n_i = n$, $i = 1, \cdots, p$, then by using the result due to Hartley and David (1954), we may have an upper bound of e as follows

$$\frac{e}{n+2} \leqq \frac{1}{n-2} + \frac{p-1}{\sqrt{2p-1}} \frac{\sqrt{2}}{(n-2)\sqrt{n-4}}.$$

Using the above inequality, we can have the value of n such that improvement is obtainable. The result is given in the following table

p	2	3	4	5~8	9~
$n \geqq$	11	8	7	6	5

Remark 2. We note that ϕ expands ϕ_0 towards infinity. We also notice that ϕ can expand ϕ_0 over minimum variance unbiased estimator x_i/n_i. So ϕ itself may be inadmissible and may be improved upon.

Remark 3. Even if means of normal distributions are unknown, we may formally apply the above arguments. However, if mean is unknown, best multiple of sample variance is inadmissible as was shown in Stein (1964).

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