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COMPLETE CONNECTIVITY OF A GRAPH

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ABSTRACT

The object of this paper is to discuss about the connectivity of a graph $G=(X, \Gamma)$ in relation to its adjacency matrix A.

Proposition 1 and Theorem 1 of this paper provide necessary and sufficient conditions for a graph G to be strongly and completely connected, respectively. Theorem 2 shows that there is a critical constant n^2-2n+2 for the number of the positive power m, to judge whether the graph G is completely connected or not.

Thus this paper gives a direct proof of Theorem 2, which is due to WIELANDT (1950) and later discussed by HOLLADAY and VARGE (1958), PERKINS (1961) and DULMAGE and MENDELSOHN (1964), on the basis of a new lemma giving a definite insight into the structure of a completely connected graph.

1. Preliminaey Results

In this paper we consider finite directed graphs which may have some loops. The notations used in this paper are the same as those used in BERGE (1963), for the most part.

A graph, $G=(X, \Gamma)$, is the pair consisting of the set of points $X=\{x_1, \cdots, x_n\}$ and the mapping Γ from X into X. We definite U as the set of all arcs of the graph.

Let us define a_{ij} by

$$a_{ij}=\begin{cases}1, & \text{if } (x_j, x_i)\in U, \\ 0, & \text{if } (x_j, x_i)\notin U,\end{cases} \tag{1}$$

and we obtain an $n\times n$ matrix $A=(a_{ij})$, which is called the adjacency matrix of the graph G.

Let E and $\mathbf{1}$ be the $n\times n$ matrix and n vector, respectively, whose every element is 1 and $\boldsymbol{e}_i$ be the n vector whose i-th element is 1 and the rest are 0.

Thus every vector and matrix we use here is Boolean vector and Boolean matrix, respectively. The sum and the multiplicity between bectors and matrices, such as $\boldsymbol{e}_i+\boldsymbol{e}_j$, $A\boldsymbol{e}_j$, $A+A$, $A\times A$, $\cdots$ means Boolean sum and Boolean multiplicity.

In order to discuss transition aspects among points in our graph, it is convenient to represent the points where we are occupying (locating) by means of a vector. We define a vector $\boldsymbol{y}=(y_1, \cdots, y_n)^t$ as follows: for $i=1, \cdots, n$ each component y_i is 1 if we are occupying the point x_i and 0 otherwise. We call $\boldsymbol{y}$ the state vector of the graph. Thus the vector $\boldsymbol{e}_i$ is the state vector corresponding one-to-one to the point x_i and we denote this one-to-one correspondence by $\boldsymbol{e}_i\sim x_i$. Similarly we have a one-to-one correspondence between any subset of X and the sum of state vectors $\boldsymbol{e}_i$, e.g., $\{x_i, x_j, \cdots, x_k\}\sim\boldsymbol{e}_i+\boldsymbol{e}_j+\cdots+\boldsymbol{e}_k$. Then, $A\boldsymbol{e}_i$ shows the set of all points where we can go from the point x_i through the arc of the graph G in one step. In general, for $\boldsymbol{y}=\boldsymbol{e}_{i1}+\boldsymbol{e}_{i2}+\cdots+\boldsymbol{e}_{ik}$ corresponding to the set $Y=\{x_{i1}, x_{i2}, \cdots, x_{ik}\}$, $A^m\boldsymbol{y}$ shows the set of the points where we can go from a point belonging to the set Y through the arcs in m steps.

A graph $G=(X, \Gamma)$ is said to be *strongly connected* if, for any points $x_i, x_j\in X$, there is a positive integer m such that $\Gamma^m x_i \ni x_j$. Or equivalently, for any $\boldsymbol{e}_i, \boldsymbol{e}_j\in E$ there is a positive integer m such that $A^m\boldsymbol{e}_i\geqq\boldsymbol{e}_j$. Here for n vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, $\boldsymbol{x}\geqq\boldsymbol{y}$ means $x_i\geqq y_i$ for all i, and otherwise we denote by $\boldsymbol{x}\ngeqq\boldsymbol{y}$.

Proposition 1. *A graph G is strongly connected if and only if its adjacency matrix is irreducible.*

Proof

Let us assume that A is reducible. Without loss of generality we may consider the case where A is of the form:

$$\left[\begin{array}{c:c} A_1 & 0 \\ \hdashline B & A_2 \end{array}\right]$$

with the square matrices A_1 and A_2. Then it is obvious that G is not strongly connected.

Now let us suppose that G is not strongly connected. We may assume that there is not a path from x_n to x_1. Namely,

$$a_{1n}=a_{1n}^{(2)}=a_{1n}^{(3)}=\cdots=a_{1n}^{(k)}=\cdots=0 \tag{2}$$

where $a_{ij}^{(k)}$ is the (i,j)-th entry of the matrix A^k.

Since

$$a_{1n}^{(2)}=\sum_{k=1}^{n} a_{1k}a_{kn}=\sum_{k=2}^{n-1} a_{1k}a_{kn}=0 \tag{3}$$

implies $a_{1k}a_{kn}=0$ for $2 \leqq k \leqq n-1$, for each k at least either a_{1k} or a_{kn} is zero.

Now let us assume that there are q zeros in the n-th column of the matrix A. By a suitable permutation of the suffices, we have

$$\left[\begin{array}{ccccc}
a_{11}a^*_{12} & \cdots & a^*_{1q} & \overbrace{0 \quad \cdots \quad 0}^{n-q-1} & 0 \\
a^*_{21}a^*_{22} & \cdots & a^*_{2q} & a^*_{2q+1} \cdots a^*_{2n-1} & 0 \\
\vdots & & \vdots & \vdots \qquad\qquad & \vdots \\
\vdots & & \vdots & \vdots \qquad\qquad & 0 \\
\vdots & & \vdots & \vdots \qquad\qquad & 1 \\
\vdots & & \vdots & \vdots \qquad\qquad & \vdots \\
a^*_{n-11} & \cdots & a^*_{n-1q} & a^*_{n-1\,q-1} \cdots\cdots & 1 \\
a_{n1} & \cdots & a^*_{nq} & a^*_{nq+1} \quad \cdots\cdots & a_{nn}
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\}q \\
\left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\}n-q-1 \\
\vphantom{0}
\end{array} \tag{4}$$

Let there be $q-p$ zeros in $\{a^*_{12}, \cdots, a^*_{1q}\}$. By a suitable permutation we have

$$\left(\begin{array}{cccccccr}
\multicolumn{1}{l}{} & & & & & & & \\
\overbrace{a_{11} \quad 1 \cdots 1}^{p} & & & & & & & \overbrace{0 \cdots 0}^{n-p} \\
 & \ddots & & & & & & \vdots \\
 & & \ddots & & & & & 0 \\
 & & & \ddots & & & & 1 \\
 & & & & \ddots & & & \vdots \\
 & & & & & \ddots & & 1 \\
 & & & & & & & a_{nn}
\end{array}\right)
\begin{array}{l}
\vphantom{0} \\
\left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\}q \\
\left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\}n-q-1 \\
\vphantom{0}
\end{array} \tag{5}$$

So, we may assume from the beginning that $A=(a_{ij})$ is of this form.

Then, since

$$a^{(3)}_{1n} = \sum_{k=1}^{n} a^{(2)}_{1k}a_{kn} = \sum_{k=1}^{n}\sum_{l=1}^{n} a_{1l}a_{lk}a_{kn} = \sum_{k=q+1}^{n}\sum_{l=1}^{p} a_{1l}a_{lk}a_{kn} = 0\,,$$

and $a_{1j}=1$ $(j=2, \cdots, p)$ and $a_{in}=1$ $(i=q+1, \cdots, n-1)$, it turns out that a_{ij} must be zero for $2 \leqq i \leqq p$, $q+1 \leqq j \leqq n-1$, that is

$$A=\left(\begin{array}{cccr}
a_{11} & \overbrace{1 \cdots 1}^{p-1} & 0 \cdots & \overbrace{0 \cdots 0}^{n-q} \\
 & & & \vdots \quad\;\; \vdots \\
 & & & 0 \cdots 0 \\
 & & & 0 \\
 & & & \vdots \\
 & & & 0 \\
 & & & 1 \\
 & & & \vdots \\
 & & & 1 \\
 & & & a_{nn}
\end{array}\right)
\begin{array}{l}
\left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\}p \\
\vphantom{\begin{array}{c}0\\0\\0\end{array}} \\
\left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\}n-q-1 \\
\vphantom{0}
\end{array}
\begin{array}{l}
\left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\\0\end{array}}\right\}q \\
\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}
\end{array} \tag{6}$$

Since $a_{1n}^{(4)}=\sum\limits_{k.1.m.} a_{1k}a_{k1}a_{1m}a_{mn}=0$, $a_{1k}=1$ $(k=2,\cdots,p)$ and $a_{mn}=1$ $(m=q+1,\cdots,n-1)$, if $a_{ij}=1$ for some i,j $(q+1\leqq j\leqq n-1, p+1\leqq i\leqq q)$ then $a_{si}=0$ for all s $(2\leqq s\leqq p)$. Accordingly, for example, if $a_{p+1\,n-1}=1$ by the permutation $(p+1,q)$, we have

$$\begin{array}{ccccc|l}
 & & & \multicolumn{2}{c}{\overbrace{\qquad\qquad}^{n-q+1}} & \\
a_{11} & 1\cdots1 & 0\cdots0 & 0\cdots & 0 & \left.\begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} p \\ \\ \\ \end{array}\right\} q \\
 & & \vdots & \vdots & \vdots & \\
 & & 0 & 0\cdots & 0 & \\
 & & & & \vdots & \\
 & & & 1 & 0 & \\
 & & & & 1 & \\
 & & & & \vdots & \\
 & & & & 1 & \\
 & & & & a_{nn} &
\end{array} \tag{7}$$

Repeating in the same manner, we finally obtain

$$\left[\begin{array}{ccccc}
a_{11} & 1\cdots1 & 0\cdots & \overset{\overset{r}{\smile}}{0} & \cdots0 \\
 & & & \vdots & \vdots \\
 & & & 0 & \cdots0 \\
 & & & & 0 \\
 & & & & \vdots \\
 & & & & 0 \\
 & & & & 1 \\
 & & & & \vdots \\
 & & & & 1 \\
 & & & & a_{nn}
\end{array}\right]\begin{array}{l} \\ \\ (s \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \tag{8}$$

where $r-1=s$ $(p+1\leqq r\leqq q+1, p\leqq s\leqq q)$, which shows that A is reducible. Q.E.D.

A graph G is said to be *completely connected* if there exists a positive integer m such that $A^m=\mathrm{E}$.

For a completely connected graph G, if k is the length of a circuit $(x_{i_1},\cdots,x_{i_k})$, then $A^k\boldsymbol{e}_j\geqq\boldsymbol{e}_j$ $(j=1,\cdots,k)$.

2. Main Results

Theorem 1. *A Graph G is completely connected if and only if the following two conditions are satisfied.*

(i) *G is strongly connected.*

(ii) *The greatest common measure of lengths of all elementary circuits of G is 1.*

Proof

Necessity: It is clear that a completely connected graph is strongly connected. By the strong connectivity of the graph, it is clear that the graph has at least one circuit. Let $C_1, C_2, \cdots, C_p$ be all elementary circuits of the graph G with length $n_1, n_2, \cdots, n_p$, respectively. Now we suppose that the greatest common divisor q of $n_1, n_2, \cdots, n_p$ is greater than 1. Then for every r and $\boldsymbol{e}_i \in E$, if $A^r\boldsymbol{e}_i \geqq \boldsymbol{e}_i$ then $A^{r+1}\boldsymbol{e}_i \ngeqq \boldsymbol{e}_i$ must hold.

For, if $A^r\boldsymbol{e}_i \geqq \boldsymbol{e}_i$ then r must be represented by the sum of n_i, i.e. there are non-negative integers a_i $(i=1, \cdots, p)$ such that $r=\sum_{i=1}^{p} a_i n_i$. Therefore, if $A^r\boldsymbol{e}_i \geqq \boldsymbol{e}_i$ and $A^{r+1}\boldsymbol{e}_i \geqq \boldsymbol{e}_i$, there must be a_i and b_i such that

$$r=\sum_{i=1}^{p} a_i n_i\,, \quad r+1=\sum_{i=1}^{p} b_i n_i\,. \tag{9}$$

Since *G.C.M.* $(n_1, n_2, \cdots, n_p)=q$, there are some positive integers u and v such that $r=uq$ and $r+1=vq$, and so $(v-u)q=1$. In order to satisfy this equation, q must be 1. This contradicts the assumption that q is greater than 1. Therefore, if $A^r\boldsymbol{e}_i \geqq \boldsymbol{e}_i$ then $A^{r+1}\boldsymbol{e}_i \ngeqq \boldsymbol{e}_i$ for every r.

On the other hand, for a strongly connected graph G, $A^r\boldsymbol{e}_i=\mathbf{1}$ implies $A^{r+1}\boldsymbol{e}_i=\mathbf{1}$ for any r.

From these facts, we know that there is not a positive integer m such that $A^m\boldsymbol{e}_i=\mathbf{1}$. This contradicts the assumption that the graph is completely connected. Thus, it is concluded that *G.C.M.* $(n_1, n_2, \cdots, n_p)=1$.

Sufficiency of the condition of Theorem 1 will be evident from the proof of Theorem 2. It should be noted that in the proof of Theorem 2, we will exclusively use the necessity of the condition of Theorem 1.

Lemma *Let $G=(X, \Gamma)$ be a strongly connected graph with n vertices. If G has two circuits C_1 and C_2 with length n_1 and n_2, respectively, such as G.C.M. $(n_1, n_2)=p$,* then all points of the subset of circuit C_1 *consisting of every p-th point of C_1 can be reached by at most $2n-(n_1-2n_2)+n_1n_2p^{-1}$ steps for any starting point.*

Proof

Starting from an arbitrary point $x_i \in X$, we can reach the circuit C_2 within $n-n_2$ steps. From the circuit C_2 we can reach circuit C_1 within $n-n_1$ steps, and after that we can reach C_1 at every n_2 step. As *G.C.M.* $(n_1, n_2)=p$, and

$$(n-n_2)+(n-n_1)+(n_1p^{-1}-1)n_2=2n-(n_1-2n_2)+n_1n_2p^{-1}\,, \tag{10}$$

so all points of the subset of circuit C_1 consisting every p-th point of C_1 can be reached within $2n-(n_1-2n_2)+n_1n_2p^{-1}$ steps. Q.E.D.

Theorem 2. *If a graph G whose adjacency matrix is A is completely connected, then there is a positive integer m such that*

(i) $A^m=E$

(ii) $m \leqq n^2-2n+2$. (11)

Proof

From Theorem 1, we may suppose that the graph is strongly connected. Moreover, according to Theorem 1, the existence of circuits $C_1, C_2, \cdots, C_h$ such that *G.C.M.* $(n_1, n_2, \cdots, n_h)=1$ is assured. Here, n_i is the length of the circuit C_i.

Let us choose a subset $\{C_i\}$ of the circuits so that *G.C.M.* $(n_1, n_2, \cdots, n_k)=1$ with the smallest number k. We define m to be the smallest number which satisfies $A^m=\mathrm{E}$.

Now let us prove the Theorem for each case of k.

Case 1: $k=1$ (i.e. the graph has a loop)

Let x_j be a point which has a loop. For arbitrary i, there exists an integer s $(0 \leqq s \leqq n-1)$ such that $A^s\boldsymbol{e}_i \geqq \boldsymbol{e}_j$. Accordingly, since $A\boldsymbol{e}_j \geqq \boldsymbol{e}_j$,

$$\begin{aligned} & A^{s-1}\boldsymbol{e}_i \geqq A\boldsymbol{e}_j = \boldsymbol{e}_j + A\boldsymbol{e}_j \\ & A^{s-2}\boldsymbol{e}_i \geqq A\boldsymbol{e}_j + A^2\boldsymbol{e}_j = \boldsymbol{e}_j + A\boldsymbol{e}_j + A^2\boldsymbol{e}_j \\ & \cdots\cdots\cdots\cdots \\ & A^{s+n-1}\boldsymbol{e}_i \geqq \boldsymbol{e}_j + A\boldsymbol{e}_j + \cdots + A^{n-1}\boldsymbol{e}_j = \sum_{m=0}^{n-1} A^m\boldsymbol{e}_j = E\boldsymbol{e}_j = \mathbf{1} \end{aligned} \tag{12}$$

Since we take $\boldsymbol{e}_i$ arbitrary, A^{s+n-1} must be E. Therefore, $m \leqq s+n-1 \leqq 2(n-1)$.

Case 2: $k=2$ (i.e. the graph has two circuits)

Let us denote the two circuits by C_1, C_2 and its lengths by n_1, n_2, respectively. In this case *G.C.M.* $(n_1, n_2)=1$, $(n_1>n_2)$.

(i) The case when $n_1 \leqq n-2$ $(n_2 \leqq n-3)$.

According to Lemma, we know by at most $2n-(n_1+2\mathrm{n}_2)+n_1n_2$ steps, all points of the circuit C_1 are occupied. Therefore, after that, by $n-n_1$ steps all points of the graph G are occupied. This gives us that

$$\begin{aligned} m & \leqq 2n-(n_1+2n_2)+n_1n_2+n-n_1 \\ & = 3n-2(n_1+n_1)+n_1n_2 \\ & \leqq 3n-2(2+3)+(n-2)(n-3) \\ & = n^2-2n-4 \end{aligned} \tag{13}$$

(ii) The case when $n_1=n$.

Let us assume $C_1 \supset C_2$ and the number of points which belong to C_1 and do not belong to C_2 be k. Then for any n $(\geqq 2)$

$$m \leqq k+(n-1)(n-k) \leqq n^2-2n+2 \tag{14}$$

(iii) The case when $n_1=n-1$.

In this case we hav $n_2 \leqq n-2$ and

$$m \leqq (n-1)+(n-2)(n-2) \leqq n^2-2n+2 \tag{15}$$

Case 3: Lastly we prove the assertion of Theorem 2 for the case when the graph has neither a loop nor a pair of circuits such that *G.C.M.* $(n_1, n_2)=1$: i.e. $k \geqq 3$. To prove this we firstly show the fact that in such a case n must be considerably large.

(i) If $k=3$, then $G.C.M.\ (n_1, n_2, n_3)=1$, $G.C.M.\ (n_1, n_2)=p_2$, $G.C.M.\ (n_1, n_3)=p_3$, $G.C.M.\ (p_2, p_3)=1$ must hold. Therefore the smallest triplet which satisfy this condition is $n_1=2\times 3$, $n_2=2\times 5$, $n_3=3\times 5$ and minimum of n is $3\times 5=15$.

(ii) If $k=4$, then $G.C.M.\ (n_1, n_2, n_3, n_4)=1$, $G.C.M.\ (n_1, n_2)=p_2$, $G.C.M.\ (n_1, n_3)=p_3$, $G.C.M.\ (n_1, n_4)=p_4$, $G.C.M.\ (p_2, p_3, p_4)=1$ must hold. Moreover, $G.C.M.\ (p_2, p_3)=q_3$, $G.C.M.\ (p_2, p_4)=q_4$, $G.C.M.\ (q_3, q_4)=1$ must be satisfied. Consequently, we know that $p_2=c_2q_3q_4$, $p_3=c_3q_3r_3$, $p_4=c_4q_4r_4$, here c_i $(i=2, 3, 4)$ are some constants, r_3 is a prime such that $r_3 \nmid p_3$, r_4 is a prime such that $r_4 \nmid p_4$. It is possible that $r_3=r_4$.

(iii) Similarly, generally if $G.C.M.\ (n_1, n_2, \cdots, n_k)=1$ and $G.C.M.\ (n_1, n_i)=p_i$ $(i=2, \cdots, k)$, then p_i $(i=2, \cdots, k)$ must have at least $k-2$ different primes as divisor. On the other hand, since $G.C.M.$ of succeeding any two natural number is 1, if we take $n_1, n_2, \cdots, n_k$ in descending order of its value, $2k\leqq n_1\leqq n$, $2(k-1)\leqq n_2\leqq n-2$, $2(k-2)\leqq n_3\leqq n-4$, $\cdots$, $2\leqq n_k\leqq n-2(k-1)$ must be satisfied.

According to Lemma, we know that by at most $2n-(n_1+2n_2)+n_1n_2p_2^{-1}$ steps, circuit C_1 is occupied every p_2-th point. After that, by at most $2n-(n_1+2n_3)+n_1n_3p_3^{-1}$ steps, circuit C_1 is occupied by every $G.C.M.\ (p_2, p_3)=G.C.M.\ (n_1, n_2, n_3)$-th point. Consequently, by at most

$$\begin{aligned}&\{2n-(n_1+2n_2)+n_1n_2p_2^{-1}\}+\cdots+\{2n-(n_1+2n_k)+n_1n_kp_k^{-1}\}\\&\quad=(k-1)(2n-n_1)-2(n_2+n_3+\cdots+n_k)+n_1(n_2p_2^{-1}+n_3p_3^{-1}+\cdots+n_kp_k^{-1})\end{aligned}\tag{16}$$

steps every point of circuit C_1 is occupied, because of the fact that $G.C.M.\ (n_1, n_2, \cdots, n_k)=1$. Therefore, after that by at most $n-n_1$ steps, every point of the graph G is occupied.

Therefore the evaluation of m is now obtained as follows.

(i) For $k=3$, we have

$$\begin{aligned}m&\leqq 2(2n-6)-2(2+4)+n\left(\frac{n-2}{2}+\frac{n-4}{3}\right)+n-6\\&=\frac{5}{6}n^2+\frac{8}{3}n-32\ ,\end{aligned}\tag{17}$$

which is smaller than $M(n)=n^2-2n+2$ for $n\geqq 12$.

Now the condition $n\geqq 12$ is not restrictive. In fact $n_1=d_1pq$, $n_2=d_2pr$, $n_3=d_3qs$, where d_i are some positive integers and p, q, r and s are primes which are different with each other. (However, it is possible that $r=s$.) Therefore, $n\geqq \max(n_1, n_2, n_3)\geqq \max(pq, pr, qs)\geqq 3\times 5=15$.

(ii) For $k\geqq 4$, we have

$$\begin{aligned}m&\leqq 2(k-1)(n-k)-4((k-1)+(k-2)+\cdots+1)\\&\quad+2^{2-k}n((n-2)+(n-4)+\cdots+(n-2k-2))\\&=2^{2-k}(k-1)n^2+(2k-2+2^{2-k}k(k-1))n-3k(1-k)\end{aligned}\tag{18}$$

which is smaller than $M(n)$ for $n>8k$ (as shown in Note).

Now the condition $n>8k$ is not restrictive. In fact, as before if $k=4$, $n\geqq 3\times 5\times 7=105$, and if $k=5$, $n\geqq 3\times 5\times 7\times 11=1155$, and generally $n\geqq 3^{k-1}>8k$ for $k\geqq 4$.

Consequentry, in this case, we have

$$m\leqq M(n)=n^2-2n+2\,. \qquad \text{Q.E.D.}$$

3. Applications

We define an $n\times n$ matrix $B=(b_{ij})$ to be non-negative (positive) if all of its elements b_{ij} are non-negative (positive). Now as an immediate consequence of Theorem 2 we have

Theorem 3. *Let B be a non-negative matrix. If B^m is not positive for $m=1, 2, \cdots, n^2-2n+2$ then B^m is not positive for all positive integer m.*

It is remarked that this result is given by WIELANDT (1950) without proof.

Corollary 1. *Let B be a non-negative matrix. If B^{n^2-2n+2} is not positive then there is no positive integer m such that B^m is positive.*

Note

$$\begin{aligned}&(n^2-2n+2)-\{2^{2-k}(k-1)n^2+(2k-2+2^{2-k}k(k-1))n-3k(1-k)\}\\&=(1-2^{2-k}(k-1))n^2-(2k-2^{2-k}k(k-1))n+3k^2-3k+2\\&\equiv an^2-bn+c \qquad (19)\end{aligned}$$

is positive for $n>ba^{-1}$ because a, b, and c are positive for $k\geqq 4$ and we have

$$\frac{b+(b^2-4ac)}{2a}<\frac{b}{a}=\frac{2k-2^{2-k}k(k-1)}{1-2^{2-k}(k-1)}<4(2k-k(k-1))<8k \qquad (20)$$

as $0<2^{2-k}(k-1)<\frac{3}{4}$ for $k\geqq 4$.

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