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# CONVERGENCE OF POWER SERIES SOLUTIONS OF A LINEAR PFAFFIAN SYSTEM AT AN IRREGULAR SINGULARITY

YASUTAKA SIBUYA**

**School of Math., Univ. of Minnesota, Minneapolis, Minn. 55455, USA**

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**ABSTRACT**

In this paper, we shall prove the existence and uniqueness of a convergent power series solution of a linear Pfaffian system at an irregular singularity. Our method is similar to the method given by W.A. HARRIS, JR., Y. SIBUYA and L. WEINBERG [2]. We do not utilize the theory of asymptotic solutions.

## 1. Introduction

We consider two n-by-n matrices $A(x, y)=\sum_{h,k=0}^{\infty} A_{hk}x^h y^k$ and $B(x, y)=\sum_{h,k=0}^{\infty} B_{hk}x^h y^k$ whose components are convergent power series in two variables $(x, y)$, where the $A_{hk}$ and $B_{hk}$ are n-by-n (complex) constant matrices. Let $p$ and $q$ be two positive integers, and set $D_1=x^{p+1}(\partial/\partial x)$, $D_2=y^{q+1}(\partial/\partial y)$, and $V=\boldsymbol{C}^n\langle\langle x, y\rangle\rangle$, where $\boldsymbol{C}^n\langle\langle x, y\rangle\rangle$ is the set of all convergent power series $\sum_{h,k=0}^{\infty} c_{hk}x^h y^k$ in $(x, y)$ whose coefficients $c_{hk}$ are $n$ dimensional constant vectors.

We introduce in $V$ a structure of a vector space over $\boldsymbol{C}$ in a natural way, and define two linear operators

$$L(u)=D_1u-A(x, y)u \quad \text{and} \quad K(u)=D_2u-B(x, y)u, \quad u\in V.$$

* Visiting Professor of Keio University from July 1977 to August 1977.

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If $A_{00} \in GL(n; \boldsymbol{C})$ and $B_{00} \in GL(n; \boldsymbol{C})$, then $L$ and $K$ are injective. The main result of this paper is the following theorem.

THEOREM A. *Suppose that*

(i) $$A_{00} \in GL(n; \boldsymbol{C}) \quad \textit{and} \quad B_{00} \in GL(n; \boldsymbol{C}), \textit{ and}$$

(ii) $$LK = KL\,.$$

*Then, for* $u \in V$, *we have* $u \in \mathrm{range}(L)$ *if and only if* $K(u) \in \mathrm{range}(L)$.

## 2. An application

As an application, we consider a linear Pfaffian system

(E) $$\begin{cases} D_1 u = A(x, y)u + f(x, y), \\ D_2 u = B(x, y)u + g(x, y), \end{cases}$$

where $f \in V$ and $g \in V$.

LEMMA 2.1: *Pfaffian system* (*E*) *is completely integrable if and only if* (i) $LK = KL$, *and* (ii) $K(f) = L(g)$

PROOF: Pfaffian system (E) is completely integrable if and only if

($C_1$) $$D_2 A(x, y) + A(x, y)B(x, y) = D_1 B(x, y) + B(x, y)A(x, y),$$

and

($C_2$) $$A(x, y)g(x, y) + D_2 f(x, y) = B(x, y)f(x, y) + D_1 g(x, y).$$

Conditions ($C_1$) and ($C_2$) can be written as

($C_1'$) $$K(A) = L(B)$$

and

($C_2'$) $$K(f) = L(g)$$

respectively. Note that

$$LK(u) = D_1 D_2 u - BD_1 u - AD_2 u - L(B)u,$$

and

$$KL(u) = D_2 D_1 u - BD_1 u - AD_2 u - K(A)u.$$

Hence ($C_1'$) and (i) are equivalent. This completes the proof of Lemma 2.1.

THEOREM B: *If Pfaffian system* (E) *is completely integrable, and if* $A_{00} \in GL(n; \boldsymbol{C})$ *and* $B_{00} \in GL(n; \boldsymbol{C})$, *then system* (E) *has a solution* $u$ *in* $V$. *Moreover, this solution is unique.*

PROOF: Condition (ii) of Lemma 2.1 implies that $K(f)\in\text{range}(L)$. Therefore, by virtue of Theorem A, it follows from condition (i) of Lemma 2.1 that $f\in\text{range}(L)$. This means that $L(u)=f$ for some $u\in V$. Hence,

$$L(g)=K(f)=KL(u)=LK(u).$$

Since $L$ is injective, we have $K(u)=g$. This proves the existence of a solution $u$ of $(E)$ in $V$. The uniqueness follows from the injectivity of $L$ and $K$.

REMARK: Theorem B is a special case (i.e. the linear case) of a theorem which was proved by R. GERARD and Y. SIBUYA [1; Théorème 3, p. 58]. Their proof was, however, based on the theory of asymptotic solutions and connection problems of nonlinear ordinary differential equations containing parameters at an irregular singular point. (Cf. Y. SIBUYA [3].) The proof of Theorem B in this paper is entirely different from their proof.

## 3. A characterization of range(L)

We shall characterize range($L$) in a way similar to the idea given by W. A. HARRIS, JR., Y. SIBUYA and L. WEINBERG [2].

Let us fix two positive numbers $\delta_0$ and $\delta$. Set

$$\mathscr{D}=\{y;\ |y|<\delta_0\},$$

and denote by $\Omega$ the set of all mappings from $\mathscr{D}$ to $\boldsymbol{C}^n$ which are holomorhic and bounded in $\mathscr{D}$. For $c\in\Omega$, the notation $|c|$ denotes $\sup_{\mathscr{D}}|c(y)|$. For a power series

$$\varphi=\sum_{m=0}^{\infty} c_m x^m \quad (c_m\in\Omega),$$

we set

$$\|\varphi\|_\delta=\sum_{m=0}^{\infty}|c_m|\delta^m,$$

$$\mathscr{B}_\delta=\{\varphi;\ \|\varphi\|_\delta<+\infty\},\ \text{and}$$

$$\mathscr{B}_{\delta,M}=\{x^M\varphi;\ \varphi\in\mathscr{B}_\delta\},$$

where $M$ is a positive integer.

Let $A(x,y)$ be the matrix which was given in § 1. Set

$$A(x,y)=\sum_{m=0}^{\infty}A_m(y)x^m \quad\text{and}\quad \|A\|_\delta=\sum_{m=0}^{\infty}|A_m|\delta^m,$$

where $|A_m|=\sup_{\mathscr{D}}|A_m(y)|$. Assume that $\|A\|_\delta<+\infty$. Choose $M$ so that

$$\frac{\|A\|_\delta}{\delta^p M}<1,$$

where $p$ is the integer which was given in § 1. We define a mapping $A$: $\mathcal{B}_{\delta,M}\to\mathcal{B}_{\delta,M}$ by $A(\varphi)(x,y)=A(x,y)\varphi(x,y)$. Then, $\|A(\varphi)\|_\delta\leqq\|A\|_\delta\|\varphi\|_\delta$.

Let us define a mapping $P$: $\mathcal{B}_{\delta,M}\to\mathcal{B}_{\delta,M+p}$ by

$$P(\varphi)=\sum_{m=M+p}^{\infty}c_m x^m,$$

where

$$\varphi=\sum_{m=M}^{\infty}c_m x^m.$$

Then

$$\|P(\varphi)\|_\delta\leqq\|\varphi\|_\delta.$$

We also define a mapping $T$: $\mathcal{B}_{\delta,M+p}\to\mathcal{B}_{\delta,M}$ by

$$T(\varphi)=\sum_{m=M}^{\infty}\frac{c_m}{m}x^m,$$

where

$$\varphi=\sum_{m=M}^{\infty}c_m x^{m+p}.$$

Then

$$\|T(\varphi)\|_\delta\leqq\frac{1}{\delta^p M}\|\varphi\|_\delta.$$

It is easy to prove that

$$D_1T(\varphi)=\varphi\quad\text{for }\varphi\in\mathcal{B}_{\delta,M+p},$$

and that

$$T(D_1\varphi)=\varphi\quad\text{for }\varphi\in\mathcal{B}_{\delta,M}\text{ if }D_1\varphi\in\mathcal{B}_{\delta,M+p}.$$

Since

$$\|TPA(\varphi)\|_\delta\leqq\frac{\|A\|_\delta}{\delta^p M}\|\varphi\|_\delta\quad\text{for }\varphi\in\mathcal{B}_{\delta,M},$$

the mapping $I-TPA$: $\mathcal{B}_{\delta,M}\to\mathcal{B}_\delta$ is an isomorphism, where $I$ is the identity map. Set

(3.1) $$\Phi=(I-TPA)^{-1}TP.$$

Then the mapping $\Phi$: $\mathcal{B}_{\delta,M}\to\mathcal{B}_{\delta,M}$ satisfies the identity

(3.2) $$L\Phi(\varphi)=\varphi-Q(\varphi)\quad(\varphi\in\mathcal{B}_{\delta,M}),$$

where

(3.3) $$Q=(I-P)(I+A\Phi).$$

In fact, note that

$$(I-TPA)\Phi=TP,$$

$$D_1(I-TPA)\Phi(\varphi)=D_1\Phi(\varphi)-PA\Phi(\varphi),\quad\text{and}$$

$$D_1TP(\varphi)=P(\varphi).$$

Hence

$$D_1\Phi(\varphi)=PA\Phi(\varphi)+P(\varphi)$$

and

$$L\Phi(\varphi)=PA\Phi(\varphi)+P(\varphi)-A\Phi(\varphi)=\varphi-(I-P)(I+A\Phi)(\varphi).$$

This proves (3. 2).

Suppose that $\varphi\in\mathcal{B}_{\delta,M}$ and $D_1\varphi\in\mathcal{B}_{\delta,M+p}$. Then

$$\Phi L(\varphi)=\varphi. \tag{3.4}$$

In fact,

$$P(D_1\varphi)=D_1\varphi \quad\text{and}\quad TP(D_1\varphi)=\varphi.$$

Hence

$$\Phi(D_1\varphi)=(I-TPA)^{-1}(\varphi).$$

Since

$$\Phi A(\varphi)=(I-TPA)^{-1}TPA(\varphi),$$

we have

$$\Phi L(\varphi)=(I-TPA)^{-1}(I-TPA)(\varphi)=\varphi.$$

LEMMA 3.1: *Suppose that $A_{00}\in GL(n;\boldsymbol{C})$. Then, if*

$$\psi=\sum_{m=M}^{M+p-1}c_m x^m\in L(\mathcal{B}_\delta),$$

*it follows that $\psi=0$.*

*Proof*: Let $\psi=L(u)$ for some $u\in\mathcal{B}_\delta$. Then $u\in\mathcal{B}_{\delta,M}$ and hence $D_1u\in\mathcal{B}_{\delta,M+p}$. Therefore, $\Phi(\psi)=u$. On the other hand, $P(\psi)=0$ and hence $\Phi(\psi)=0$. This proves that

$$\psi=L(0)=0.$$

LEMMA 3.2: *Suppose that $A_{00}\in GL(n;\boldsymbol{C})$. Then, for $\varphi\in\mathcal{B}_{\delta,M}$, we have $\varphi\in L(\mathcal{B}_\delta)$ if and only if $Q(\varphi)=0$.*

*Proof*: If $Q(\varphi)=0$, then (3.2) implies that $\varphi=L\Phi(\varphi)\in L(\mathcal{B}_\delta)$. If $\varphi=L(u)$ for some $u\in\mathcal{B}_\delta$, then

$$Q(\varphi)=\varphi-L\Phi(\varphi)=L(u-\Phi(\varphi))\in L(\mathcal{B}_\delta).$$

Hence, by virtue of Lemma 3.1, we have $Q(\varphi)=0$.

LEMMA 3.3: *Suppose that $A_{00}\in GL(n;\boldsymbol{C})$. Then, there exists a mapping $S$: $\mathcal{B}_\delta\to\mathcal{B}_\delta$ such that*

(i) $$D_1S(\mathcal{B}_\delta)\subset\mathcal{B}_\delta;$$

(ii) $$(I-LS)(\mathcal{B}_\delta)\subset\mathcal{B}_{\delta,M}.$$

REMARK: An easy (formal) computation would determine $S(\varphi)$ as a polynomial in $x$ with coefficients in $\Omega$.

Lemma 3.4: *Suppose that* $A_{00}\in GL(n;\boldsymbol{C})$. *Then, for* $\varphi\in\mathcal{B}_\delta$, *we have* $\varphi\in L(\mathcal{B}_\delta)$ *if and only if*

(3.5) $$Q(I-LS)(\varphi)=0.$$

*Proof*: Suppose that (3.5) is satisfied. Then $(I-LS)(\varphi)\in L(\mathcal{B}_\delta)$, and hence $\varphi\in L(\mathcal{B}_\delta)$. *If* $\varphi\in L(\mathcal{B}_\delta)$, then $(I-LS)(\varphi)\in L(\mathcal{B}_\delta)$. Hence, (3.5) must be satisfied.

## 4. An important lemma.

Throughout this section, we assume that

( i ) $$A_{00}\in GL(n;\boldsymbol{C}) \quad\text{and}\quad B_{00}\in GL(n;\boldsymbol{C}), \text{ and}$$

( ii ) $$\|A\|_\delta<+\infty \quad\text{and}\quad \|B\|_\delta<+\infty.$$

We also utilize notations and results of §3.

Lemma 4.1: *If* $\psi=\sum_{m=M}^{M+p-1} c_m x^m$, *and if* $D_2\psi\in\mathcal{B}_\delta$ *and*

(4.1) $$K(\psi)\in L(\mathcal{B}_\delta),$$

*then* $\psi=0$.

*Proof*: Since $K(\psi)\in\mathcal{B}_{\delta,M}$ and $A_{00}\in GL(n;\boldsymbol{C})$, we have

$$QK(\psi)=0.$$

This means that

(4.2) $$Q(D_2\psi)-QB(\psi)=0.$$

We shall prove first that

(4.3) $$Q(D_2\psi)=D_2\psi.$$

Note that $D_2\psi=\sum_{m=M}^{M+p-1}(D_2c_m)x^m$. Hence $P(D_2\psi)=0$. Therefore $\Phi(D_2\psi)=0$, and hence (4.3) follows from (3.3).

By utilizing (4.3), we write (4.2) as

(4.4) $$D_2\psi=QB(\psi).$$

This means that

(4.5) $$D_2c_m=\sum_{k=M}^{M+p-1}\gamma_{mk}c_k, \qquad m=M,\cdots,M+p-1,$$

where the $\gamma_{mk}$ are n-by-n matrices whose components are in $\Omega$. Set

$$\Gamma(y)=\begin{bmatrix} \gamma_{MM} & \cdots & \gamma_{MM+p-1} \\ \cdots & \cdots & \cdots \\ \gamma_{M+p-1M} & \cdots & \gamma_{M+p-1M+p-1} \end{bmatrix}.$$

We shall prove that $\Gamma(0)\in GL(np\,;\boldsymbol{C})$.

The quantities $A, B, L, K, \Phi, Q, \varphi$ etc depend on $y$. We denote by $A_0, B_0, L_0, K_0, \Phi_0, Q_0, \varphi_0$ etc those quantities at $y=0$. Then,

$$K_0(\varphi_0)=-B_0(\varphi_0),$$

$$L_0(B_0(\varphi_0))=B_0(L_0(\varphi_0)),\ \text{or}\ B_0^{-1}L_0B_0(\varphi_0)=L_0(\varphi_0),$$

and

$$L_0\Phi_0(\varphi_0)=\varphi_0-Q_0(\varphi_0)\,.$$

Hence, if $Q_0B_0(\psi_0)=0$, we have

$$B_0(\psi_0)=L_0\Phi_0(B_0(\psi_0)),\ \text{or}$$

$$\psi_0=B_0^{-1}L_0B_0B_0^{-1}\Phi_0B_0(\psi_0)=L_0B_0^{-1}\Phi_0B_0(\psi_0).$$

This means that

$$\psi_0=L_0(u_0),$$

where

$$u_0=B_0^{-1}\Phi_0B_0(\psi_0)$$

Since $\Phi_0L_0(u_0)=u_0$, we have $\Phi_0(\psi_0)=u_0=0$, and hence $\psi_0=0$. This proves that $\Gamma(0)\in GL(np\,;\boldsymbol{C})$.

Finally, it follows from (4.5) that $\psi=0$.

## 5. Proof of Theorem A.

Set $K(u)=L(v)$. Note that

$$K(I-LS)(u)=L(v-KS(u)),$$

and hence

$$\begin{aligned} KQ(I-LS)(u)&=K(I-LS)(u)-LK\Phi(I-LS)(u) \\ &=L(v-KS(u)-K\Phi(I-LS)(u)). \end{aligned}$$

Therefore,

$$Q(I-LS)(u)=0\quad (\text{cf. Lemma 4.1}),$$

and

$$u\in L(\mathscr{B}_\delta)\quad (\text{cf. Lemma 3.4}).$$

This completes the proof of Theorem A. In this argument, we must choose $\delta_0$ and $\delta$ sufficiently small.

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