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CONVERGENCE OF POWER SERIES SOLUTIONS OF A LINEAR PFAFFIAN SYSTEM AT AN IRREGULAR SINGULARITY

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ABSTRACT

In this paper, we shall prove the existence and uniqueness of a convergent power series solution of a linear Pfaffian system at an irregular singularity. Our method is similar to the method given by W.A. HARRIS, JR., Y. SIBUYA and L. WEINBERG [2]. We do not utilize the theory of asymptotic solutions.

1. Introduction

We consider two n-by-n matrices $A(x, y) = \sum_{\substack{h,k=0\\h,k=0}}^{\infty} A_{hk} x^h y^k$ and $B(x, y) = \sum_{\substack{h,k=0\\h,k=0}}^{\infty} B_{hk} x^h y^k$ whose components are convergent power series in two variables (x, y), where the A_{hk} and B_{hk} are n-by-n (complex) constant matrices. Let p and q be two positive integers, and set $D_1 = x^{p+1}(\partial/\partial x)$, $D_2 = y^{q+1}(\partial/\partial y)$, and $V = C^n \ll x, y \gg$, where $C^n \ll x, y \gg$ is the set of all convergent power series $\sum_{\substack{h,k=0\\h,k=0}}^{\infty} c_{hk} x^h y^k$ in (x, y) whose coefficients c_{hk} are n dimensional constant vectors.

We introduce in V a structure of a vector space over C in a natural way, and define two linear operators

 $L(u) = D_1 u - A(x, y)u \text{ and } K(u) = D_2 u - B(x, y)u, u \in V.$

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If $A_{00} \in GL(n; C)$ and $B_{00} \in GL(n; C)$, then L and K are injective. The main result of this paper is the following theorem.

THEOREM A. Suppose that

(i)
$$A_{00} \in GL(n; \mathbf{C})$$
 and $B_{00} \in GL(n; \mathbf{C})$, and

(ii) LK = KL.

Then, for $u \in V$, we have $u \in \operatorname{range}(L)$ if and only if $K(u) \in \operatorname{range}(L)$.

2. An application

As an application, we consider a linear Pfaffian system

(E)
$$\begin{cases} D_1 u = A(x, y)u + f(x, y), \\ D_2 u = B(x, y)u + g(x, y), \end{cases}$$

where $f \in V$ and $g \in V$.

LEMMA 2.1: Pfaffian system (E) is completely integrable if and only if

(i) LK = KL, and (ii) K(f) = L(g)

 PROOF : Pfaffian system (E) is completely integrable if and only if

(C₁)
$$D_2A(x, y) + A(x, y)B(x, y) = D_1B(x, y) + B(x, y)A(x, y),$$

and

(C₂)
$$A(x, y)g(x, y) + D_2f(x, y) = B(x, y)f(x, y) + D_1g(x, y).$$

Conditions (C_1) and (C_2) can be written as

$$(C_1') K(A) = L(B)$$

and
$$(C_2')$$
 $K(f) = L(g)$

respectively. Note that

 $LK(u) = D_1 D_2 u - B D_1 u - A D_2 u - L(B) u,$

and

$$KL(u) = D_2 D_1 u - B D_1 u - A D_2 u - K(A) u.$$

Hence (C_1') and (i) are equivalent. This completes the proof of Lemma 2.1.

THEOREM B: If Pfaffian system (E) is completely integrable, and if $A_{00} \in GL(n; C)$ and $B_{00} \in GL(n; C)$, then system (E) has a solution u in V. Moreover, this solution is unique. Convergence of Power Series Solutions of a Linear Pfaffian System

PROOF: Condition (ii) of Lemma 2.1 implies that $K(f) \in \operatorname{range}(L)$. Therefore, by virtue of Theorem A, it follows from condition (i) of Lemma 2.1 that $f \in \operatorname{range}(L)$. This means that L(u) = f for some $u \in V$. Hence,

$$L(g) = K(f) = KL(u) = LK(u).$$

Since L is injective, we have K(u)=g. This proves the existence of a solution u of (E) in V. The uniqueness follows from the injectivity of L and K.

REMARK: Theorem B is a special case (i.e. the linear case) of a theorem which was proved by R. GERARD and Y. SIBUYA [1; Théorème 3, p. 58]. Their proof was, however, based on the theory of asymptotic solutions and connection problems of nonlinear ordinary differential equations containing parameters at an irregular singular point. (Cf. Y. SIBUYA [3].) The proof of Theorem B in this paper is entirely different from their proof.

3. A characterization of range(L)

We shall characterize range(L) in a way similar to the idea given by W.A. HARRIS, JR., Y. SIBUYA and L. WEINBERG [2].

Let us fix two positive numbers δ_0 and δ . Set

$$\mathcal{D} = \{y; |y| < \delta_0\},\$$

and denote by Ω the set of all mappings from \mathcal{D} to \mathbb{C}^n which are holomorphic and bounded in \mathcal{D} . For $c \in \Omega$, the notation |c| denotes $\sup_{\alpha} |c(y)|$. For a power series

$$\varphi = \sum_{m=0}^{\infty} c_m x^m \quad (c_m \in \mathcal{Q}),$$

we set

$$\begin{aligned} ||\varphi||_{\delta} &= \sum_{m=0}^{\infty} |c_{m}| \hat{o}^{m}, \\ \mathcal{B}_{\delta} &= \{\varphi; ||\varphi||_{\delta} < +\infty\}, \text{ and} \\ \mathcal{B}_{\delta,M} &= \{x^{M}\varphi; \varphi \in \mathcal{B}_{\delta}\}, \end{aligned}$$

where M is a positive integer.

Let A(x, y) be the matrix which was given in §1. Set

$$A(x, y) = \sum_{m=0}^{\infty} A_m(y) x^m \text{ and } ||A||_{\delta} = \sum_{m=0}^{\infty} |A_m| \delta^m,$$

where $|A_m| = \sup_{\mathcal{D}} |A_m(y)|$. Assume that $||A||_{\delta} < +\infty$. Choose M so that

$$\frac{||A||_{\delta}}{\delta^p M} \! < \! 1$$

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where p is the integer which was given in §1. We define a mapping $A: \mathcal{B}_{\delta,M} \to \mathcal{B}_{\delta,M}$ by $A(\varphi)(x,y) = A(x,y)\varphi(x,y)$. Then, $||A(\varphi)||_{\delta} \leq ||A||_{\delta}||\varphi||_{\delta}$.

Let us define a mapping $P: \mathcal{B}_{\delta, M} \rightarrow \mathcal{B}_{\delta, M+p}$ by

$$P(\varphi) = \sum_{m=M+p}^{\infty} c_m x^m,$$

where

$$\varphi = \sum_{m=M}^{\infty} c_m x^m.$$

Then

 $||P(\varphi)||_{\delta} \leq ||\varphi||_{\delta}.$

We also define a mapping $T: \mathcal{B}_{\delta, M+p} \rightarrow \mathcal{B}_{\delta, M}$ by

$$T(\varphi) = \sum_{m=M}^{\infty} \frac{c_m}{m} x^m,$$

where

$$\varphi = \sum_{m=M}^{\infty} c_m x^{m+p}.$$

Then

$$||T(\varphi)||_{\delta} \leq \frac{1}{\delta^{p}M} ||\varphi||_{\delta}.$$

It is easy to prove that

$$D_1T(\varphi) = \varphi \quad \text{for } \varphi \in \mathcal{B}_{\delta, M+p}$$

and that

$$T(D_1\varphi) = \varphi$$
 for $\varphi \in \mathcal{B}_{\delta, M}$ if $D_1\varphi \in \mathcal{B}_{\delta, M+p}$

Since

$$||TPA(\varphi)||_{\delta} \leq \frac{||A||_{\delta}}{\delta^{p}M} ||\varphi||_{\delta} \quad \text{for } \varphi \in \mathcal{B}_{\delta, M},$$

the mapping $I-TPA: \mathcal{B}_{\delta,M} \to \mathcal{B}_{\delta}$ is an isomorphism, where I is the identity map. Set

Then the mapping $\Phi: \mathcal{B}_{\delta, M} \rightarrow \mathcal{B}_{\delta, M}$ satisfies the identity

(3.2)
$$L \Phi(\varphi) = \varphi - Q(\varphi) \quad (\varphi \in \mathcal{B}_{\delta, M}),$$

where

(3.3)
$$Q = (I - P)(I + A\Phi).$$

In fact, note that

$$(I - TPA)\Phi = TP,$$

 $D_1(I - TPA)\Phi(\varphi) = D_1\Phi(\varphi) - PA\Phi(\varphi),$ and
 $D_1TP(\varphi) = P(\varphi).$

Hence

$$D_1 \Phi(\varphi) = P A \Phi(\varphi) + P(\varphi)$$

and

$$L\Phi(\varphi) = PA\Phi(\varphi) + P(\varphi) - A\Phi(\varphi) = \varphi - (I - P)(I + A\Phi)(\varphi)$$

This proves (3.2).

Suppose that $\varphi \in \mathcal{B}_{\delta,M}$ and $D_1 \varphi \in \mathcal{B}_{\delta,M-p}$. Then

 $(3.4) \qquad \qquad \phi L(\varphi) = \varphi.$

In fact,

 $P(D_1\varphi) = D_1\varphi$ and $TP(D_1\varphi) = \varphi$.

Hence

 $\Phi(D_1\varphi) = (I - TPA)^{-1}(\varphi).$

Since

$$\Phi A(\varphi) = (I - TPA)^{-1}TPA(\varphi),$$

we have

$$\Phi L(\varphi) = (I - TPA)^{-1}(I - TPA)(\varphi) = \varphi.$$

LEMMA 3.1: Suppose that $A_{00} \in GL(n; C)$. Then, if

$$\psi = \sum_{m=M}^{M+p-1} c_m x^m \in L(\mathcal{B}_{\delta}),$$

it follows that $\phi = 0$.

Proof: Let $\psi = L(u)$ for some $u \in \mathcal{B}_{\delta}$. Then $u \in \mathcal{B}_{\delta, M}$ and hence $D_1 u \in \mathcal{B}_{\delta, M+p}$. Therefore, $\Phi(\psi) = u$. On the other hand, $P(\psi) = 0$ and hence $\Phi(\psi) = 0$. This proves that

 $\phi = L(0) = 0.$

LEMMA 3.2: Suppose that $A_{00} \in GL(n; \mathbb{C})$. Then, for $\varphi \in \mathcal{B}_{\delta, M}$, we have $\varphi \in L(\mathcal{B}_{\delta})$ if and only if $Q(\varphi) = 0$.

Proof: If $Q(\varphi)=0$, then (3.2) implies that $\varphi = L\Phi(\varphi) \in L(\mathcal{B}_{\delta})$. If $\varphi = L(u)$ for some $u \in \mathcal{B}_{\delta}$, then

$$Q(\varphi) = \varphi - L \Phi(\varphi) = L(u - \Phi(\varphi)) \in L(\mathcal{B}_{\delta}).$$

Hence, by virtue of Lemma 3.1, we have $Q(\varphi)=0$.

LEMMA 3.3: Suppose that $A_{00} \in GL(n; \mathbb{C})$. Then, there exists a mapping S: \mathcal{B}_{δ} $\rightarrow \mathcal{B}_{\delta}$ such that

$$(i) D_1 S(\mathcal{B}_{\delta}) \subset \mathcal{B}_{\delta};$$

(ii)
$$(I-LS)(\mathcal{B}_{\delta}) \subset \mathcal{B}_{\delta,M}$$

REMARK: An easy (formal) computation would determine $S(\varphi)$ as a polynomial

in x with coefficients in Ω .

LEMMA 3.4: Suppose that $A_{00} \in GL(n; \mathbb{C})$. Then, for $\varphi \in \mathcal{B}_{\delta}$, we have $\varphi \in L(\mathcal{B}_{\delta})$ if and only if

$$(3.5) \qquad \qquad Q(I-LS)(\varphi) = 0.$$

Proof: Suppose that (3.5) is satisfied. Then $(I-LS)(\varphi) \in L(\mathcal{B}_{\delta})$, and hence $\varphi \in L(\mathcal{B}_{\delta})$. If $\varphi \in L(\mathcal{B}_{\delta})$, then $(I-LS)(\varphi) \in L(\mathcal{B}_{\delta})$. Hence, (3.5) must be satisfied.

4. An important lemma.

Throughout this section, we assume that

(i)
$$A_{00} \in GL(n; \mathbf{C})$$
 and $B_{00} \in GL(n; \mathbf{C})$, and

$$||A||_{\delta} < +\infty \quad \text{and} \quad ||B||_{\delta} < +\infty.$$

We also utilize notations and results of §3.

LEMMA 4.1: If
$$\zeta' = \sum_{m=M}^{M+p-1} c_m x^m$$
, and if $D_2 \zeta' \in \mathcal{B}_{\delta}$ and
(4.1) $K(\zeta') \in L(\mathcal{B}_{\delta}),$

then $\phi = 0$.

Proof: Since $K(\psi) \in \mathcal{B}_{\delta, M}$ and $A_{00} \in GL(n; \mathbb{C})$, we have

 $QK(\phi) = 0.$

This means that

 $(4.2) \qquad \qquad Q(D_2\psi) - QB(\psi) = 0.$

We shall prove first that

 $(4.3) Q(D_2\psi) = D_2\psi.$

Note that $D_2 \psi = \sum_{m=M}^{M+p-1} (D_2 c_m) x^m$. Hence $P(D_2 \psi) = 0$. Therefore $\Psi(D_2 \psi) = 0$, and hence (4.3) follows from (3.3).

By utilizing (4.3), we write (4.2) as

$$(4.4) D_2 \phi = QB(\phi)$$

This means that

(4.5)
$$D_2 c_m = \sum_{k=M}^{M+p-1} \gamma_{mk} c_k, \qquad m = M, \cdots, M+p-1,$$

where the γ_{mk} are n-by-n matrices whose components are in Ω . Set

$$I'(y) = \begin{bmatrix} \tilde{i}' M M & \cdots & \tilde{i}' M M \cdot p - 1 \\ \cdots & \cdots & \cdots \\ \tilde{i}' M \cdot p \cdots 1 M & \cdots & \tilde{i}' M \cdot p - 1 M \cdot p \cdots \end{bmatrix}.$$

We shall prove that $\Gamma(0) \in GL(np; C)$.

The quantities $A, B, L, K, \Phi, Q, \varphi$ etc depend on y. We denote by $A_0, B_0, L_0, K_0, \Phi_0, Q_0, \varphi_0$ etc those quantities at y=0. Then,

$$\begin{split} &K_0(\varphi_0) = -B_0(\varphi_0), \\ &L_0(B_0(\varphi_0)) = B_0(L_0(\varphi_0)), \text{ or } B_0^{-1}L_0B_0(\varphi_0) = L_0(\varphi_0), \end{split}$$

and

$$L_0 \Phi_0(\varphi_0) = \varphi_0 - Q_0(\varphi_0)$$
.

Hence, if $Q_0B_0(\phi_0)=0$, we have

$$B_{0}(\dot{\varphi_{0}}) = L_{0} \Phi_{0}(B_{0}(\dot{\varphi_{0}})), \text{ or}$$

$$\dot{\varphi_{0}} = B_{0}^{-1} L_{0} B_{0} B_{0}^{-1} \Phi_{0} B_{0}(\dot{\varphi_{0}}) = L_{0} B_{0}^{-1} \Phi_{0} B_{0}(\dot{\varphi_{0}}).$$

This means that

$$\psi_0 = L_0(u_0),$$

where

$$u_0 = B_0^{-1} \Phi_0 B_0(\phi_0)$$

Since $\Phi_0 L_0(u_0) = u_0$, we have $\Phi_0(\phi_0) = u_0 = 0$, and hence $\phi_0 = 0$. This proves that $I'(0) \in GL(np; \mathbf{C})$.

Finally, it follows from (4.5) that $\psi = 0$.

5. Proof of Theorem A.

Set K(u) = L(v). Note that

$$K(I-LS)(u) = L(v-KS(u)),$$

and hence

$$\begin{split} KQ(I-LS)(u) = K(I-LS)(u) - LK \phi(I-LS)(u) \\ = L(v-KS(u) - K \phi(I-LS)(u)). \end{split}$$

Therefore,

Q(I-LS)(u) = 0 (cf. Lemma 4.1),

and

 $u \in L(\mathcal{B}_{\delta})$ (cf. Lemma 3.4).

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This completes the proof of Theorem A. In this argument, we must choose δ_0 and δ sufficiently small.

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