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APPROXIMATION FOR A CLASS OF SIGNAL FUNCTIONS BY SAMPLING SERIES REPRESENTATION

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ABSTRACT

On the line of BUTZER & SPLETTSTÖßER (1976), a theorem on sampling series approximations for a class of signal functions which are not band-limited or duration-limited is shown.

1. Introduction

Recently BUTZER and SPLETTSTÖßER (1976) have shown a sampling theorem of the form:

$$f(t)=\lim_{W\to\infty}\sum_{-N}^{N} f\left(\frac{k}{W}\right)\frac{\sin\pi(Wt-k)}{\pi(Wt-k)} \tag{1-1}$$

under the condition that $f(t)$, $-\infty<t<\infty$ is duration-limited, that is $f(t)$ has a bounded support, among other conditions. The aim of this paper is to show (1-1) under the more general condition:

$$|f(t)|\leq C|t|^{-\alpha} \qquad (\alpha>2) \tag{1-2}$$

for $|t|>T$, T and C being some positive constants. More precisely we are going to show the following main result.

Theorem 1. Let $f(t)$, $-\infty<t<\infty$ be a function of $L^1(-\infty,\infty)$ such that its Fourier transform

$$\hat{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\lambda t}f(t)dt,$$

is of $L^1(-\infty,\infty)$, and furthermore assume for $|t|>T$,

$$|f(t)|\leqq C|t|^{-\alpha} \qquad (\alpha>2), \tag{1-2}$$

where T and C are positive constants.

Then for $N=N(W)$ such that

$$\frac{N(W)}{W} \longrightarrow \infty \qquad \text{as} \qquad W \longrightarrow \infty, \tag{1-3}$$

we have

$$f(t)=\lim_{W\to\infty}\sum_{k=-N}^{N} f\left(\frac{k}{W}\right)\frac{\sin\pi(Wt-k)}{\pi(Wt-k)}. \tag{1-4}$$

The proof is an adaptation of BUTZER & SPLETTSTÖßER's paper. We also give error estimates for (1-1) as they did.

2. Lemmas

We need several lemmas.

Lemma 1. If $f(t)\in L^1(-\infty,\infty)$ and $f(t)=0(|t|^{-\alpha})$ $(\alpha>2)$ for $|t|>T$, then Fourier transform $\hat{f}(\lambda)$ of $f(t)$ satisfies the Lipschitz condition of order 1.

Proof.
$$\sqrt{2\pi}\,|\hat{f}(\lambda+h)-\hat{f}(\lambda)|=\left|\int_{-\infty}^{\infty}[e^{i(\lambda+h)t}-e^{i\lambda t}]f(t)dt\right|$$
$$\leqq |h|\int_{-\infty}^{\infty}|tf(t)|dt,$$

in which the integral is finite.

Lemma 2. Assume the condition in Lemma 1. Then if $\hat{f}(\lambda)\in L^1(-\infty,\infty)$

$$\left|\int_{|\lambda|>\pi W}\hat{f}(\lambda)e^{i\frac{\lambda k}{W}}d\lambda\right|\leqq\frac{CW^\alpha}{|k|^\alpha}+2L(\pi W)^2\frac{1}{|k|} \tag{2-1}$$

for $|k|>WT$, where C and L are some positive constants.

Proof. Since $\left|f\left(\frac{k}{W}\right)\right|\leqq C\frac{W^\alpha}{|k|^\alpha}$ $(\alpha>2)$ for $|k|>WT$,

$$\left|\int_{|\lambda|>\pi W}\hat{f}(\lambda)e^{i\frac{\lambda k}{W}}d\lambda\right|=\left|\int_{-\infty}^{\infty}\hat{f}(\lambda)e^{i\frac{\lambda k}{W}}d\lambda-\int_{-\pi W}^{\pi W}\hat{f}(\lambda)e^{i\frac{\lambda k}{W}}d\lambda\right|$$
$$\leqq\frac{CW^\alpha}{|k|^\alpha}+\left|\int_{-\pi W}^{\pi W}\hat{f}(\lambda)e^{i\frac{\lambda k}{W}}d\lambda\right|, \tag{2-2}$$

for $|k|>WT$. Since by Lemma 1, $\hat{f}(\lambda)\in\mathrm{Lip}\,1$, then we have the lemma by Lemma 3 of [1].

3. Proof of Theorem 1

We now turn out to prove Theorem 1. Let us consider Fourier series expansion of the periodic extension of $\hat{f}(\lambda)$ on $[-\pi W, \pi W)$. Since Dini-Lipschitz condition (See [2] p. 45, Theorem 49 or [4] p. 30.) that $\hat{f}(\lambda+u)-\hat{f}(\lambda)=0(|\log|u||^{-1})$ as $u\to 0$ uniformly for $\lambda\in[-\pi W', \pi W']$, $0<W'<W$, is satisfied because of Lemma 1, the partial sum of Fourier series of the periodic extension $\hat{f}(\lambda)$ on $[-\pi W, \pi W)$ converges to $\hat{f}(\lambda)$ uniformly for $\lambda\in[-\pi W'', \pi W'']$, $0<W''<W'<W$. Moreover we may write

$$\hat{f}(\lambda)=\sum_{k=-\infty}^{\infty}\frac{\sqrt{2\pi}}{2\pi W}f\left(\frac{k}{W}\right)e^{-i\frac{\lambda k}{W}}-\sum_{k=-\infty}^{\infty}\frac{1}{2\pi W}\left\{\int_{|u|>\pi W}e^{i\frac{uk}{W}}f(u)du\right\}e^{-i\frac{\lambda k}{W}}, \quad (3\text{-}1)$$

because both the series on the right hand side converge uniformly for $\lambda\in[-\pi W'', \pi W'']$ under the condition (1-2) and Lemma 2. Let us consider

$$R_W(t)=f(t)-\sum_{k=-N}^{N}f\left(\frac{k}{W}\right)\frac{\sin\pi(Wt-k)}{\pi(Wt-k)}, \quad (3\text{-}2)$$

where $N=N(W)$ is large. Using the relation (3-1) and the uniform convergence of the series in (3-1), we have just as in [1],

$$\begin{aligned}R_W(t)=&\sum_{-N}^{N}f\left(\frac{k}{W}\right)\frac{\sin\pi W''\left(t-\frac{k}{W}\right)-\sin\pi(Wt-k)}{\pi(Wt-k)}\\&+\sum_{|k|>N}f\left(\frac{k}{W}\right)\frac{\sin\pi W''\left(t-\frac{k}{W}\right)}{\pi(Wt-k)}\\&+\frac{1}{\sqrt{2\pi}}\int_{|\lambda|>\pi W''}e^{i\lambda t}\hat{f}(\lambda)d\lambda\\&-\frac{1}{\sqrt{2\pi}}\sum_{-\infty}^{\infty}\frac{1}{2\pi W}\int_{-\pi W''}^{\pi W''}e^{i\lambda\left(t-\frac{k}{W}\right)}d\lambda\int_{|u|>\pi W}e^{i\frac{uk}{W}}\hat{f}(u)du\\=&J_1(W'')+J_2(W'')+J_3(W'')-J_4(W''),\end{aligned} \quad (3\text{-}3)$$

for $0<W''<W'<W$.

From [1] we know $\lim_{W''\to W}J_1(W'')=0$

and

$$\begin{aligned}&\lim_{W''\to W}|J_3(W'')-J_4(W'')|\\&\leq\frac{1}{\sqrt{2\pi}}\int_{|\lambda|>\pi W}\left|e^{i\lambda t}-\sum_{-\infty}^{\infty}\frac{\sin\pi(Wt-k)}{\pi(Wt-k)}e^{i\frac{\lambda k}{W}}\right||\hat{f}(\lambda)|d\lambda\\&\leq\sqrt{\frac{2}{\pi}}\int_{|\lambda|>\pi W}|\hat{f}(\lambda)|d\lambda.\end{aligned} \quad (3\text{-}4)$$

Now let us consider the remainder term $J_2(W'')$. We have for $N > \max\{WT, 2Wt\}$

$$|J_2(W'')| \leqq \sum_{|k|>N} \left| \frac{\sin \pi W'' \left(t - \dfrac{k}{W}\right)}{\pi(Wt-k)} \right| \left| f\left(\frac{k}{W}\right) \right|$$

$$\leqq \sum_{|k|>N} \frac{C}{2\pi} \frac{1}{|k|} \left(\frac{W}{|k|}\right)^{\alpha} \leqq \frac{4C}{\pi} \left(\frac{W}{N}\right)^{\alpha}. \tag{3-5}$$

Then altogether we have

$$|R_W(t)| \leqq \frac{4C}{\pi}\left(\frac{W}{N}\right)^{\alpha} + \sqrt{\frac{2}{\pi}} \int_{|\lambda|>\pi W} |\hat{f}(\lambda)|\, d\lambda, \tag{3-6}$$

which converge to zero as $N/W \to \infty$ $(W \to \infty)$. This completes the proof.

4. Error estimates

In this section we discouss about the error $R_W(t)$ of the approximation.

Theorem 2. Let $f(t)$ be a function which satisfies the assumption of Theorem 1. Moreover assume that for $r \geqq 1$, $f(t)$ is r-times differentiable for every t and satisfies the following conditions,

(i) $f^{(n)}(t) \in L^1(-\infty, \infty)$ and $\lim_{|t| \to \infty} |f^{(n)}(t)| = 0$ $(n = 0, 1, \cdots, r-1)$,

(ii) $\lim_{t \to \infty} |f^{(r)}(t)| = K_1$, and $\lim_{t \to -\infty} |f^{(r)}(t)| = K_2$,

(iii) $\int_{-\infty}^{\infty} |df^{(r)}(t)| = K_3$.

If $N = N(W)$ satisfies $(N^{\alpha}/W^{r+\alpha}) \to \infty$ as $W \to \infty$ for $\alpha > 2$, then we have

$$|R_W(t)| \leqq \left[\frac{4C}{\pi} + \frac{2(r+1)(K_1+K_2+K_3)}{\pi^{r+1}}\right] \frac{1}{W^r}. \tag{4-1}$$

for large W so that $N^{\alpha}/W^{r+\alpha} \geqq 1$.

Proof. By using integration by parts we have

$$\sqrt{2\pi}\, |\hat{f}(\lambda)| = \left| \frac{1}{(i\lambda)^r} \left\{ \frac{e^{-i\lambda u}}{-i\lambda} f^{(r)}(u) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{e^{-i\lambda u}}{i\lambda} df^{(r)}(u) \right\} \right|,$$

$$\leqq \frac{1}{|\lambda|^{r+1}} (K_1 + K_2 + K_3). \tag{4-2}$$

Substituting (4-2) into (3-6), we have

$$|R_W(t)| \leqq \frac{4C}{\pi}\left(\frac{W}{N}\right)^{\alpha} + \frac{2(r+1)(K_1+K_2+K_3)}{\pi^{r+1}} \frac{1}{W^r}, \tag{4-3}$$

for $\alpha>2$. We have the theorem because $(W/N)^{\alpha}\leqq W^{-r}$ for large W such that $N^{\alpha}/W^{r+\alpha}\geqq 1$.

Theorem 3. Let $f(t)$ be a function which satisfies the assumption of Theorem 1, and $f(t)$ be r-times differentiable for each t, $r\geqq 1$. Moreover assume the condition (i) of Theorem 2, and assume that $f^{(r)}(t)$ satisfies the integral Lipschitz condition of the order β $(\beta>0)$:

$$\int_{-\infty}^{\infty}|f^{(r)}(t+h)-f^{(r)}(t)|dt\leqq I_r|h|^{\beta}, \tag{4-4}$$

where I_r is some positive constant.

If $(N^{\alpha}/W^{r+\alpha+\beta-1})\to\infty$ $(W\to\infty)$ for $\alpha>2$, then

$$|R_W(t)|\leqq\left[\frac{4C}{\pi}+\frac{I_r(r+\beta-1)}{\pi^r}\right]\frac{1}{W^{r+\beta-1}} \tag{4-5}$$

Proof. Using integration by parts again, we have

$$\hat{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\frac{1}{(i\lambda)^r}\int_{-\infty}^{\infty}f^{(r)}(t)e^{-i\lambda t}dt \tag{4-6}$$

$$-\hat{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\frac{1}{(i\lambda)^r}\int_{-\infty}^{\infty}f^{(r)}\left(t+\frac{\pi}{\lambda}\right)e^{-i\lambda t}dt. \tag{4-7}$$

Then we obtain

$$\begin{aligned}2|\hat{f}(\lambda)|&\leqq\frac{1}{\sqrt{2\pi}}\frac{1}{|\lambda|^r}\int_{-\infty}^{\infty}\left|f^{(r)}(t)-f^{(r)}\left(t+\frac{\pi}{\lambda}\right)\right|dt\\&\leqq\frac{1}{\sqrt{2\pi}}\frac{I_r\pi^{\beta}}{|\lambda|^{r+\beta}}\end{aligned} \tag{4-8}$$

By substituting (4-8) into (3-6) and observing $(N^{\alpha}/W^{r+\alpha+\beta-1})\to\infty(W\to\infty)$ for $\alpha>2$, we have the theorem.

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Added in proof: In the course of proofreading, we found that the results in [1] have been recently published: P. L. BUTZER and W. SPLETTSTÖßER, the same title as in [1], Information and Controll, 34 pp. 55–65 (1977).

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