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APPROXIMATION FOR A CLASS OF SIGNAL FUNCTIONS BY SAMPLING SERIES REPRESENTATION

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ABSTRACT

On the line of BUTZER & SPLETTSTÖBER (1976), a theorem on sampling series approximations for a class of signal functions which are not band-limited or duration-limited is shown.

1. Introduction

Recently BUTZER and SPLETTSTÖBER (1976) have shown a sampling theorem of the form:

$$f(t) = \lim_{W \to \infty} \sum_{-N}^{N} f\left(\frac{k}{W}\right) \frac{\sin \pi (Wt - k)}{\pi (Wt - k)}$$
(1-1)

under the condition that f(t), $-\infty < t < \infty$ is duration-limited, that is f(t) has a bounded support, among other conditions. The aim of this paper is to show (1-1) under the more general condition:

$$|f(t)| \leq C|t|^{-\alpha} \qquad (\alpha > 2) \tag{1-2}$$

for |t| > T, T and C being some positive constants. More precisely we are going to show the following main result.

Theorem 1. Let f(t), $-\infty < t < \infty$ be a function of $L^1(-\infty, \infty)$ such that its Fourier transform

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda t} f(t) dt$$

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is of $L^1(-\infty,\infty)$, and furthermore assume for |t| > T,

$$|f(t)| \leq C|t|^{-\alpha} \qquad (\alpha > 2), \tag{1-2}$$

where T and C are positive constants.

Then for N = N(W) such that

$$\frac{N(W)}{W} \longrightarrow \infty \quad \text{as} \quad W \longrightarrow \infty, \tag{1-3}$$

we have

$$f(t) = \lim_{W \to \infty} \sum_{k=-N}^{N} f\left(\frac{k}{W}\right) \frac{\sin \pi (Wt - k)}{\pi (Wt - k)}.$$
(1-4)

The proof is an adaptation of BUTZER & SPLETTSTÖBER'S paper. We also give error estimates for (1-1) as they did.

2. Lemmas

We need several lemmas.

Lemma 1. If $f(t) \in L^1(-\infty, \infty)$ and $f(t) = 0(|t|^{-\alpha})$ ($\alpha > 2$) for |t| > T, then Fourier transform $\hat{f}(\lambda)$ of f(t) satisfies the Lipschitz condition of order 1.

Proof. $\sqrt{2\pi} |\hat{f}(\lambda+h) - \hat{f}(\lambda)| = \left| \int_{-\infty}^{\infty} [e^{i(\lambda+h)t} - e^{i\lambda t}] f(t) dt \right|$

$$\leq |h| \int_{-\infty}^{\infty} |tf(t)| dt$$
,

in which the integral is finite.

Lemma 2. Assume the condition in Lemma 1. Then if $\hat{f}(\lambda) \in L^1(-\infty, \infty)$

$$\left| \int_{|\lambda| > \pi W} \hat{f}(\lambda) e^{i\frac{\lambda k}{W}} d\lambda \right| \leq \frac{CW^{\alpha}}{|k|^{\alpha}} + 2L(\pi W)^2 \frac{1}{|k|}$$
(2-1)

for |k| > WT, where C and L are some positive constants.

Proof. Since
$$\left| f\left(\frac{k}{W}\right) \right| \leq C \frac{W^{\alpha}}{|k|^{\alpha}} \quad (\alpha > 2) \text{ for } |k| > WT,$$

 $\left| \int_{|\lambda| > \pi W} \hat{f}(\lambda) e^{i\frac{\lambda k}{W}} d\lambda \right| = \left| \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\frac{\lambda k}{W}} d\lambda - \int_{-\pi W}^{\pi W} \hat{f}(\lambda) e^{i\frac{\lambda k}{W}} d\lambda \right|$
 $\leq \frac{CW^{\alpha}}{|k|^{\alpha}} + \left| \int_{-\pi W}^{\pi W} \hat{f}(\lambda) e^{i\frac{\lambda k}{W}} d\lambda \right|, \qquad (2-2)$

for |k| > WT. Since by Lemma 1, $\hat{f}(\lambda) \in \text{Lip 1}$, then we have the lemma by Lemma 3 of [1].

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3. Proof of Theorem 1

We now turn out to prove Theorem 1. Let us consider Fourier series expansion of the periodic extension of $\hat{f}(\lambda)$ on $[-\pi W, \pi W)$. Since Dini-Lipschitz condition (See [2] p. 45, Theorem 49 or [4] p. 30.) that $\hat{f}(\lambda+u)-\hat{f}(\lambda)=0(|\log |u||^{-1})$ as $u \to 0$ uniformly for $\lambda \in [-\pi W', \pi W']$, 0 < W' < W, is satisfied because of Lemma 1, the partial sum of Fourier series of the periodic extension $\hat{f}(\lambda)$ on $[-\pi W, \pi W]$ converges to $\hat{f}(\lambda)$ uniformly for $\lambda \in [-\pi W'', \pi W'']$, 0 < W'' < W' < W. Moreover we may write

$$\hat{f}(\lambda) = \sum_{k=-\infty}^{\infty} \frac{\sqrt{2\pi}}{2\pi W} f\left(\frac{k}{W}\right) e^{-i\frac{\lambda k}{W}} - \sum_{k=-\infty}^{\infty} \frac{1}{2\pi W} \left\{ \int_{|u| > \pi W} e^{i\frac{uk}{W}} f(u) du \right\} e^{-i\frac{\lambda k}{W}}, \quad (3-1)$$

because both the series on the right hand side converge uniformly for $\lambda \in [-\pi W'', \pi W'']$ under the condition (1-2) and Lemma 2. Let us consider

$$R_{W}(t) = f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{W}\right) \frac{\sin \pi (Wt - k)}{\pi (Wt - k)},$$
(3-2)

where N=N(W) is large. Using the relation (3-1) and the uniform convergence of the series in (3-1), we have just as in [1],

$$R_{W}(t) = \sum_{-N}^{N} f\left(\frac{k}{W}\right) \frac{\sin \pi W''\left(t - \frac{k}{W}\right) - \sin \pi (Wt - k)}{\pi (Wt - k)}$$

$$+ \sum_{|k| = N} f\left(\frac{k}{W}\right) \frac{\sin \pi W''\left(t - \frac{k}{W}\right)}{\pi (Wt - k)}$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{|\lambda| > \pi W''} e^{i\lambda t} \hat{f}(\lambda) d\lambda$$

$$- \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \frac{1}{2\pi W} \int_{-\pi W''}^{\pi W''} e^{i\lambda \left(t - \frac{k}{W}\right)} d\lambda \int_{|u| > \pi W} e^{i\frac{uk}{W}} \hat{f}(u) du$$

$$= J_{1}(W'') + J_{2}(W'') + J_{3}(W'') - J_{4}(W''), \qquad (3-3)$$

for 0 < W'' < W' < W.

From [1] we know $\lim_{W'' \to W} J_1(W'') = 0$

and

$$\lim_{W'' \to W} |J_{\mathfrak{z}}(W'') - J_{\mathfrak{z}}(W'')| \leq \frac{1}{\sqrt{2\pi}} \int_{|\lambda| > \pi W} \left| e^{i\lambda t} - \sum_{-\infty}^{\infty} \frac{\sin \pi (Wt - k)}{\pi (Wt - k)} e^{i\frac{\lambda k}{W}} \right| |\hat{f}(\lambda)| d\lambda$$

$$\leq \sqrt{\frac{2}{\pi}} \int_{|\lambda| > \pi W} |\hat{f}(\lambda)| d\lambda. \qquad (3-4)$$

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Now let us consider the remainder term $J_2(W'')$. We have for $N > \max \{WT, 2Wt\}$

$$J_{2}(W^{\prime\prime}) \leq \sum_{|k|=N} \left| \frac{\sin \pi W^{\prime\prime} \left(t - \frac{k}{W} \right)}{\pi (Wt - k)} \right| \left| f\left(\frac{k}{W} \right) \right|$$
$$\leq \sum_{|k|=N} \frac{C}{2\pi} \frac{1}{|k|} \left(\frac{W}{|k|} \right)^{\alpha} \leq \frac{4C}{\pi} \left(\frac{W}{N} \right)^{\alpha}.$$
(3-5)

Then altogether we have

$$|R_W(t)| \leq \frac{4C}{\pi} \left(\frac{W}{N}\right)^{\alpha} + \sqrt{\frac{2}{\pi}} \int_{|\lambda| \leq \pi W} |\hat{f}(\lambda)| d\lambda, \qquad (3-6)$$

which converge to zero as $N/W \rightarrow \infty$ ($W \rightarrow \infty$). This complets the proof.

4. Error estimates

In this section we discouss about the error $R_W(t)$ of the approximation.

Theorem 2. Let f(t) be a function which satisfies the assumption of Theorem 1. Moreover assume that for $r \ge 1$, f(t) is *r*-times differentiable for every *t* and satisfies the following conditions,

(i)
$$f^{(n)}(t) \in L^{1}(-\infty, \infty)$$
 and $\lim_{|t|\to\infty} |f^{(n)}(t)| = 0$ $(n=0, 1, \dots, r-1),$
(ii) $\lim_{t\to\infty} |f^{(r)}(t)| = K_1,$ and $\lim_{t\to-\infty} |f^{(r)}(t)| = K_2,$
(iii) $\int_{-\infty}^{\infty} |df^{(r)}(t)| = K_3.$

If N=N(W) satisfies $(N^{\alpha}/W^{r+\alpha}) \to \infty$ as $W \to \infty$ for $\alpha > 2$, then we have

$$|R_W(t)| \leq \left[\frac{4C}{\pi} + \frac{2(r+1)(K_1 + K_2 + K_3)}{\pi^{r+1}}\right] \frac{1}{W^r}.$$
(4-1)

for large W so that $N^{\alpha}/W^{r_{+}\alpha} \ge 1$.

Proof. By using integration by parts we have

$$\begin{split} \sqrt{2\pi} |\hat{f}(\lambda)| &= \left| \frac{1}{(i\lambda)^r} \left\{ \frac{e^{-i\lambda u}}{-i\lambda} f^{(r)}(u) \right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{e^{-i\lambda u}}{i\lambda} df^{(r)}(u) \right\} \right|, \\ &\leq \frac{1}{|\lambda|^{r+1}} (K_1 + K_2 + K_3). \end{split}$$
(4-2)

Substituting (4-2) into (3-6), we have

$$|R_W(t)| \le \frac{4C}{\pi} \left(\frac{W}{N}\right)^{\alpha} + \frac{2(r+1)(K_1 + K_2 + K_3)}{\pi^{r+1}} - \frac{1}{W^r}, \qquad (4-3)$$

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for $\alpha > 2$. We have the theorem because $(W/N)^{\alpha} \leq W^{-r}$ for large W such that $N^{\alpha}/W^{r+\alpha} \geq 1$.

Theorem 3. Let f(t) be a function which satisfies the assumption of Theorem 1, and f(t) be *r*-times differentiable for each t, $r \ge 1$. Moreover assume the condition (i) of Theorem 2, and assume that $f^{(r)}(t)$ satisfies the integral Lipschitz condition of the order β ($\beta > 0$):

$$\int_{-\infty}^{\infty} |f^{(r)}(t+h) - f^{(r)}(t)| dt \leq I_r |h|^{\beta},$$
(4-4)

where I_r is some positive constant.

If $(N^{\alpha}/W^{r+\alpha+\beta-1}) \rightarrow \infty$ $(W \rightarrow \infty)$ for $\alpha > 2$, then

$$|R_{W}(t)| \leq \left[\frac{4C}{\pi} + \frac{I_{r}(r+\beta-1)}{\pi^{r}}\right] \frac{1}{W^{r+\beta-1}}$$
(4-5)

Proof. Using integration by parts again, we have

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \frac{1}{(i\lambda)^r} \int_{-\infty}^{\infty} f^{(r)}(t) e^{-i\lambda t} dt$$
(4-6)

$$-\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \frac{1}{(i\lambda)^r} \int_{-\infty}^{\infty} f^{(r)} \left(t + \frac{\pi}{\lambda}\right) e^{-i\lambda t} dt.$$
(4-7)

Then we obtain

$$2|\hat{f}(\lambda)| \leq \frac{1}{\sqrt{2\pi}} \frac{1}{|\lambda|^r} \int_{-\infty}^{\infty} \left| f^{(r)}(t) - f^{(r)}\left(t + \frac{\pi}{\lambda}\right) \right| dt$$
$$\leq \frac{1}{\sqrt{2\pi}} \frac{I_r \pi^{\beta}}{|\lambda|^{r+\beta}}$$
(4-8)

By substituting (4-8) into (3-6) and observing $(N^{\alpha}/W^{r+\alpha+\beta-1}) \rightarrow \infty (W \rightarrow \infty)$ for $\alpha > 2$, we have the theorem.

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Added in proof: In the course of proofreading, we found that the results in [1] have been recently published: P. L. BUTZER and W. SPLETTSTÖBER, the same title as in [1], Information and Controll, 34 pp. 55-65 (1977).

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