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A NON-UNIFORM ESTIMATE IN THE CENTRAL LIMIT THEOREM FOR m-DEPENDENT RANDOM VARIABLES

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ABSTRACT

Let $\{X_i^{(n)}, i=1, 2, \cdots, k_n\}$ be a sequence of series of random variables and suppose that $\{X_i^{(n)}\}$ is m-dependent for each series, where m is unbounded, namely, $m=m_n$ may depend on n which is the number of the series. In this paper, the author gives a non-uniform estimate of the remainder term in the central limit theorem for these random variables. Furthermore an L_p version of the central limit theorem is stated as a direct consequence of the non-uniform estimate.

1. Introduction

Let $\{X_i^{(n)}, i=1, 2, \cdots, k_n\}$ be a sequence of series of random variables with $EX_i^{(n)}=0$ for all i and n. Suppose that $k_n \to \infty$ as $n \to \infty$ and that, for each n, the series $\{X_i^{(n)}\}$ is m-dependent, where $m=m_n$ may depend on n which is the number of the series. Write $S_n=\sum_{i=1}^{k_n} X_i^{(n)}$, $B_n^2=ES_n^2$, $\bar{B}_n^2=\sum_{i=1}^{k_n} E(X_i^{(n)})^2$, $F_n(x)=P(S_n \leqq B_n x)$ and $\Delta_n(x)=|F_n(x)-\Phi(x)|$, $\Phi(x)$ being the standard normal distribution.

The central limit theorem for m-dependent random variables has a long history, and recently the case of m-dependent random variables with unbounded m has been studied by BERK (1973), SHERGIN (1976) and others. In this paper, we shall show a non-uniform estimate of convergence of $F_n(x)$ to $\Phi(x)$. The technique of the proof is closely related to those employed by PETROV (1970), EGOROV (1970) and SHERGIN (1976). Furthermore, we shall state an L_p version of the central limit theorem for random variables considered here as a direct consequence of the non-uniform estimate. The L_p version of the central limit theorem for m-dependent random variables has been studied by ERICKSON (1973, 1974).

A Theorem

Our results are the followings. In what follows C denotes a positive constant which may differ from one inequality to another.

Theorem. *Let* $0<\delta\leqq 1$ *and* $\varepsilon_n=B_n^{-2}m_n^{(3\delta+2)/\delta}$. *Suppose that* $E|X_i^{(n)}|^{2+\delta}<\infty$ *and the following conditions are satisfied:*

(a) $B_n^2\to\infty$,

(b) $\bar{B}_n^2=O(B_n^2)$,

(c) $\sum_{i=1}^{k_n} E|X_i^{(n)}|^{2+\delta}=O(B_n^2)$,

(d) $k_n=O(B_n^2)$,

(e) $B_n^8 m_n^{-6}\leqq k_n^7$ *for large* n,

(f) $\varepsilon_n\to 0$.

Then we have, for all x,

$$\Delta_n(x)\leqq\frac{C}{(1+|x|)^{2+\delta}}\varepsilon_n^{\gamma_\delta},$$

where $\gamma_\delta=\dfrac{\delta(\delta+2)}{2(\delta^2+4\delta+2)}=\dfrac{\delta}{(3\delta+2)}\left(1+\dfrac{\delta^2}{2(\delta^2+4\delta+2)}\right)=\dfrac{\delta}{2(\delta+1)}\left(1-\dfrac{\delta}{\delta^2+4\delta+2}\right)$.

Corollary. *Under the same conditions as in Theorem, we have, for* $1\leqq p\leqq\infty$,

$$||(1+|x|)^{2-1/p}\Delta_n(x)||_p\leqq C\varepsilon_n^{\gamma_\delta},$$

where $||\cdot||_p=\left(\int|\cdot|^p dx\right)^{1/p}$ *for* $1\leqq p<\infty$, *and* $||\cdot||_\infty=\sup_x|\cdot|$.

This corollary follows directly from the above theorem and the inequality $||(1+|x|)^{2-1/p}\Delta_n(x)||_p\leqq||(1+|x|)^2\Delta_n(x)||_\infty^{(p-1)/p}||(1+|x|)\Delta_n(x)||_1^{1/p}$.

Proof

We first state some lemmas. The first lemma is a non-uniform version of a lemma due to PETROV (1970).

Lemma 1. *Let* X *and* Y *be random variables, and let* $F(x)$ *and* $H(x)$ *be the distribution function of* X *and* $X+Y$, *respectively. Let* $\alpha>0$. *If*

$$|F(x)-\Phi(x)|\leqq\frac{K}{(1+|x|)^\alpha},\tag{1}$$

K being a positive constant, then we have, for any $0<\varepsilon<1/2$ *and for all* x,

$$|H(x)-\Phi(x)| \leqq \frac{C}{(1+|x|)^{\alpha}}\left(K+\varepsilon+\frac{E|Y|^{\alpha}}{\varepsilon^{\alpha}}\right).$$

Proof. From (1), it is obvious that $|F(x)-\Phi(x)| \leqq K$, so that the lemma of PETROV gives us

$$|H(x)-\Phi(x)| \leqq K+C\varepsilon+\frac{E|Y|^{\alpha}}{\varepsilon^{\alpha}}.$$

Therefore, it is sufficient to show that for $|x| \geqq 1$,

$$|H(x)-\Phi(x)| \leqq \frac{C}{|x|^{\alpha}}\left(K+\varepsilon+\frac{E|Y|^{\alpha}}{\varepsilon^{\alpha}}\right). \tag{2}$$

Replacing ε by $|x|\varepsilon$ in the proof of the lemma of PETROV, we have

$$F(x-|x|\varepsilon)-P(|Y|>|x|\varepsilon) \leqq H(x) \leqq F(x+|x|\varepsilon)+P(|Y|>|x|\varepsilon)$$

and so

$$F(x-|x|\varepsilon)-\frac{E|Y|^{\alpha}}{\varepsilon^{\alpha}|x|^{\alpha}} \leqq H(x) \leqq F(x+|x|\varepsilon)+\frac{E|Y|^{\alpha}}{\varepsilon^{\alpha}|x|^{\alpha}}. \tag{3}$$

On the other hand, we have

$$|F(x \pm |x|\varepsilon)-\Phi(x)| \leqq |F(x \pm |x|\varepsilon)-\Phi(x \pm |x|\varepsilon)|+|\Phi(x \pm |x|\varepsilon)-\Phi(x)|. \tag{4}$$

Noticing that $0<\varepsilon<1/2$ and $|F(x)-\Phi(x)| \leqq K|x|^{-\alpha}$ by (1), we have

$$|F(x \pm |x|\varepsilon)-\Phi(x \pm |x|\varepsilon)| \leqq \frac{2^{\alpha}K}{|x|^{\alpha}} \tag{5}$$

and

$$|\Phi(x \pm |x|\varepsilon)-\Phi(x)| \leqq \varepsilon|x|e^{-x^2/8}. \tag{6}$$

Thus, (2) follows from (3)−(6) and the lemma is proved.

Lemma 2. *Let* $\alpha>0$ *and* $q>0$. *Then we have*

$$|\Phi(qx)-\Phi(x)| \leqq \frac{C}{(1+|x|)^{\alpha}} \max\left\{|q-1|, \frac{1}{q^{\alpha}}\left|1-\frac{1}{q}\right|\right\}.$$

It is quite easy to show this lemma.

Lemma 3. *Let* $\{Y_n\}$ *and* $\{Z_n\}$ *be sequences of random variables with mean zero and finite variances, and suppose that* $W_n=Y_n+Z_n$, $EW_n^2 \to \infty$ *and* $\omega_n^2=EZ_n^2/EW_n^2 \to 0$. *Then* $Q_n^2=EY_n^2/EW_n^2 \to 1$ *and for sufficiently large n,*

$$\max\left\{|1-Q_n|, \frac{1}{Q_n^{\alpha}}\left|1-\frac{1}{Q_n}\right|\right\} \leqq C\omega_n \qquad \text{for any } \alpha>0.$$

This is a slight modification of a lemma of SHERGIN (1976), and the validity of it is easily shown.

Now, let us return to the proof of the theorem. Let $\{T_n\}$ be a sequence of interger numbers such that $T_n > 3m_n$ and $T_n \leqq k_n$ for large n. For each n, fix an index set $\{s_i\}$ such that

$$(i-1)T_n + m_n < s_i \leqq iT_n - m_n, \qquad i=1, 2, \cdots, h_n - 1 = [k_n/T_n].$$

Write

$$Z_i^{(n)} = \sum_{j=s_i+1}^{s_i+m_n} X_j^{(n)} \qquad (i=1, 2, \cdots, h_n-1),$$

$$Y_i^{(n)} = \sum_{j=s_{i-1}+m_n+1}^{s_i} X_j^{(n)} \qquad (i=2, 3, \cdots, h_n-1),$$

$$Y_1^{(n)} = \sum_{j=1}^{s_1} X_j^{(n)}, \quad Y_{h_n}^{(n)} = \sum_{j=s_{h_n-1}+m_n+1}^{k_n} X_j^{(n)},$$

$$Y_n = \sum_{i=1}^{h_n} Y_i^{(n)}, \qquad Z_n = \sum_{i=1}^{h_n-1} Z_i^{(n)}.$$

Then we have $S_n = Y_n + Z_n$. Note that each of $\{Y_i^{(n)}\}$ and $\{Z_i^{(n)}\}$ is a sequence of independent random variables. Put $T_n = [m_n^2 A_n]$, $A_n \to \infty$. Then, SHERGIN (1976) has shown the following estimates under our conditions:

$$\frac{EZ_n^2}{B_n^2} \leqq \frac{C}{A_n}, \tag{7}$$

$$\sum_{i=1}^{h_n-1} E|Z_i^{(n)}|^{2+\delta} \leqq \frac{Cm_n^\delta B_n^2}{A_n}, \tag{8}$$

$$\sum_{i=1}^{h_n} E|Y_i^{(n)}|^{2+\delta} \leqq Cm_n^{1+\delta/2} T_n^{\delta/2} B_n^2. \tag{9}$$

Further, we note that, for sufficiently large n,

$$EY_n^2 = E(S_n - Z_n)^2 > \frac{B_n^2}{2}, \tag{10}$$

because of (7). Letting $q_n = B_n/(EY_n^2)^{1/2}$, we have

$$\left|P\left(\frac{Y_n}{B_n} \leqq x\right) - \Phi(x)\right| \leqq \left|P\left(\frac{Y_n}{(EY_n^2)^{1/2}} \leqq q_n x\right) - \Phi(q_n x)\right| + |\Phi(q_n x) - \Phi(x)|,$$

where $q_n \to 1$ by Lemma 3 and (7). We consider only the case of n such that $q_n > 1/2$. Then, making use of a non-uniform estimate of BIKELIS (1966) for indendent random variables together with Lemma 2 with $\alpha = 2+\delta$, (7), (9) and (10), we have

$$\left|P\left(\frac{Y_n}{B_n} \leqq x\right) - \Phi(x)\right|$$

$$\leqq \frac{C \sum_{i=1}^{h_n} E|Y_i^{(n)}|^{2+\delta}}{(1+|q_n x|)^{2+\delta} (EY_n^2)^{1+\delta/2}} + \frac{C}{(1+|x|)^{2+\delta}} \max\left\{|q_n - 1|, \frac{1}{q_n^{2+\delta}}\left(1 + \frac{1}{q_n}\right)\right\}$$

$$\leqq \frac{C}{(1+|x|)^{2+\delta}}\left(\frac{m_n^{1+3\delta/2}A_n^{\delta/2}}{B_n^{\delta}}+A_n^{-1/2}\right)$$

$$=\frac{C}{(1+|x|)^{2+\delta}}(\varepsilon_n^{\delta/2}A_n^{\delta/2}+A_n^{-1/2}). \tag{11}$$

Since

$$\frac{S_n}{B_n}=\frac{Y_n}{B_n}+\frac{Z_n}{B_n},$$

we have, using Lemma 1 with $\alpha=2+\delta$ and (11),

$$\Delta_n(x)\leqq\frac{C}{(1+|x|)^{2+\delta}}\left(\varepsilon_n^{\delta/2}A_n^{\delta/2}+A_n^{-1/2}+\varepsilon+\frac{E|Z_n|^{2+\delta}}{\varepsilon^{2+\delta}B_n^{2+\delta}}\right). \tag{12}$$

Since $Z_i^{(n)}$, $i=1,2,\cdots,h_n-1$ are independent and $EZ_i^{(n)}=0$, we have

$$E|Z_n|^{2+\delta}\leqq Ch_n^{\delta/2}\sum_{i=1}^{h_n-1}E|Z_i^{(n)}|^{2+\delta}, \tag{13}$$

because of Marcinkiewicz-Zygmund inequality (see KAWATA (1972) p. 576). Further, the condition (d) yields that

$$h_n\leqq\frac{k_n}{T_n}=\frac{k_n}{m_n^2A_n}\leqq\frac{CB_n^2}{m_n^2A_n}. \tag{14}$$

Therefore we have from (8) and (12)−(14),

$$\Delta_n(x)\leqq\frac{C}{(1+|x|)^{2+\delta}}(\varepsilon_n^{\delta/2}A_n^{\delta/2}+A_n^{-1/2}+\varepsilon+\varepsilon^{-(2+\delta)}A_n^{-1-\delta/2}). \tag{15}$$

We set

$$A_n=\varepsilon_n^{-\frac{\delta}{(\delta+1)}\left(1+\frac{1}{\delta^2+4\delta+2}\right)},\qquad \varepsilon=\varepsilon_n^{\frac{\delta}{2(\delta+1)}\left(1-\frac{\delta}{\delta^2+4\delta+2}\right)}$$

in (15). We recall the requirement that $T_n=[m_n^2A_n]\leqq k_n$. For the above choise of A_n, this requirement is assured by the condition (e). We thus conclude the theorem.

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