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ON THE INVARIANT MEASURE FOR THE TRANSFORMATIONS ASSOCIATED WITH SOME REAL CONTINUED-FRACTIONS

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ABSTRACT

We introduce two types of real continued-fraction expansions, one of which is the real part of the complex continued-fraction expansion of HURWITZ. For the transformations associated to these expansions we shall determine the precise form of invariant measures according to the method of P. LÉVY for the case of simple continued-fraction. Moreover, we shall clarify the mathematical meaning of the method of P. LÉVY.

§ 0 Introduction

In the investigation of properties of a measurable transformation given on a space, a measure invariant under the transformation, if it exists, provides a valuable clue. Hence, one often takes the following approach in such an investigation. First, one asks whether the transformation has an invariant measure possessing reasonable properties. Next, if there is sucn an invariant measure, one tries to determine its concrete form. Of course, it is, in general, difficult to obtain the precise form of an invariant measure since one has to obtain it predictly from the precise description of each transformation concerned. On the other hand, for this very reason, the derivation of the concrete form of an invariant measure, if it can be carried out, is extremely useful for the quantitative analysis of the given transformation.

In this connection, we recall that there is remarkable history associated with the transformation induced by the well-known simple continued-fraction expansion. For this transformation GAUSS pointed out as if it is obvious apriori that the measure having the density function of the form $\frac{1}{\log 2}\frac{1}{1+x}$ is invariant. Indeed, if one is given the function $\frac{1}{\log 2}\frac{1}{1+x}$, then it is easy to prove that it is the density of a measure invariant under the simple continued-fraction transformation. However, history played a trick and left us with no clue as to how GAUSS actually arrived at this function $\frac{1}{\log 2}\frac{1}{1+x}$. Much later, KUZMIN [3] and LÉVY [4] showed, in their respective papers, ways to arrive at the density function $\frac{1}{\log 2}\frac{1}{1+x}$ for the invariant measure and filled this missing gap, although we have no way of knowing whether the reasoning used by GAUSS was the same as those employed by KUZMIN and LÉVY.

In this paper, we formulate and then solve a couple of problems. The first problem is to search for effective methods for determining precisely the invariant measure for simple continued-fraction transformation and other related transformations. The second problem is to clarify the mathematical structure lying behind the method used by LÉVY in his derivation of the density function $\frac{1}{\log 2}\frac{1}{1+x}$.

With these objectives in mind, we structure this paper in the following manner: In §1 we simplify LÉVY's argument give in [4] to derive the concrete form of the invariant measure for the transformation associated with simple continued-fraction expansion. The method we employ in this section, however, is based on a rather technical and seemingly restrictive assumption, which we shall leave unexplained at that point. For this reason, we shall call the method employed in §1 "the method based on pure chance discovery".

In §2 we introduce a new type of (real) continued-fraction expansion. This expansion corresponds to the real part of the complex continued-fraction expansion introduced by HURWITZ in [2]. In §3, we introduce still another type of real continued-fraction expansion. The relationship between these two continued-fraction expansions can be explained in the following way: Each continued-fraction expansion induces in a natural way an endomorphism on the space of infinite sequences of positive integer (symbolic space). The natural extension of one of these endomorphisms turns out to be the inverse of the natural extension of the other one. For these reason, we shall call the transformation defined in §3 the backward transformation associated with transformation defined in §2.

In §4, we will determine the precise form of the density function for the invariant measures for the continued-fraction transformations defined in §2 and §3. Our method in §4 is different from the "method of pure-chance discovery" employed in §1. We hope, in fact, that our procedure in §4 will clarify the mathematical meaning and give justification to seemingly ad-hoc "method of pure chance discovery".

In order to justify this claim, we shall show in §5 that the reason why the function $g(x)$ seems to emerge suddenly and in somewhat unnatural manner is because for the case of simple continued-fraction expansion the transformation induced by it and its backward transformation coincide with each other, and that it is this fact which makes it difficult to explain the naturalness of the emergence of the function $g(x)$.

In concluding these introductory remarks, we would like to thank Professors TAKUJI ONOYAMA, YUJI ITO and YOICHIRO TAKAHASHI for their interest on the problem and valuable advice.

§1 The Transformation Associated with Continued-Fraction and the Invariant Measure of GAUSS

As it is well-known, the transformation T associated with simple continued-fraction expansions is defined as follows:

For $x \in [0, 1)$,

$$Tx = \begin{cases} \dfrac{1}{x} - \left[\dfrac{1}{x}\right] & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Here, $[a]$ for any number a denotes its integer part. If we let, for $x \in [0, 1)$

$$a(x) = \begin{cases} \left[\dfrac{1}{x}\right] & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

and define

$$a_n(x) = a(T^{n-1}x) \qquad \text{for } n \geqq 1, \tag{1}$$

then the simple continued-fraction expansion of x is given by

$$\begin{aligned} x &= \frac{1}{|a_1(x)} + \frac{1}{|a_2(x)} + \cdots\cdots + \frac{1}{|a_n(x) + T^n x} \\ &= \frac{1}{|a_1(x)} + \frac{1}{|a_2(x)} + \cdots\cdots + \frac{1}{|a_n(x)} + \frac{1}{|a_{n+1}(x)} + \cdots\cdots \end{aligned} \tag{2}$$

If x is a rational number, then $T^n x = 0$ for all but a finite number of n's. Frequently, it is convenient to exclude the set of all rationals in $[0, 1)$ when we consider the metric properties of the simple continued-fraction transformation. In the sequel, we shall disregard the rationals from consideration without stating so explicitly.

Let us associate to each $x \in [0, 1)$ the sequence $(a_1(x), a_2(x), a_3(x), \cdots, a_n(x), \cdots)$, where $a_n(x)$'s are determined by (1), and denote by

$$V = \{(a_1(x), a_2(x), \cdots, a_n(x), \cdots) | x \in [0, 1)\}.$$

V is, in fact, equal to the cartesian product $\boldsymbol{N}^{\boldsymbol{N}}$, where $\boldsymbol{N}$ is the set of all positive integers. For a sequence $(a_1, a_2, \cdots, a_n, \cdots)$ of positive integers, let

$$(3)\qquad \begin{cases} p_{-1}=1, & p_0=0, \quad p_n=a_n p_{n-1}+p_{n-2}, \quad (n\geqq 1) \\ q_{-1}=0, & q_0=1, \quad q_n=a_n q_{n-1}+q_{n-2}, \quad (n\geqq 1), \end{cases}$$

then one can show that

$$(4)\qquad p_n q_{n-1}-p_{n-1}q_n=(-1)^{n-1} \qquad (n\geqq 1),$$

$$(5)\qquad \frac{p_n}{q_n}=\frac{1|}{|a_1}+\frac{1|}{|a_2}+\frac{1|}{|a_3}+\cdots\cdots+\frac{1|}{|a_n} \qquad (n\geqq 1),$$

$$(6)\qquad \frac{q_{n-1}}{q_n}=\frac{1|}{|a_n}+\frac{1|}{|a_{n-1}}+\cdots\cdots+\frac{1|}{|a_1} \qquad (n\geqq 1).$$

For a sequence $(a_1, a_2, \cdots, a_n, \cdots)$ of positive integers, define a function $\psi_{a_1,a_2,\cdots,a_n}(x)$ on $[0, 1)$ by

$$(7)\qquad \psi_{a_1,a_2,\cdots,a_n}(x)=\frac{1|}{|a_1}+\cdots\cdots+\frac{1|}{|a_{n-1}}+\frac{1|}{|a_n+x},$$

it can be shown that

$$(8)\qquad \psi_{a_1,a_2,\cdots,a_n}(x)=\frac{p_n+p_{n-1}x}{q_n+q_{n-1}x}$$

where p_n and q_n are defined by (3).

We denote by $[a_1, a_2, \cdots, a_n]$ the sub-interval of $[0, 1)$ determined by the sequence $(a_1, a_2, \cdots, a_n)$. Namely,

$$[a_1, a_2, \cdots, a_n]=\{x:\ a_1(x)=a_1,\ a_2(x)=a_2,\ \cdots,\ a_n(x)=a_n\}.$$

Then, $[a_1, a_2 \cdots, a_n]$ is the image of the interval $[0, 1)$ under the map $\psi_{a_1,a_2,\cdots,a_n}$. If we denote by m the Lebesgue measure defined on $[0, 1)$, then from (4) it follows that

$$(9)\qquad m([a_1, a_2, \cdots, a_n])=|\psi_{a_1,a_2,\cdots,a_n}(1)-\psi_{a_1,a_2,\cdots,a_n}(0)|$$

$$=\frac{1}{q_n(q_n+q_{n-1})}.$$

In this section, we simplify LÉVY's argument in [4] to show how the form of the density $\frac{1}{\log 2}\frac{1}{1+x}$ can be determined directly.

Now, if we let $m_n(x)=m(T^{-n}[0, x))$, then it follows from (2), (4) and (8) that

$$m_n(x)=\sum_{a_1,a_2,\cdots,a_n}|\psi_{a_1,a_2,\cdots,a_n}(x)-\psi_{a_1,a_2,\cdots,a_n}(0)|$$

$$=\sum_{a_1,a_2,\cdots,a_n}\frac{x}{q_n(q_n+q_{n-1}x)}.$$

Therefore, if we denote by $f_n(x)$ the Radon-Nikodym derivative $\frac{dm_n}{dm}(x)$, then

$$f_n(x)=\sum_{a_1,a_2,\cdots,a_n}\frac{1}{(q_n+q_{n-1}x)^2}$$

$$=\sum_{a_1,a_2,\cdots,a_n}\frac{1+\dfrac{q_{n-1}}{q_n}}{\left(1+\dfrac{q_{n-1}}{q_n}x\right)^2 q_n(q_n+q_{n-1})}$$

We shall show that as n tends to ∞, $f_n(x)$ converges to the desired density function $\dfrac{1}{\log 2}\dfrac{1}{1+x}$.

Since $\sum_{a_1,a_2,\cdots,a_n}\dfrac{1}{q_n(q_n+q_{n-1})}=1$ by (9), we can define, for each n, a discrete probability measure G_n which assigns to each point $\dfrac{q_{n-1}}{q_n}$ a mass $\dfrac{1}{q_n(q_n+q_{n-1})}$. Then, we can write

$$f_n(x)=\sum_{a_1,a_2,\cdots,a_n}\frac{1+\dfrac{q_{n-1}}{q_n}}{\left(1+\dfrac{q_{n-1}}{q_n}x\right)^2}G_n\left(\left\{\frac{q_{n-1}}{q_n}\right\}\right). \tag{10}$$

$$=\int_0^1\frac{1+y}{(1+yx)^2}dG_n(y).$$

For an arbitrary positive interger p, it follows from (3) and (6) that

$$\int_{\frac{1}{p+x}}^{\frac{1}{p}}dG_n(y)=\sum_{\frac{1}{p+x}\leq\frac{q_{n-1}}{q_n}\leq\frac{1}{p}}\frac{1}{q_n(q_n+q_{n-1})} \tag{11}$$

$$=\sum_{\substack{a_n=p\\0\leq\frac{q_{n-2}}{q_{n-1}}\leq x}}\frac{1}{(pq_{n-1}+q_{n-2})((p+1)q_{n-1}+q_{n-2})}$$

$$=\sum_{\substack{a_n=p\\0\leq\frac{q_{n-2}}{q_{n-1}}\leq x}}\frac{\left(1+\dfrac{q_{n-2}}{q_{n-1}}\right)}{\left(p+\dfrac{q_{n-2}}{q_{n-1}}\right)\left(p+1+\dfrac{q_{n-2}}{q_{n-1}}\right)q_{n-1}(q_{n-1}+q_{n-2})}$$

$$=\int_0^x\frac{1+y}{(p+y)(p+1+y)}dG_{n-1}(y)\qquad(0\leqq x<1).$$

(*) If we can make the assumption that the sequence of the probability measure $\{G_n\}$ converges vaguely to some probability measure G which has a continuous density function $g(y)$, then we can obtain from (11), by letting $n\to\infty$ and then differentiating with respect to x, the relation

$$\frac{1}{(p+x)^2}g\left(\frac{1}{p+x}\right)=\frac{1+x}{(p+x)(p+1+x)}g(x),$$

from which it follows that

$$\left(1+\frac{1}{p+x}\right)g\left(\frac{1}{p+x}\right)=(1+x)g(x).$$

Since we are assuming that g is continuous, this implies that $(1+x)g(x)$ is a constant. If we now let $n\to\infty$ in (10), then

$$\lim_{n\to\infty} f_n(x)=\int_0^1\frac{1+y}{(1+yx)^2}g(y)dy=C\int_0^1\frac{1}{(1+yx)^2}dy=C\frac{1}{1+x}.$$

Here, C is a normalizing constant, i.e., $C=1/\log 2$.

§ 2 Continued-Fraction Transformation of HURWITZ.

In § 2 and § 3, we discuss properties of a new continued-fraction transformation, and in § 4 we derive the density function of the invariant measure for that transformation. We hope that the argument we use in § 4 will help clarify the meaning of the "concrete method" that we introduced in § 1.

HURWITZ treated in [2] certain type of continued-fraction expansion for complex numbers. Restricted to the real numbers, this expansion takes the following form.

Let X be the interval $[-1/2, 1/2)$ and define for any real number x, $[x]_1=[x+1/2]$. We then define a transformation S on X by

$$(12)\qquad Sx=\begin{cases}\dfrac{1}{x}-\left[\dfrac{1}{x}\right]_1 & \text{for } x\neq 0\\ 0 & \text{for } x=0.\end{cases}$$

If we write

$$a(x)=\begin{cases}\left[\dfrac{1}{x}\right]_1 & \text{for } x\neq 0\\ \infty & \text{for } x=0,\end{cases}$$

and

$$(13)\qquad a_n(x)=a(S^{n-1}x),$$

then, just as in § 1, the continued-fraction expansion of the following form is valid for every $x\in[-1/2, 1/2)$:

$$(14)\qquad x=\frac{1|}{|a_1(x)}+\frac{1|}{|a_2(x)}+\cdots\cdots+\frac{1|}{|a_n(x)+S^n(x)}$$

$$=\frac{1|}{|a_1(x)}+\frac{1|}{|a_2(x)}+\cdots\cdots+\frac{1|}{|a_n(x)}+\frac{1|}{|a_{n+1}(x)}$$

We shall exclude, as we did in § 1, the set of all rational number from $[-1/2, 1/2)$. We associate to each $x\in[-1/2, 1/2)$ the sequnce of intergers $(a_1(x), a_2(x),$

$\cdots, a_n(x), \cdots)$ determined by (13), and let

$$W=\{(a_1(x), a_2(x), \cdots, a(x_n), \cdots) | x \in [-1/2, 1/2)\}.$$

Let us call a finite sequence $(a_1, a_2, \cdots, a_n)$ of intergers S-admissible if $|a_i| \geqq 2$ for each $i=1, 2, \cdots, n$, and furthermore, if $a_i=2$ then $a_{i+1} \geqq 2$, while if $a_i=-2$ then $a_{i+1} \leqq -2$. Call an infinite sequence $(a_1, a_2, \cdots, a_n, \cdots)$ S-admissible if for every pair $i, j (1 \leqq i \leqq j)$ the finite sequence $(a_i, a_{i+1}, \cdots, a_j)$ is S-admissible. From the definition of the transformations S and of $a_n(x)$ it follows that

$$W=\{(a_1, a_2, \cdots, a_n, \cdots) | (a_1, \cdots, a_n, \cdots) \text{ is } S\text{-admissible}\}.$$

The transformation S induces in the obvious way the shift transformation σ on the symbolic space W.

For an S-admissible sequence $(a_1, a_2, \cdots, a_n, \cdots)$, one can define sequences $\{p_n\}$ and $\{q_n\}$ in the same way as (3) in § 1 and one can show that for these sequences, too, the properties (4), (5) and (6) hold. One can also define the $\psi_{a_1, a_2, \cdots a_n}$ by

$$\psi_{a_1, a_2, \cdots, a_n}(x) = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \cdots\cdots + \frac{1|}{|a_n + x}$$

for an S-admissible sequence $(a_1, a_2, \cdots, a_n)$. The domain of definition of $\psi_{a_1, a_2 \cdots, a_n}$ depends now on the sequence $(a_1, a_2, \cdots, a_n)$ and is given by

$$S^n[a_1, a_2, \cdots, a_n] = \begin{cases} \left(-\frac{1}{2}, \frac{1}{2}\right) & \text{if } a_n \neq \pm 2 \\ \left(0, \frac{1}{2}\right) & \text{if } a_n = 2 \\ \left(-\frac{1}{2}, 0\right) & \text{if } a_n = -2. \end{cases} \tag{15}$$

Here, $[a_1, a_2 \cdots, a_n]$ denotes, as in § 1, the sub-interval of $[-1/2, 1/2)$ determined by the S-admissible sequence $(a_1, a_2, \cdots, a_n)$, i. e.,

$$[a_1, a_2, \cdots, a_n] = \left\{ x \in \left[-\frac{1}{2}, \frac{1}{2}\right) \middle| a_1(x) = a_1, a_2(x) = a_2, \cdots, a_n(x) = a_n \right\}.$$

By using (14) one can show that

$$[a_1, a_2, \cdots, a_n] = \begin{cases} \left\{x : x = \psi_{a_1, \cdots, a_n}(t), -\frac{1}{2} < t < \frac{1}{2}\right\} & \text{if } a_n \neq \pm 2 \\ \left\{x : x = \psi_{a_1, \cdots, a_n}(t), 0 < t < \frac{1}{2}\right\} & \text{if } a_n = 2 \\ \left\{x : x = \psi_{a_1, \cdots, a_n}(t), -\frac{1}{2} < t < 0\right\} & \text{if } a_n = -2. \end{cases} \tag{16}$$

In our discussion in § 1 the quantity q_{n-1}/q_n played a significant role. By (6), q_{n-1}/q_n is equal to $\frac{1|}{|a_n} + \frac{1|}{|a_{n-1}} + \cdots\cdots + \frac{1|}{|a_1}$. However, the sequence $(a_n, a_{n-1}, \cdots,$

a_1) may no longer be S-admissible even if $(a_1, a_2, \cdots, a_n)$ is. So, it will be necessary for us to represent q_{n-1}/q_n in a different way. The following lemma will play a key role in § 3 and § 4.

Lemma 2.1 *If* $(a_1, a_2, \cdots a_n)$ *is S-admissible, then*

$$\alpha = \sup_{\substack{(a_1,\cdots,a_n) \\ 1\leq n<\infty}} \frac{q_{n-1}}{q_n} = \frac{\sqrt{5}-1}{2} = \frac{1|}{|2} + \frac{1|}{|-3} + \frac{1|}{|3} + \frac{1|}{|-3} + \frac{1|}{|3} + \cdots\cdots$$

and

$$\beta = \sup_{\substack{(a_1,\cdots,a_n),\ a_n\neq 2 \\ 1\leq n<\infty}} \frac{q_{n-1}}{q_n} = \frac{3-\sqrt{5}}{2} = \frac{1|}{|3} + \frac{1|}{|-3} + \frac{1|}{|3} + \frac{1|}{|-3} + \cdots\cdots.$$

Proof. Let us define

$$\alpha_n = \sup_{(a_1,\cdots,a_n)} \frac{q_{n-1}}{q_n} \quad \text{and} \quad \beta_n = \sup_{\substack{(a_1,\cdots,a_n) \\ a_n\neq 2}} \frac{q_{n-1}}{q_n},$$

then one can easily show that

$$-\beta_n = \inf_{\substack{(a_1,\cdots,a_n) \\ a_n\neq -2}} \frac{q_{n-1}}{q_n},$$

so that

$$\alpha_n = \sup_{(a_1,\cdots,a_{n-1})} \frac{1}{2+\dfrac{q_{n-2}}{q_{n-1}}} = \frac{1}{2-\beta_{n-1}}, \tag{17}$$

$$\beta_n = \sup_{(a_1,\cdots,a_{n-1})} \frac{1}{3+\dfrac{q_{n-2}}{q_{n-1}}} = \frac{1}{3-\beta_{n-1}}. \tag{18}$$

By induction on n, one can show that both $\{\alpha_n\}$ and $\{\beta_n\}$ are monotone increasing in n and that $|\alpha_n|$, $|\beta_n|<1$ for each n. Therefore, $\alpha=\lim_{n\to\infty}\alpha_n$ and $\beta=\lim_{n\to\infty}\beta_n$ both exist and from (17) and (18) it follows that $\alpha=1/(2-\beta)$ and $\beta=1/(3-\beta)$ and therefore $\alpha=(\sqrt{5}-1)/2$ and $\beta=(3-\sqrt{5})/2$. It is clear that α and β are the desired supremums.

The continued-fraction transformation S of HURWITZ satisfies, because of (15), the condition (b) of NAKADA [5]. The same argument as in § 3 of [5] can be used to show that other conditions of NAKADA [5] are satisfied as well. In particular, we can prove that $\sigma(n)\leq\left(\frac{\sqrt{5}-1}{2}\right)^{2n}$ and that $\gamma(n)=0$.

Using these facts one can establish the following theorem:

Theorem 2.1 (i) *The transformation* S *admits an invariant measure* μ *which is equivalent with the Lebesgue measure* m *on* $X=[-1/2, 1/2)$ *and satisfies* $1/K< d\mu/dm<K$ *for some constant* $K>0$.

(ii) *For an arbitrary Borel set E*

$$|m(S^{-n}E)-\mu(E)|\leqq C_1 m(E)\left(\frac{\sqrt{5}-1}{2}\right)^{2n}.$$

(iii) *For any fundamental interval* $F=[a_1, a_2, \cdots, a_n]$, *where* $(a_1, \cdots, a_n)$ *is S-admissible, and for any Borel set E*

$$|\mu(S^{-k}E\cap F)-\mu(E)\mu(F)|\leqq C_2\mu(E)\mu(F)\left(\frac{\sqrt{5}-1}{2}\right)^{2(k-n)},$$

where C_1 *and* C_2 *are constants independent of sets E, F.*

From Theorem 2.1 it follows easily that the system (X, S, μ) is a weak Bernoulli endomorphism.

§ 3 Backward Transformation

In this section we consider a new expansion associated with the continued-fraction expansion considered in § 2. This new expansion induces a transformation which is the "reverse" of the transformation considered in § 2. For this reason, we call the new transformation the "backward transformation".

Let $\alpha=(\sqrt{5}-1)/2$ and $\beta=(3-\sqrt{5})/2$. α and β have the meaning described in Lemma 2.1. Denote by X^* the interval $[-\alpha, \alpha)$ and define for any real number x,

(19) $$[x]_2=\begin{cases} [x+\alpha] & \text{if } x\geqq 0 \\ -[-x+\beta] & \text{if } x<0, \end{cases}$$

define a transformation S^* or X^* by

(20) $$S^*x=\begin{cases} \dfrac{1}{x}-\left[\dfrac{1}{x}\right]_2 & \text{if } x\neq 0 \\ 0 & \text{if } x=0, \end{cases}$$

and let

$$a^*(x)=\begin{cases} \left[\dfrac{1}{x}\right]_2 & \text{if } x\neq 0 \\ \infty & \text{if } x=0, \end{cases}$$

and

(21) $$a_n^*(x)=a^*(S^{*n-1}(x)) \qquad n\geqq 1.$$

Then, just as in § 1 and § 2, the following continued-fraction expansion is valid:

For any arbitrary irrational $x\in[-\alpha, \alpha)$

(22) $$x=\frac{1}{|a_1^*(x)}+\frac{1}{|a_1^*(x)}+\cdots\cdots+\frac{1}{|a_n^*(x)+S^{*n}(x)}$$

$$=\frac{1}{|a_1^*(x)}+\frac{1}{|a_2^*(x)}+\cdots\cdots+\frac{1}{|a_n^*(x)}+\frac{1}{|a_{n+1}^*(x)}+\cdots\cdots.$$

Define the symbolic spaced W^* by

$$W^*=\{(a_1^*(x), a_2^*(x), \cdots, a_n^*(x), \cdots)|x\in[\alpha, \alpha)\}.$$

Let us call finite sequence of integers $(a_1, a_2, \cdots, a_n)$ S^*-admissible if $|a_i|\geqq 2$ for each $i=1, 2, \cdots, n$, and furthermore, if $a_i\geqq 2$ then $a_{i+1}\neq -2$, while if $a_i\leqq -2$ then $a_{i+1}\neq 2$. An infinite sequence $(a_1, a_2, \cdots, a_n, \cdots)$ will be called S^*-admissible if for every pair i, j $(1\leqq i\leqq j)$ the finite sequence $(a_i, a_{i+1}, \cdots, a_j)$ is admissible. From the definition of the transformation S^* and $a_n^*(x)$ it follows that

$$W^*=\{(a_1, a_2, \cdots. a_n, \cdots)|(a_1, a_2, \cdots, a_n, \cdots) \text{ is } S^*\text{-admissible}\} \tag{23}$$

and that the transformation S^* induces the shift operator σ^* on W^*.

For an S^*-admissibble sequence $(a_1, a_2, \cdots, a_n)$, one can define quantities p_n and q_n in the same way as in (3) of §1, and the properties (4), (5) and (6) are valid for these sequence also. A function $\psi^*_{a_1, a_2, \cdots, a_n}$ similar to $\psi_{a_1, a_2, \cdots, a_n}$ of sections §1 and §2 can be defined in the same way as before for an S^*-admissible sequence $(a_1, a_2, \cdots, a_n)$ and the sub-interval

$$[a_1, a_2, \cdots, a_n]^*=\{x\in[-\alpha, \alpha)|a_1^*(x)=a_1, \cdots, a_n^*(x)=a_n\}$$

of $X^*=[-\alpha, \alpha)$ can be represented in terms of the function $\psi^*_{a_1, a_2, \cdots, a_n}$ as

$$[a_1, a_2, \cdots, a_n]^*=\begin{cases}\{x:\ x=\psi^*_{a_1, \cdots, a_n}(y),\ -\beta\leq y<\alpha\} & \text{if } a_n\geqq 2\\ \{x:\ x=\psi^*_{a_1, \cdots, a_n}(y),\ -\alpha\leq y<\beta\} & \text{if } a_n\leqq -2.\end{cases} \tag{24}$$

You note that a sequence $(a_1, a_2, \cdots, a_n)$ is S-admissible, then the sequence $(a_n, a_{n-1}, \cdots, a_1)$ will be S^*-admissible. If we denote by $(\tilde{W}, \tilde{\sigma})$ the natural extension of (W, σ), and by $(\tilde{W}^*, \tilde{\sigma}^*)$ the natural extension of (W^*, σ^*), then we can obtain the following:

Lemma 3.1 *System $(\tilde{W}, \tilde{\sigma}^{-1})$ and $(\tilde{W}^*, \tilde{\sigma}^*)$ are isomorphic dynamical systems.*

Proof. Define a map φ from $\tilde{W}$ onto $\tilde{W}^*$ by

$$(\cdots, a_{-2}, a_{-1}, a_1, a_2, \cdots)=(\cdots, a_2, a_1, a_{-1}, a_{-2}, \cdots)$$

for every $(\cdots, a_{-2}, a_{-1}, a_1, a_2, \cdots)\in\tilde{W}$. Then, it is easy to check that $\varphi\tilde{\sigma}^{-1}\varphi^{-1}=\tilde{\sigma}^*$.

Remark. The natural extension $(\tilde{W}, \tilde{\sigma}, \tilde{\lambda})$ of the endomorphism (W, σ, λ) which was induced by the endomorphism (X, S, μ) is weak Bernoulli by Theorem 2.1. Therefore, the system $(\tilde{W}^*, \tilde{\sigma}^*, \tilde{\lambda}^*)$ is also a weak Bernoulli automorphism, where $\tilde{\lambda}^*$ denotes the measure $\varphi\tilde{\lambda}^*$.

Lemma 3.2 *Call the restriction of $\tilde{\lambda}^*$ to W^* by λ^*, and denote by μ^* the measure induced on X^* by λ^* in the natural way. Then,*

1) *μ^* is S^*-invariant.*
2) *μ^* is equivalent with the Lebesgue measure on X^*.*

Proof. 1) follows immediately from Lemma 2.1. To prove 2), it suffices to show that there exists a constant $K_1>1$ such that

(25) $$K_1^{-1}m([a_1, a_2, \cdots, a_n]^*) < \mu^*([a_1, a_2, \cdots, a_n]^*) < K_1 m([a_1, a_2, \cdots, a_n]^*)$$

for an arbitrary S^*-admissible sequence $(a_1, a_2, \cdots, a_n)$ for any $n \geqq 1$, where m denote the Lebesgue measure on X^*. We note that

(26) $$\mu^*([a_1, a_2, \cdots, a_n]^*) = \mu([a_n, a_{n-1}, \cdots, a_1])$$

and that by Theorem 2.1 there exists some constant K such that

(27) $$K^{-1}m([a_n, \cdots, a_1]) < \mu([a_n, \cdots, a_1]) < Km([a_n, \cdots, a_1]).$$

It follows from (16) and (24) that $m([a_n, \cdots, a_1])$ and $m([a_1, a_2 \cdots, a_n]^*)$ can be represented as

(28) $$m([a_n, a_{n-1}, \cdots, a_1])$$
$$= \begin{cases} \dfrac{1}{\left\{q_n(a_n, a_{n-1}, \cdots, a_1) + \frac{1}{2}q_{n-1}(a_n, \cdots, a_2)\right\}\left\{(q_n(a_n, \cdots, a_1) - \frac{1}{2}q_{n-1}(a_n, \cdots, a_2)\right\}} & \text{if } a_1 \neq \pm 2 \\ \dfrac{1}{2\left\{q_n(a_n, \cdots, a_1) + \frac{1}{2}q_{n-1}(a_n, \cdots, a_2)\right\}q_n(a_n, \cdots, a_1)} & \text{if } a_1 = 2 \\ \dfrac{1}{2\left\{q_n(a_n, \cdots, a_1) - \frac{1}{2}q_{n-1}(a_n, \cdots, a_2)\right\}q_n(a_n, \cdots, a_1)} & \text{if } a_1 = -2 \end{cases}$$

and

(29) $$m([a_1, \cdots, a_n]^*)$$
$$= \begin{cases} \dfrac{1}{\{q_n(a_1, \cdots, a_n) + \alpha q_{n-1}(a_1, \cdots, a_{n-1}\}\{q_n(a_1, \cdots, a_n) - \beta q_{n-1}(a_1, \cdots, a_{n-1})\}} & \text{if } a_n \geqq 2 \\ \dfrac{1}{\{q_n(a_1, \cdots, a_n) + \beta q_{n-1}(\alpha_1, \cdots, \alpha_{n-1})\}\{q_n(\alpha_1, \cdots, \alpha_n) - \alpha q_{n-1}(\alpha_1, \cdots, a_n)\}} & \text{if } a_n \leqq -2. \end{cases}$$

In formulae (28) and (29) and here-in-after, we denote by $p_n(a_1, \cdots, a_n)$, $p_n(a_n, \cdots, a_1)$, $q_n(a_1, \cdots, a_n)$, $q_n(a_n, \cdots, a_1)$, etc. in order to distinguish p_n's and q_n's which are determined by an S^*-admissible sequence $(a_1, \cdots, a_n)$ and an S-admissible sequence $(a_n, \cdots, a_1)$.

Since $|q_n| > |q_{n-1}|$ hold for either case (i. e. for the case of S^*-admissible sequence and for S-admissible sequence), it follows from (28) and (29) that there exists constants K_2, $K_3 > 1$ such that

(30) $$K_2^{-1}q_n(a_n, \cdots a_1)^{-2} < m([a_n, \cdots, a_1]) < K_2 q_n(a_n, \cdots, a_1)^{-2}$$

(31) $$K_3^{-1}q_n(a_1, \cdots, a_n)^{-2} < m([a_1, \cdots, a_n]^*) < K_3 q_n(a_1, \cdots, a_n)^{-2}.$$

On the other hand, since $q_n(a_1, \cdots, a_n) = q_n(a_n, \cdots, a_1)$ follows from (6), we obtain (25) for some K_1 by using (26), (27), (30) and (31).

§4 The Density Functions for Invariant Measures for S and S^*

We determine the precise form of the density functions for the invariant measures for S and S^*.

Theorem 4.1 *The invariant measure μ equivalent with the Lebesgue measure m for the transformation S has the following density function:*

$$\frac{d\mu}{dm}(x) = \begin{cases} C_3 \dfrac{1}{(1+x\alpha)(1-x\beta)} & \text{for } x>0 \\ C_3 \dfrac{1}{(1-x\alpha)(1+x\beta)} & \text{for } x<0, \end{cases}$$

here C_3 denotes the normalizing constant.

Theorem 4.2 *The invariant measure μ^* equivalent with the Lebesgue measure m for the transformation S^* has the following density function:*

$$\frac{d\mu^*}{dm}(x) = \begin{cases} C_3 \dfrac{1}{2\left(1-\frac{1}{2}x\right)} & \text{for } x \in (-\alpha, -\beta) \\ C_3 \dfrac{1}{\left(1-\frac{1}{4}x^2\right)} & \text{for } x \in (-\beta, \beta) \\ C_3 \dfrac{1}{2\left(1+\frac{1}{2}x\right)} & \text{for } x \in (\beta, \alpha). \end{cases}$$

Proofs of Theorems 4.1 and 4.2 Let $(a_1, a_2, \cdots a_n)$ be an S-admissible sequence, and denote by $p_n(a_1, \cdots, a_n)$ and $q_n(a_1, \cdots a_n)$ the p_n and q_n determined by (3). Then, the Lebesgue measure of the interval $[a_1, a_2, \cdots, a_n]$ is given by (28). We know also that the sequence $(a_n, \cdots, a_1)$ is S^*-admissible and that $q_n(a_n, \cdots, a_1) = q_n(a_1, \cdots, a_n)$ and $p_n(a_n, \cdots, a_1) = q_{n-1}(a_2, \cdots, a_n)$. Next, for each fixed n, we consider the set

$$\left\{ \frac{q_{n-1}}{q_n} = \frac{1}{a_n} + \frac{1}{a_{n-1}} + \cdots\cdots + \frac{1}{a_1} ;\ (a_1, \cdots, a_n) \text{ is } S\text{-admissible} \right\}.$$

From Lemma 2.1 it follows that $(q_{n-1}/q_n) \in X^* = [-\alpha, \alpha)$. Let G_n be the probability measure on X^* which assigns to each point q_{n-1}/q_n the mass $m([a_1, \cdots, a_n])$. Namely,

$$(32)\qquad G_n\left(\left\{\frac{q_{n-1}}{q_n}\right\}\right)=\begin{cases}\dfrac{1}{\left(q_n+\frac{1}{2}q_{n-1}\right)\left(q_n-\frac{1}{2}q_{n-1}\right)} & \text{if } a_n\neq\pm 2\\[2ex] \dfrac{1}{2\left(q_n+\frac{1}{2}q_{n-1}\right)q_n} & \text{if } a_n=2\\[2ex] \dfrac{1}{2\left(q_n-\frac{1}{2}q_{n-1}\right)q_n} & \text{if } a_n=-2\end{cases}$$

Now, for the sub-interval $[a_n, a_{n-1}, \cdots, a_1]^*$ of X^* which is determined by the S^*-admissible sequence $(a_n, a_{n-1}, \cdots, a_1)$, we have

$$(33)\qquad G_{n+k}([a_n, \cdots, a_1]^*)=\sum_{\substack{(b_1,\cdots,b_{n+k});\ S\text{-admissible},\\ b_{k+i}=a_i,\ i=1,\cdots,n}} m([b_1, b_2, \cdots, b_k, b_{k+1}, \cdots, b_{n+k}])$$

$$=m(S^{-k}[a_1, a_2, \cdots, a_n]) \qquad \text{for every } k\geqq 1.$$

From Theorem 2.1 it follows that

$$\lim_{k\to\infty} G_{n+k}([a_n, \cdots, a_1]^*)=\mu([a_1, \cdots, a_n]).$$

The definition of μ^* given in Lemma 3.2 implies that

$$\mu([a_1, \cdots, a_n])=\mu^*([a_n, \cdots, a_1]^*).$$

Therefore, $\lim_{n\to\infty} G_n=\mu^*$, and by Lemma 3.2, the density $\dfrac{d\mu^*}{dm}(y)=g(y)$ exists.

In order to determine the density function $g(y)$ precisely, we consider the following several cases:

Case (i): $-\beta<a<x<\beta$.

Suppose $p\neq\pm 2$, then

$$\int_{\frac{1}{p+x}}^{\frac{1}{p+a}} dG_n(y)=\sum_{\frac{1}{p+x}\leqq\frac{q_{n-1}}{q_n}\leqq\frac{1}{p+a}} G_n\left(\frac{q_{n-1}}{q_n}\right)$$

$$=\sum_{\substack{a_n=p \text{ and}\\ a\leqq\frac{q_{n-2}}{q_{n-1}}\leqq x}} \frac{1}{\left(q_n+\frac{1}{2}q_{n-1}\right)\left(q_n-\frac{1}{2}q_{n-1}\right)}$$

$$=\sum_{\substack{a_n=p \text{ and}\\ a\leqq\frac{q_{n-2}}{q_{n-1}}\leqq x}} \frac{\left(1+\frac{1}{2}\frac{q_{n-2}}{q_{n-1}}\right)\left(1-\frac{1}{2}\frac{q_{n-2}}{q_{n-1}}\right)}{\left(p+\frac{1}{2}+\frac{q_{n-2}}{q_{n-1}}\right)\left(p-\frac{1}{2}+\frac{q_{n-2}}{q_{n-1}}\right)}\ \frac{1}{\left(q_{n-1}+\frac{1}{2}\frac{q_{n-2}}{q_{n-1}}\right)\left(q_{n-1}-\frac{1}{2}\frac{q_{n-2}}{q_{n-1}}\right)}$$

$$=\int_a^x \frac{\left(1+\frac{1}{2}y\right)\left(1-\frac{1}{2}y\right)}{\left(p+\frac{1}{2}+y\right)\left(p-\frac{1}{2}+y\right)}\, dG_{n-1}(y).$$

By letting $n\to\infty$, we obtain

$$\int_{\frac{1}{p+x}}^{\frac{1}{p+a}} g(y)dy=\int_a^x \frac{\left(1+\frac{1}{2}y\right)\left(1-\frac{1}{2}y\right)}{\left(p+\frac{1}{2}+y\right)\left(p-\frac{1}{2}+y\right)} g(y)dy. \tag{34}$$

Similar argument shows that for the cases $p=2$ and $p=-2$, one can get

$$\int_{\frac{1}{p+x}}^{\frac{1}{p+a}} g(y)dy=\begin{cases}\displaystyle\int_a^x \frac{\left(1+\frac{1}{2}y\right)\left(1-\frac{1}{2}y\right)}{2(p+y)\left(p+\frac{1}{2}+y\right)} g(y)dy & \text{if } p=2\\ \displaystyle\int_a^x \frac{\left(1+\frac{1}{2}y\right)\left(1-\frac{1}{2}y\right)}{2(p+y)\left(p-\frac{1}{2}+y\right)} g(y)dy & \text{if } p=-2.\end{cases} \tag{35}$$

Case (ii): $\beta<a<x<\alpha$.

Nothing that if $a\leqq q_{n-2}/q_{n-1}\leqq x$ then $a_{n-1}=2$ and hence $a_n\geqq 2$, we get

$$\int_{\frac{1}{p+x}}^{\frac{1}{p+a}} g(y)dy=\begin{cases}\displaystyle\int_a^x \frac{2\left(1+\frac{1}{2}y\right)}{\left(p+\frac{1}{2}+y\right)\left(p-\frac{1}{2}+y\right)} g(y)dy & \text{if } p\geqq 3\\ \displaystyle\int_a^x \frac{1-\frac{1}{2}y}{(p+y)\left(p+\frac{1}{2}+y\right)} g(y)dy & \text{if } p=2.\end{cases} \tag{36}$$

Case (iii): $-\alpha<a<x<-\beta$.

$$\int_{\frac{1}{p+x}}^{\frac{1}{p+a}} g(y)dy=\begin{cases}\displaystyle\int_a^x \frac{2\left(1-\frac{1}{2}y\right)}{\left(p+\frac{1}{2}+y\right)\left(p-\frac{1}{2}+y\right)} g(y)dy & \text{if } p\leqq -3\\ \displaystyle\int_a^x \frac{1-\frac{1}{2}y}{(p+y)\left(p-\frac{1}{2}+y\right)} g(y)dy & \text{if } p=-2.\end{cases} \tag{37}$$

Differentiating both sides of equations (34), (35), (36) and (37) with respect to x, we obtain

$$\left(1-\frac{1}{4}\left(\frac{1}{p+x}\right)^2\right)g\left(\frac{1}{p+x}\right)=\left(1-\frac{1}{4}x^2\right)g(x) \quad \text{if } -\beta<x<\beta, \quad p\neq\pm 2 \tag{38}$$

$$2\left(1+\frac{1}{2}\,\frac{1}{p+x}\right)g\left(\frac{1}{p+x}\right)=\left(1-\frac{1}{4}x^2\right)g(x) \quad \text{if } -\beta<x<\beta, \quad p=2 \tag{39}$$

$$(40)\qquad 2\left(1-\frac{1}{2}\frac{1}{p+x}\right)g\left(\frac{1}{p+x}\right)=\left(1-\frac{1}{4}x^2\right)g(x) \qquad \text{if} \quad -\beta<x<\beta, \quad p=-2$$

$$(41)\qquad \left(1-\frac{1}{4}\left(\frac{1}{p+x}\right)^2\right)g\left(\frac{1}{p+x}\right)=2\left(1+\frac{1}{2}x\right)g(x) \qquad \text{if} \quad \beta<x<\alpha, \quad p\geqq 3$$

$$(42)\qquad \left(1+\frac{1}{2}\frac{1}{p+x}\right)g\left(\frac{1}{p+x}\right)=\left(1+\frac{1}{2}x\right)g(x) \qquad \text{if} \quad \beta<x<\alpha, \quad p=2$$

$$(43)\qquad \left(1-\frac{1}{4}\left(\frac{1}{p+x}\right)^2\right)g\left(\frac{1}{p+x}\right)=2\left(1-\frac{1}{2}x\right)g(x) \qquad \text{if} \quad -\alpha<x<\beta, \quad p\leqq -3.$$

$$(44)\qquad \left(1-\frac{1}{2}\frac{1}{p+x}\right)g\left(\frac{1}{p+x}\right)=\left(1-\frac{1}{2}x\right)g(x) \qquad \text{if} \quad -\alpha<x<-\beta, \ p=-2.$$

If we define

$$h(x)=\begin{cases} 2\left(1-\frac{1}{2}x\right)g(x) & \text{if } x\in(-\alpha,-\beta) \\ \left(1-\frac{1}{4}x^2\right)g(x) & \text{if } x\in(-\beta,\beta) \\ 2\left(1+\frac{1}{2}x\right)g(x) & \text{if } x\in(\beta,\alpha), \end{cases}$$

then, from (38)—(44), it follows that

$$h(x)=h(S^*x).$$

Since S^* is ergodic, h must be constant almost everywhere.
Thus, we obtain

$$g(x)=\begin{cases} C_3\dfrac{1}{2\left(1-\frac{1}{2}x\right)} & \text{if } x\in(-\alpha,-\beta) \\ C_3\dfrac{1}{\left(1-\frac{1}{4}x^2\right)} & \text{if } x\in(-\beta,\beta) \\ C_3\dfrac{1}{2\left(1+\frac{1}{2}x\right)} & \text{if } x\in(\beta,\alpha) \end{cases}$$

where C_3 is a normalizing constant. This completes the proof of Theorem 4.2.

For the remainder of the proof of Theorem 4.1, let $m_n(x)=m(S^{-n}[0,x))$ and $f_n(x)=(dm_n(x))/dx$ and calculate $\lim_{n\to\infty} f_n(x)$, in case $x>0$. It follows from (8) and (28)

$$f_n(x)=\sum_{\substack{(a_1,\cdots,a_n);\\ S\text{-admissible}}} \frac{1}{(q_n+q_{n-1}x)^2}$$

$$= \sum_{\substack{(a_1, \cdots, a_n): \\ S\text{-admissible} \\ -\beta < \frac{q_{n-1}}{q_n} < \beta}} \frac{\left(q_n+\frac{1}{2}q_{n-1}\right)\left(q_n+\frac{1}{2}q_{n-1}\right)}{(q_n+q_{n-1}x)^2\left(q_n+\frac{1}{2}q_{n-1}\right)\left(q_n-\frac{1}{2}q_{n-1}\right)}$$

$$+ \sum_{\frac{q_{n-1}}{q_n} > \beta} \frac{2\left(q_n+\frac{1}{2}q_{n-1}\right)q_n}{(q_n+q_{n-1}x)^2 2q_n\left(q_n+\frac{1}{2}q_{n-1}\right)}$$

$$= \int_{-\beta}^{\beta} \frac{\left(1+\frac{1}{2}y\right)\left(1-\frac{1}{2}y\right)}{(1+xy)^2} dG_n(y) + \int_{\beta}^{\alpha} \frac{2\left(1+\frac{1}{2}y\right)}{(1+xy)^2} dG_n(y).$$

Letting $n \to \infty$, we obtain

$$\frac{d\mu}{dm} = \lim_{n\to\infty} f_n(x) = C_3 \int_{-\beta}^{\alpha} \frac{1}{(1+xy)^2} dy = C_3 \frac{1}{(1+x\alpha)(1-x\beta)} \qquad (x>0).$$

In case $x<0$, we let $m_n(x)=m(S^{-n}(x,0))$ and use the same argument to get

$$\frac{d\mu}{dm} = C_3 \frac{1}{(1-x\alpha)(1+x\beta)} \qquad (x<0).$$

§ 5 Concluding Remarks

In this section, we try to establish the logical connection between the " method of pure-chance discovery " of § 1 and the method of proof we employed in § 2—§ 4.
1) If we wish to employ the " method of pure-chance discovery " in the derivation of results of § 2—§ 4, we can adopt the following procedure:

First of all, assume that the measure G_n defined on $X^*=[-\alpha, \alpha)$ by (32) satisfies the hypotheses ($*$) made in § 1. Then under this assumption one can prove in exactly the same manner as we employed in § 4 that the function $g(x)$ satisfies the identities (38)—(42). This will enable one to determine the form of the function $g(x)$ and the fact that $\frac{d\mu}{dm} = C\frac{1}{(1-x\beta)(1+x\alpha)}$. If one follows this procedure, there will remain a problem of showing that, in fact, the measure μ with the density $d\mu/dm$ is the invariant measure for the transformation S on $[-1/2, 1/2)$. On the other hand, for the transformation S^* one can prove directly that the identities (34)—(37) satisfied by the function $g(x)$ are nothing but the identities to be satisfied by the density function of any invariant measure for S^*. Alternatively, one can verify that the function $g(x)$ having this particular form is indeed the density function of an invariant measure for S^*.
2) Conversery, one possible " explanation " of the method of pure-chance discovery of § 1 can be given if we look at it from the view-point of § 2—§ 4. Namely, if we consider the backward transformation associated with the transformation in-

duced by the simple continued-fraction expansion, then we see that the backward transformation in fact coincides with the simple continued-fraction transformation itself. Therefore, the hypotheses ($*$) made on the sequence of discrete probability measures $\{G_n\}$ can be interpreted as the assumtion on the existence of an invariant measure for the backward transformation, and we are led to the reasonable conclusion that the function $g(x)$ and $d\mu/dm(x)$ coincide with each other since the spaces X and X^* are identical in the present situation.

3) If we emphasize "the method of pure-chance discovery" in §2—§4, it is possible to take the following approach. Define the discrete measure G_n concentrated on points g_{n-1}/g_n by $G_n(g_{n-1}/g_n)=1/q_n^2$. Make the hypotheses that the sequence $\{G_n\}$ converges vaguely to a probability measure G having a continuous density function $g(x)$. One can then prove as in §1 that $g(x)$=constant, and using this, fact, one can explicitly determine the form of the Radon-Nikodym derivative $d\mu/dm$ for the invariant measure μ. This approach will simplify the computation considerably. However, the characterization of $g(x)$ as the density function for the invariant measure for the backward transformation will no longer be valid. Even for the case of the simple continued-fraction transformation of §1 one can use this simplified procedure.

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