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ON MOTION OF AN IDEAL FLUID, WHICH IS FILLED UP IN A ROTATING VESSEL—II

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ABSTRACT

Here, it is given second example to illustrate the study about flow of an ideal fluid, which is contained in a rotating vessel, as described in the author's previous paper under the same title. We take up the case of two-dimensional flow of an ideal fluid, which is contained in between two eccentric circular walls, rotating with given angular acceleration. The result of analysis is given as a proposed coefficient of virtual mass of fluid, together with some numerical examples about it.

1. Introduction

The author has reported, in the previous paper under the same title[2], an account about general theory of motion of an ideal fluid, which is filled up in a rotaing vessel. Also, he has shown an example to illustrate this general theory, in which the vessel was taken to be of fan-shaped figure, restricting ourselves to case of two-dimensional motion. In the present report, the author gives here another example to illustrate the general theory. It refers to the case of an ideal fluid contained in between two eccentric cicular walls, these walls being kept in rotational motion with nonuniform angular velocity ω. The result of analysis is studied numerically, about eight different configurations of side walls, obtaining values of numerical coefficients as so-called "coefficients of virtual mass" in the author's proposed form.

2. Statement of our Problem

We consider, here again, the two dimensional motion of an ideal (non-viscous, incompressible) fluid, and take up the case of fluid motion which is contained in between two eccentric circular (rigid) walls. These rigid walls are assumed to be making a rotational motion about fixed axis $\boldsymbol{O}_a$, with an angular velocity ω. In order to treat hydrodynamically, this case of non-stationary fluid motion, we use following notations:

- $\boldsymbol{x}, \boldsymbol{y}$=coordinates of any point P in $(\boldsymbol{x}, \boldsymbol{y})$ plane, the frame of coordinates $(\boldsymbol{O}_a\boldsymbol{xy})$ being assumed to be making a rotational motion about the axis $\boldsymbol{O}_a$.
- x, y=coordinates of the same point P, with regard to rectangular axes, whose origin O is situated at a distance d, from the axis of rotation $\boldsymbol{O}_a$, that is, $\overline{O_aO}=d$.
- ω=angular velocity of rotation of frame of coordinate axes $\boldsymbol{O}_a\boldsymbol{xy}$.
- ξ, η=curvilinear coordinates of point $P(x, y)$, expressed by means of bi-polar coordinates.

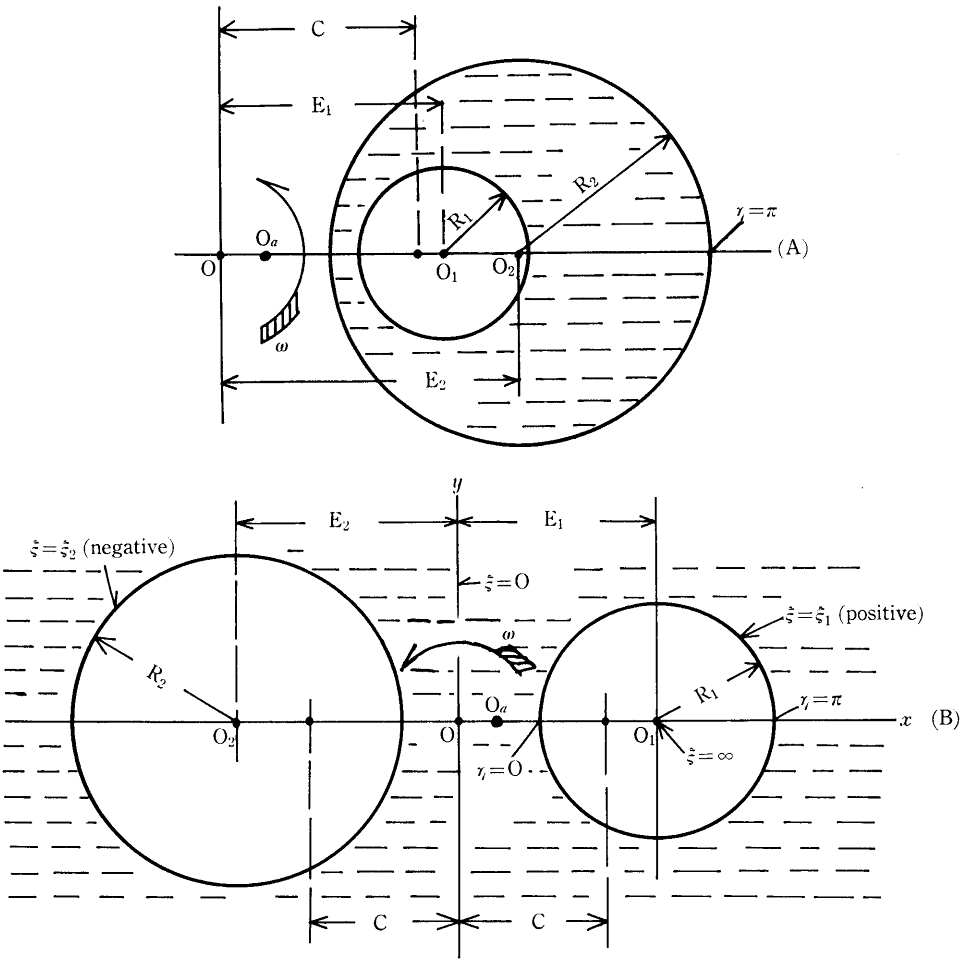


Fig. 1. Fluid Region contained in between two Eccentric Circular Cylinders.

c=distance from origin O, of radical center of system of bi-polar coordinates.
h=coefficient of linear element for the case of bi-polar coordinates (ξ, η).
E_i=position of center of the circle $\xi=\xi_i$.
R_i=radius of circle $\xi=\xi_i$ (i is taken to be 1 and 2 for representing two eccentric circles $\xi=\xi_1$ or $\xi=\xi_2$).
θ=angular coordinate of a point $P(x, y)$, which lie on the circle $\xi=\xi_i$.
ϕ=velocity potential of fluid motion, giving absolute velocity of flow.
V_c=linear velocity of a point P, corresponding to rotational motion with angular velocity ω, around the axis of rotation $\boldsymbol{O}_a$.
r_1, φ=polar coordinates of point P, with regard to axis of rotation $\boldsymbol{O}_a$.
V_n=normal component, in direction of decrease of ξ, due to rotation (on the circular wall at which $\xi=\xi_i$).
p=fluid pressure.
ρ=density of the fluid.

Now, referring to Fig. 1, which shows us fluid region contained in between two eccentric circular walls, any point $P(x, y)$ may be represented by means of bi-polar coordinates (ξ, η), the mutual relation between coordinates $(\boldsymbol{x}, \boldsymbol{y})$, (x, y) being expressed by[1]

$$\left.\begin{aligned} x=\boldsymbol{x}+d&=\frac{c \operatorname{sh} \xi}{\operatorname{ch} \xi+\cos \eta}, \\ y=\ \boldsymbol{y}\ &=\frac{c \sin \eta}{\operatorname{ch} \xi+\cos \eta}. \end{aligned}\right\} \tag{1}$$

The line element ds is given by

$$\begin{aligned} (ds)^2=(d\boldsymbol{x})^2+(d\boldsymbol{y})^2&=(dx)^2+(dy)^2 \\ &=h^2[(d\xi)^2+(d\eta)]^2 \end{aligned}$$

where we have put

$$h=\frac{c}{\operatorname{ch} \xi+\cos \eta} . \tag{2}$$

The velocity potential ϕ, which represents absolute velocity of fluid flow, is taken to satisfy the Laplace equation

$$\Delta_2\phi=0 \tag{3}$$

wherein the two-dimensional Laplacian $\Delta_2\phi$ is given by

$$\begin{aligned} \Delta_2\phi &\equiv \frac{\partial^2\phi}{\partial \boldsymbol{x}^2}+\frac{\partial^2\phi}{\partial \boldsymbol{y}^2} \equiv \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} \\ &\equiv \frac{1}{h^2}\left(\frac{\partial^2\phi}{\partial \xi^2}+\frac{\partial^2\phi}{\partial \eta^2}\right) . \end{aligned}$$

It is to be noted that in the present instance, ϕ is function of ξ and η, and also of time t.

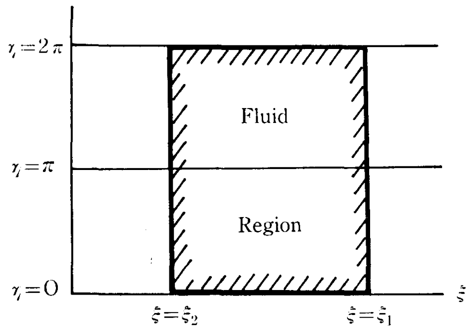


Fig. 2. Region in $\zeta=\xi+i\eta$ Plane.

Two eccentric circular walls are taken to be given by $\xi=\xi_1$ and $\xi=\xi_2$. Value of ξ_2 is taken to be positive (Fig. 1 (A)) or negative (Fig. 1 (B)). Always we take $\xi_2<\xi_1$. The fluid region may be represented in (ξ, η) plane as a rectangular region composed of $\xi_2\leq\xi\leq\xi_1$ and $0\leq\eta\leq 2\pi$, as shown in Fig. 2.

The general solution of the eq. (3) may be given by

$$\phi=\sum_n [A_n \sin n\eta+B_n \cos n\eta]\cdot[\text{sh } n\xi+C_n \text{ ch } n\xi] \tag{4}$$

where A_n, B_n and C_n are arbitrary constants with regard to ξ, η, but they may be functions of time t. The summation $\sum$ are to be made for $n=1, 2, \cdots, \infty$. Thus, we observe that our problem may be stated as follows: to determine arbitrary constants A_n, B_n and C_n in such way that the boundry conditions

$$-\frac{1}{h}\frac{\partial\phi}{\partial\xi}=V_n \tag{5}$$

are satisfied at two moving walls $\xi=\xi_1$ and $\xi=\xi_2$, V_n being component normal to wall surfaces, of the linear velocity of rotation V_c, of wall itself. (see Fig. 3)

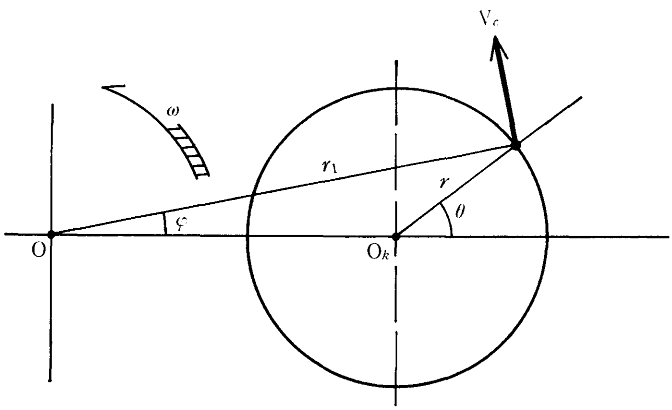


Fig. 3. Relations of r_1, φ and θ [$r=R_k$ at the Circular Wall].

Having thus obtained actual value of velocity potential ϕ, the value of hydraulic pressure p can be given by the formula

$$-\frac{1}{\rho}p=-\frac{1}{\rho}p_s\omega^2+\phi_s\frac{d\omega}{dt}+C(t) \tag{6}$$

which was deduced from the fundamental equation of motion in hydrodynamics, as referred to moving axes, as was given in author's previous paper[2]. In this eq. (6), we have put

$$-\frac{1}{\rho}p_s=\frac{1}{2}\left[\left(\frac{\partial\phi_s}{\partial \boldsymbol{x}}\right)^2+\left(\frac{\partial\phi_s}{\partial \boldsymbol{y}}\right)^2\right]+\boldsymbol{y}\frac{\partial\phi_s}{\partial \boldsymbol{x}}-\boldsymbol{x}\frac{\partial\phi_s}{\partial \boldsymbol{y}} .$$

ϕ_s being value of velocity potential ϕ for the case in which we take $\omega=1$ (stationary state of rotation).

3. The Solution of our Problem

Along the circumference of circle $\xi=\xi_k$ $(k=1, 2)$ we have,

$$\begin{aligned} R_k e^{i\theta} &= x+iy-E_k \\ &= c\frac{\mathrm{sh}\,\xi_k+i\sin\eta}{\mathrm{ch}\,\xi_k+\cos\eta}-E_k . \end{aligned} \tag{7}$$

From this relation we have

$$R_k\cos\theta=\frac{c\,\mathrm{sh}\,\xi_k}{\mathrm{ch}\,\xi_k+\cos\eta}-E_k ,$$

$$R_k\sin\theta=\frac{c\sin\eta}{\mathrm{ch}\,\xi_k+\cos\eta} ,$$

whence we deduce that

$$\left.\begin{aligned} E_k &= c\frac{\mathrm{ch}\,\xi_k}{\mathrm{sh}\,\xi_k} , \\ R_k &= c\frac{1}{\mathrm{sh}\,\xi_k} . \end{aligned}\right\} \tag{8}$$

Next, we evaluate value of V_c of linear velocity of rotation as follows, referring to Fig. 3.

$$\begin{aligned} V_c=\omega r_1 &= \omega[\boldsymbol{x}^2+\boldsymbol{y}^2]^{1/2} \\ &= \omega[(x-d)^2+y^2]^{1/2} \end{aligned}$$

wherein we have, by eq. (1),

$$r_1^2=x^2+y^2-2dx+d^2=\frac{K_1(\xi)+K_2\cos\eta}{\text{ch}\,\xi+\cos\eta}\ .$$

Putting here

$$K_1(\xi)=(c^2+d^2)\,\text{ch}\,\xi-2cd\,\text{sh}\,\xi\ ,$$
$$K_2=d^2-c^2\ .$$

Thus, we obtain

$$V_c=\omega r_1=\omega\left[\frac{K_1(\xi)+K_2\cos\eta}{\text{ch}\,\xi+\cos\eta}\right]^{1/2}. \tag{9}$$

On the other hand, we have, also referring to Fig. 3,

$$\cos\varphi=\frac{y}{r_1}=\frac{c\sin\eta}{\text{ch}\,\xi+\cos\eta}\left[\frac{\text{ch}\,\xi+\cos\eta}{K_1(\xi)+K_2\cos\eta}\right]^{1/2},$$

$$\sin\varphi=\frac{x-d}{r_1}$$
$$=\frac{c\,\text{sh}\,\xi-d(\text{ch}\,\xi+\cos\eta)}{\text{ch}\,\xi+\cos\eta}\left[\frac{\text{ch}\,\xi+\cos\eta}{K_1(\xi)+K_2\cos\eta}\right]^{1/2}.$$

The component V_n of linear velocity of rotation Vc, taken in direction normal to circumference of circle $\xi=\xi_k$ is given by

$$V_n=V_c\sin(\theta-\varphi)\ . \tag{10}$$

After putting values of $\sin\varphi$ and $\cos\varphi$ as given above into this eq. (10), we obtain, after rearrangement

$$V_n=\frac{cV_c}{R_k}(E_k-d)\frac{\sin\eta}{[\text{ch}\,\xi_k+\cos\eta]^{1/2}[K_1(\xi_k)+K_2\cos\eta]^{1/2}}\ .$$

Thus, finally we have, by eq. (9)

$$V_n=\omega cf_k\frac{\text{ch}\,\xi_k\sin\eta}{\text{ch}\,\xi_k+\cos\eta} \tag{11}$$

where we have put

$$f_k=1-\frac{d}{E_k}\ . \tag{12}$$

Knowing thus the value of V_n, the boundary conditions as imposed by eq. (5) are written as follows, using expression (4) of velocity potential ϕ:

$$\omega cf_k\frac{\text{ch}\,\xi_k\sin\eta}{\text{ch}\,\xi_k+\cos\eta}=-\frac{1}{h}\frac{\partial\phi}{\partial\xi}$$
$$=-\frac{1}{c}(\text{ch}\,\xi_k+\cos\eta)\left[\sum_n n(A_n\sin n\eta+B_n\cos n\eta)\cdot\right.$$
$$\left.(\text{ch}\,n\xi_k+C_n\,\text{sh}\,n\xi_k)\right], \tag{13}$$

which may also be rewritten

$$-\omega c^2 f_k \frac{\operatorname{ch}\xi_k \sin\eta}{(\operatorname{ch}\xi_k+\cos\eta)^2}$$
$$=\sum_n n(A_n \sin n\eta + B_n \cos n\eta)\cdot(\operatorname{ch} n\xi_k + C_n \operatorname{sh} n\xi_k) \tag{14}$$

This eq. (14) shows us that arbitrary constants A_n, B_n and C_n can be determined in manner of Fourier coefficients. It is apparent that we have

$$B_n=0.$$

Also, we have

$$\omega c^2 f_k \operatorname{ch}\xi_k \int_0^{2\pi} \frac{\sin\eta \sin n\eta}{(\operatorname{ch}\xi_k+\cos\eta)^2} d\eta = -n\pi(\operatorname{ch} n\xi_k + C_n \operatorname{sh} n\xi_k)A_n \tag{15}$$
$$(n=1, 2, \cdots).$$

This eq. (15) leads us to the evaluation of definite integral

$$I=\frac{1}{2\pi}\int_0^{2\pi} \frac{\cos m\eta + i \sin m\eta}{[\lambda+\cos\eta]^s} d\eta$$

where m and s are positive integers. These values of I are to be found by evaluation of following contour integral in complex z-plane,

$$I_m^{(s)}=\frac{1}{2\pi i}\int_C \frac{z^m}{\left[\lambda+\frac{1}{2}(z+z^{-1})\right]^s} \frac{dz}{z}$$

for $z=e^{i\theta}$ ($\theta=0$ to 2π) around a unit circle, inside of which there exist a pole at $z=-\varepsilon$;

$$\varepsilon=\frac{1}{\lambda+\sqrt{\lambda^2-1}}.$$

λ being a positive constant such that $1<\lambda$. Evaluating this contour integral $I_m^{(s)}$, by means of calculus of residues, we obtain for $s=2$,

$$I_m^{(2)}=(-)^m\left[\frac{(m+1)\varepsilon^{m+2}}{(1-\varepsilon^2)^2}+\frac{2\varepsilon^{m+4}}{(1-\varepsilon^2)^3}\right]. \tag{16}$$

Returning to expressions in real variables, we obtain,

$$K_m^{(2)}=\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos m\eta}{(\lambda+\cos\eta)^2}d\eta=(-)^m\frac{4(m+1)\varepsilon^{m+2}}{(1-\varepsilon^2)^2}+(-)^m\frac{8\varepsilon^{m+4}}{(1-\varepsilon^2)^3}. \tag{17}$$

Applying this expression (17) to our case of eq. (15), we have

$$\frac{1}{\pi}\int_0^{2\pi}\frac{\sin\eta \sin n\eta}{(\operatorname{ch}\xi_k+\cos\eta)^2}d\eta=K_{n-1}^{(2)}(\varepsilon_k)-K_{n+1}^{(2)}(\varepsilon_k)=L_n(\varepsilon_k)$$

where we put

$$\varepsilon_k = \frac{1}{\lambda_k + \sqrt{\lambda_k^2 - 1}}, \qquad \lambda_k = \mathrm{ch}\,\xi_k,$$

or

$$\varepsilon_k = \frac{1}{\mathrm{ch}\,\xi_k + |\mathrm{sh}\,\xi_k|} \quad .$$

Actual values of $L_n(\varepsilon_k)$ are thus found to be (for $n=1, 2, \cdots$)

$$L_n(\varepsilon_k) = (-)^{n-1} \frac{4n}{(1-\varepsilon_k^2)} (\varepsilon_k)^{n-1} \tag{18}$$

Using these values of $L_n(\varepsilon_k)$, eq. (15) becomes

$$\omega c^2 f_k \,\mathrm{ch}\,\xi_k L_n(\varepsilon_k) = -nA_n(\mathrm{ch}\,n\xi_k + C_n\,\mathrm{sh}\,n\xi_k) \tag{19}$$

for $k=1$ and 2. Eliminating A_n from eq. (19) for $k=1$ and 2, we obtain

$$(\mathrm{ch}\,n\xi_2 + C_n\,\mathrm{sh}\,n\xi_2) f_1\,\mathrm{ch}\,\xi_1 L_n(\varepsilon_1) = (\mathrm{ch}\,n\xi_1 + C_n\,\mathrm{sh}\,n\xi_1) f_2\,\mathrm{ch}\,\xi_2 L_n(\varepsilon_2)$$

from which we find, as values of C_n and A_n, as follows:

$$C_n = -\frac{M_n}{N_n} \tag{20}$$

where we put

$$M_n = F_n(\xi_1)\,\mathrm{ch}\,n\xi_2 - F_n(\xi_2)\,\mathrm{ch}\,n\xi_1,$$

$$N_n = F_n(\xi_1)\,\mathrm{sh}\,n\xi_2 - F_n(\xi_2)\,\mathrm{sh}\,n\xi_1,$$

$$F_n(\xi_1) = f_1\,\mathrm{ch}\,\xi_1 L_n(\varepsilon_1),$$

$$F_n(\xi_2) = f_2\,\mathrm{ch}\,\xi_2 L_n(\varepsilon_2),$$

$$A_n = -\frac{\omega c^2 f_k\,\mathrm{ch}\,\xi_k\, L_n(\varepsilon_k)}{n(\mathrm{ch}\,n\xi_k + C_n\,\mathrm{sh}\,n\xi_k)} \quad . \tag{21}$$

Thus, our solution (4) is completely determined.

4. Driving Torque caused by Fluid Pressure acting upon Wall Surfaces

When the fluid pressure p acts normally to wall surface $\xi=\xi_k$, it will give rise to torque T_k, whose amount is given by (per unit depth)

$$T_k = (-)^k \int r_1 p R_k \sin(\theta - \varphi) d\theta .$$

The integral is to be taken around whole circumference of the circle $\xi=\xi_k$. The factor $(-)^k$ is introduced to take into account the fact that T_k is taken positive for counter-clock wise torque. The fluid pressure p is to be found from eq. (6), in which we put value of velocity potential ϕ in eq. (4), with coefficients A_n, B_n and C_n as given by (20) and (21).

Here we shall be mainly interested in that part of torque T_k which correspond to term in angular acceleration $d\omega/dt$. For this purpose, we put

$$-\frac{1}{\rho}p=\phi_s\frac{d\omega}{dt}. \tag{22}$$

ϕ_s being value of ϕ as found before for the case of $\omega=1$. In this case the driving torque T_k due to fluid pressure p will be given by

$$T_k=(-)^k\rho R_k U_k\frac{d\omega}{dt}$$

where we put

$$U_k=\int_0^{2\pi} r_1\phi_s \sin(\theta-\varphi)d\theta. \tag{23}$$

Now we have, by differentiating the equation

$$R_k\cos\theta=\frac{c\,\mathrm{sh}\,\xi_k}{\mathrm{ch}\,\xi_k+\cos\eta}-E_k,$$

the relation

$$\frac{d\theta}{d\eta}=-\frac{\mathrm{sh}\,\xi_k}{(\mathrm{ch}\,\xi_k+\cos\eta)},$$

from which we also deduce that

$$r_1\sin(\theta-\varphi)\frac{d\theta}{d\eta}=-c\frac{E_k-d}{R_k}\frac{\sin\xi_k\sin\eta}{(\mathrm{ch}\,\xi_k+\cos\eta)^2},$$

while we have, at $\xi=\xi_k$,

$$\phi_s=\sum_n\frac{A_n}{\omega}\sin n\eta[\mathrm{sh}\,n\xi_k+C_n\,\mathrm{ch}\,n\xi_k].$$

Putting these values into eq. (23) and noting that the integration about θ for $\theta=0$ to 2π, correspond to integration about η for $\eta=2\pi$ to 0, we arrive at the following expression for the torque T_k:

$$\begin{aligned}U_k&=\frac{E_k-d}{R_k}c\,\mathrm{sh}\,\xi_k\sum_{n=1}^{\infty}(\mathrm{sh}\,n\xi_k+C_n\,\mathrm{ch}\,n\xi_k)\cdot\frac{A_n}{\omega}\int_0^{2\pi}\frac{\sin n\eta\sin\eta}{(\mathrm{ch}\,\xi_k+\cos\eta)^2}d\eta\\&=\frac{E_k-d}{R_k}c\pi\,\mathrm{sh}\,\xi_k\sum_{n=1}^{\infty}\frac{A_n}{\omega}(\mathrm{sh}\,n\xi_k+C_n\,\mathrm{ch}\,n\xi_k)\cdot L_n(\varepsilon_k).\end{aligned}$$

Thus we have

$$U_k = -\pi c^3 \frac{(E_k-d)^2}{E_k R_k} \text{ ch } \xi_k \text{ sh } \xi_k \cdot \sum_{n=1}^{\infty} [L_n(\varepsilon_k)]^2 \frac{\tanh n\xi_k + C_n}{1+\tanh n\xi_k \cdot C_n}$$

for $k=1$ and 2. Thus finally, we obtain, as final formula for the driving torque T_k,

$$T_k = (-)^{k+1} \rho\pi c^4 \left(\frac{d\omega}{dt}\right) f_k^2 (\text{ch } \xi_k)^2 Q_k, \tag{24}$$

wherein we put for shortness

$$Q_k = \sum_{n=1}^{\infty} \frac{1}{n} [L_n(\varepsilon_k)]^2 D_n^{(k)}. \tag{25}$$

$D_n^{(k)}$ are numerical coefficients, whose values, as shown below, were obtained by above mentioned analysis,

$$\left.\begin{aligned} D_n^{(1)} &= -\frac{F_n(\varepsilon_2)}{F_n(\varepsilon_1)} \frac{A_n \text{ ch } n(\xi_1-\xi_2)-1}{\text{sh } n(\xi_1-\xi_2)}, \\ D_n^{(2)} &= \frac{\text{ch } n(\xi_1-\xi_2)-A_n}{\text{sh } n(\xi_1-\xi_2)}, \end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned} F_n(\xi_1) &= f_1 \text{ ch } \xi_1 L_n(\xi_1), \\ F_n(\xi_2) &= f_2 \text{ ch } \xi_2 L_n(\xi_2), \end{aligned}\right\} \tag{27}$$

$$\begin{aligned} A_n &= \frac{\text{ch } \xi_1}{\text{ch } \xi_2} \frac{L_n(\varepsilon_1)}{L_n(\varepsilon_2)} \\ &= \frac{\text{ch } \xi_1}{\text{ch } \xi_2} \left(\frac{1-\varepsilon_2^2}{1-\varepsilon_1^2}\right) \left[\frac{\text{ch } \xi_2 + |\text{sh } \xi_2|}{\text{ch } \xi_1 + |\text{sh } \xi_1|}\right]^{n+1}. \end{aligned} \tag{28}$$

We see that if $|\xi_2|<|\xi_1|$, we have $A_n<1$, while we have $A_n=1$ if $|\xi_1|=|\xi_2|$. It is to be noted that for the special case in which $\xi_2=-\xi_1$, we have

$$-D_n^{(1)} = D_n^{(2)} = \frac{\text{sh } n\xi_1}{\text{ch } n\xi_1}.$$

Also, we see that $D_n^{(1)}$ and $D_n^{(2)}$ tend to unity when we make $n\to\infty$. Referring to actual values of coefficients $L_n(\xi_k)$, as given by eq. (18) and observing that we have $\varepsilon_k<1$, we infer that the infinite series (25) are absolutely convergent.

In order to obtain a convenient factor F_s which gives us a judgement about amount of total torque

$$T_s = T_1 + T_2, \tag{29}$$

let us consider the case of solid disk with density ρ, which is contained in between two circles $\xi=\xi_1$ and $\xi=\xi_2$ (ξ_1 and ξ_2 being positive such that $\xi_2<\xi_1$). The area of this disk is

$$A_s = \pi R_2^2 - \pi R_1^2 = \pi c^2 \left[\frac{1}{(\text{sh } \xi_2)^2} - \frac{1}{(\text{sh } \xi_1)^2}\right]$$

and moment of inertia of this disk (of unit thickness) about center of rotation $\boldsymbol{O}_a$ is given by

$$M_s = \rho(E_2-d)^2 A_2 - \rho(E_1-d)^2 A_1$$

$$= \rho\pi[f_2^2 E_2^2 R_2^2 - f_1^2 E_1^2 R_1^2]$$

$$= \pi\rho c^4\left[f_2^2\frac{(\text{ch}\,\xi_2)^2}{(\text{sh}\,\xi_2)^4} - f_1^2\frac{(\text{ch}\,\xi_1)^2}{(\text{sh}\,\xi_1)^4}\right] = \pi\rho c^4 N_s\,.$$

It is to be understood that M_s is merely a fictitious quantity obtained by assuming that masses are concentrated at centers of each circles, and only used as reference purposes. N_s is a numerical coefficient (of no dimension) corresponding to M_s. Final results may be given in the following form.

$$T_s = T_1 + T_2$$

$$= \rho c^4\pi\left(\frac{d\omega}{dt}\right)[f_1^2(\text{ch}\,\xi_1)^2 Q_1 - f_2^2(\text{ch}\,\xi_2)^2 Q_2]\,. \tag{31}$$

Or, expression as the ratio of T_s to M_s,

$$F_s = \frac{T_s}{M_s} = \left(\frac{d\omega}{dt}\right)\frac{f_1^2(\text{ch}\,\xi_1)^2 Q_1 - f_2^2(\text{ch}\,\xi_2)^2 Q_2}{N_s}\,, \tag{32}$$

where we have put,

$$N_s = f_2^2\frac{(\text{ch}\,\xi_2)^2}{(\text{sh}\,\xi_2)^4} - f_1^2\frac{(\text{ch}\,\xi_1)^2}{(\text{sh}\,\xi_1)^4}\,. \tag{33}$$

We may, for convenience, name the factor F_s (of no-dimension) a coefficient of virtual mass of the fluid motion.

On the other hand, for the case in which we have $\xi_2 = -\xi_1$, we find it convenient to use the factor

$$N_s' = f_2^2\frac{(\text{ch}\,\xi_2)^2}{(\text{sh}\,\xi_2)^4} + f_1^2\frac{(\text{ch}\,\xi_1)^2}{(\text{sh}\,\xi_1)^4}$$

$$= (f_1^2 + f_2^2)\frac{(\text{ch}\,\xi_1)^2}{(\text{sh}\,\xi_1)^4}\,. \tag{34}$$

5. Summary of Results of Numerical Estimation

In order to show you numerical values relating to our formula above obtained, we carried out numerical estimation about cases of eight different configurations as shown in Fig. 4, (A) to (G).

It was found more convenient for carrying out numerical calculation, in case of ξ_1 and ξ_2 both positive and $\xi_2 < \xi_1$, to rearrange the above equation into following form:

$$Q_1=\sum_{n=1}^{\infty}\frac{1}{n}[L_n(\varepsilon_1)L_n(\varepsilon_2)]G_n^{(1)}\ , \tag{35}$$

where we put

$$G_n^{(1)}=\frac{1-A_n\,\mathrm{ch}\,n(\xi_1-\xi_2)}{\mathrm{sh}\,n(\xi_1-\xi_2)}\left(\frac{\mathrm{ch}\,\xi_2}{\mathrm{sh}\,\xi_1}\right)\ .$$

and

$$Q_2=\sum_{n=1}^{\infty}\frac{1}{n}[L_n(\varepsilon_2)]^2G_n^{(2)} \tag{36}$$

and also we put,

$$G_n^{(2)}=\frac{\mathrm{ch}\,n(\xi_1-\xi_2)-A_n}{\mathrm{sh}\,n(\xi_1-\xi_2)}\ .$$

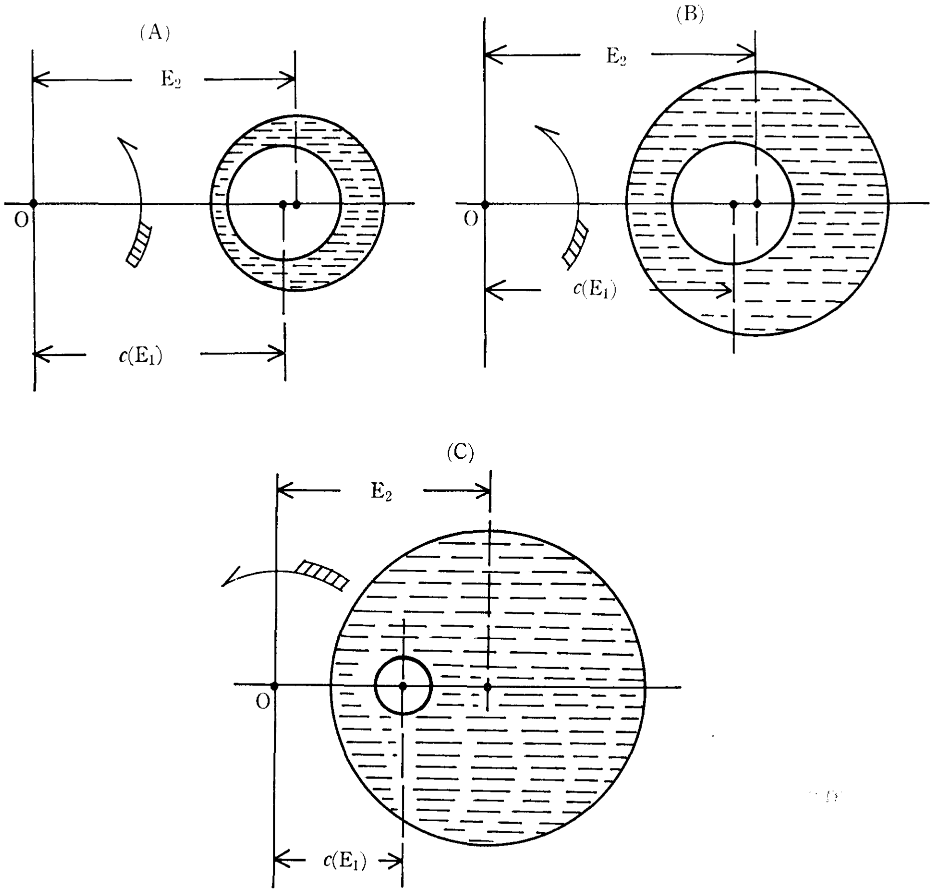
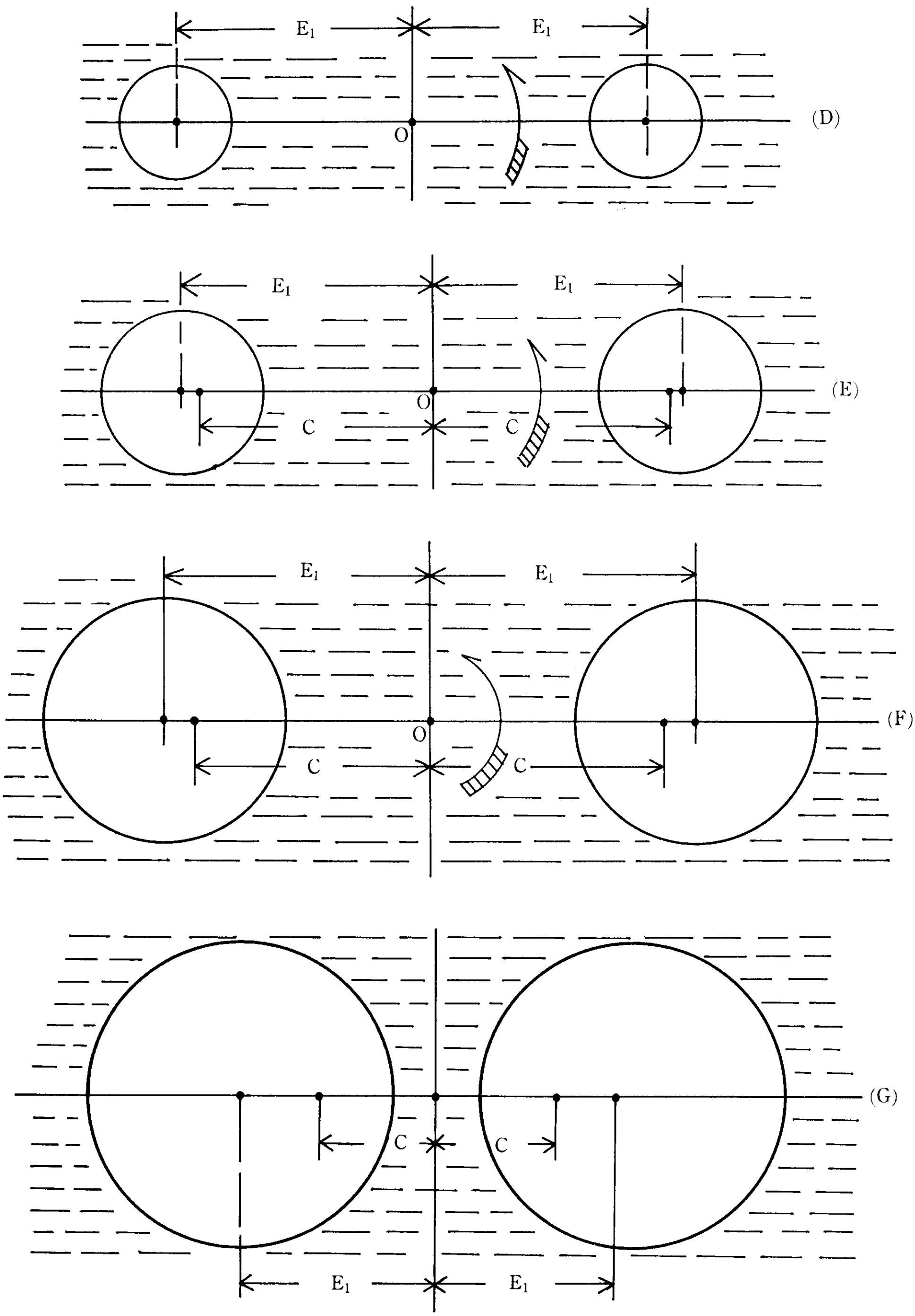


Fig. 4. Configurations of Fluid Region which were taken up as Numerical Examples.

(*Continued*) Fig. 4



The summation of infinite series in eqs. (35) and (36) were made for $n=1$ to $n=6$ (for cases of $\xi_k=\log_e 4$, $\log_e 6$ and $\log_e 8$), and for $n=1$ to $n=10$ (for case of $\xi_1=\log_e 2$).

(A)

$$\xi_1=\log_e 8,\quad \mathrm{ch}\,\xi_1=4.06250,\quad \mathrm{sh}\,\xi_1=3.93750,$$

$$E_1=1.03175\,c,\quad R_1=0.25397\,c,$$

$$\xi_2=\log_e 6,\quad \mathrm{ch}\,\xi_2=3.08333,\quad \mathrm{sh}\,\xi_2=2.91667,$$

$$E_2=1.05714\,c,\quad R_2=0.34286\,c,$$

$$F_s=-1.00705,\quad T_1/T_2=-0.54872.$$

(B)

$$\xi_1=\log_e 8,$$

$$\mathrm{ch}\,\xi_1=4.06250,\quad \mathrm{sh}\,\xi_1=3.93750,$$

$$\mathrm{ch}\,\xi_2=2.12500,\quad \mathrm{sh}\,\xi_2=1.87500,$$

$$E_1=1.03175\,c,\quad R_1=0.25397\,c,$$

$$E_2=1.13333\,c,\quad R_2=0.53333\,c,$$

$$F_s=-1.01395,\quad T_1/T_2=-0.22347.$$

(C)

$$\xi_1=\log_e 8,\quad \xi_2=\log_e 2,$$

$$\mathrm{ch}\,\xi_1=4.06250,\quad \mathrm{sh}\,\xi_1=3.93750,$$

$$E_1=1.03175\,c,\quad R_1=0.25397\,c,$$

$$E_2=1.66667\,c,\quad R_2=1.33333\,c,$$

$$F_s=-1.00968,\quad T_1/T_2=-0.03124.$$

(D)

$$\xi_1=-\xi_2=\log_e 8,$$

$$E_1=-E_2=1.03175\,c,\quad R_1=R_2=0.25397\,c,$$

$$F_s=-0.97017,\quad T_1/T_2=1.$$

(E)

$$\xi_1=-\xi_2=\log_e 6,$$

$$E_1=-E_2=1.05714\,c,\quad R_1=R_2=0.34286\,c,$$

$$F_s=-0.95096,\quad T_1/T_2=1.$$

(F)

$$\xi_1=-\xi_2=\log_e 4,$$

$$E_1=-E_2=1.13333\,c,\quad R_1=R_2=0.53333\,c,$$

$$F_s=-0.89573,\quad T_1/T_2=1.$$

(G)

$$\xi_1=-\xi_2=\log_e 2,$$

$$E_1 = -E_2 = 1.66667\,c, \quad R_1 = R_2 = 1.33333\,c,$$

$$F_s = -0.73836, \quad T_1/T_2 = 1.$$

Minus sign in F_s means that the torque $T_s = T_1 + T_2$ acts in the sense to oppose angular acceleration $d\omega/dt$ of rotation of the vessel. It will be seen that the values of coefficient F_s for the case of (A), (B), and (C), are only slightly larger than unity, whereas in the cases of (D) to (G), the values of F_s are less than unity, and varies in a considerable range.

6. Case of two Concentric Circular Walls

The case of two concentric circular cylindrical walls, which are rotating about an axis, can be deduced from the analytical results about two eccentric circular walls, as given above, by taking the limiting case. But it can easily be obtained by direct analysis, as follows:.

Let the axis of rotation be O_1, and let center of two concentric circles be O, being situated at a distance d from O_1, taken along the real axis. Moreover, we assume that radii of two concentric circles are R_1 and R_2, as shown in Fig. 5.

Using polar coordinates (r, θ) with O as origin, the velocity potential ϕ may be given by

$$\phi = \sum_{n=1}^{\infty} [A_n \sin n\theta + B_n \cos n\theta] r^n + \sum_{n=1}^{\infty} [C_n \sin n\theta + D_n \cos n\theta] r^{-n}$$

where A_n, B_n, C_n and D_n are arbitrary constants. As we see from Fig. 5, we have

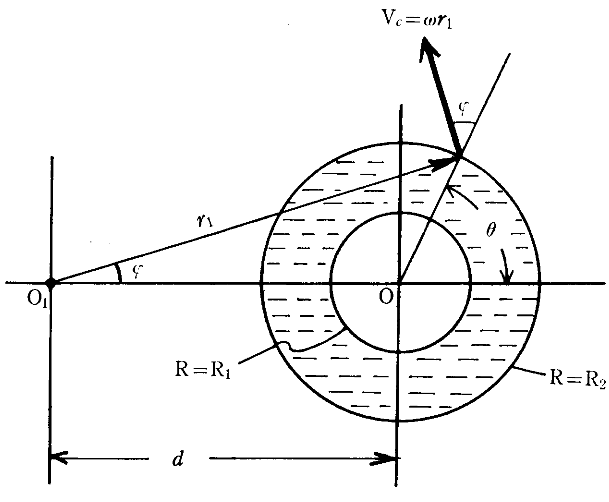


Fig. 5. Case of two Concentric Circular Walls.

$$r_1 \cos \varphi = x = d + r \cos \theta,$$

$$r_1 \sin \varphi = y = r \sin \theta.$$

Hence the normal component V_n of linear rotational velocity $V_c = \omega r_1$, is given by

$$V_n = \omega r_1 \cos \varphi_1 = \omega r_1 \sin (\theta - \varphi) = \omega d \cos \theta.$$

So that the boundary conditions at two walls $r = r_i$ $(i = 1, 2)$ become

$$\omega d \sin \theta = \partial \phi / \partial r \qquad (r = R_i)$$

from which we deduce that

$$A_1 = \omega d,$$

other constants being equal to zero. The driving torque T_i is given by

$$T_i = (-)^{k+1} \pi \rho d^2 R_i^2 \left(\frac{d\omega}{dt} \right)$$

for $i = 1$ and 2. Hence the total torque T_s becomes

$$T_s = T_1 + T_2 = -\pi \rho d^2 \left(\frac{d\omega}{dt} \right) [R_2^2 - R_1^2].$$

The fictituous mass M_s, as used above becomes, in the present instance

$$M_s = \rho \pi d^2 (R_2^2 - R_1^2).$$

So that the factor F_s is given by

$$F_s = T_s / M_s = -1,$$

that is, the value of the factor F_s is always equal to (-1).

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