慶應義塾大学学術情報リポジトリ
Keio Associated Repository of Academic resouces

Title	On the weak conditions for mean ergodic theorems
Sub Title	
Author	Anzai, Kazuo
Publisher	慶応義塾大学工学部
Publication year	1977
Jtitle	Keio engineering reports Vol.30, No.9 (1977. 7) ,p.85- 92
JaLC DOI	
Abstract	In this paper, we shall deal with the weak conditions for a mean ergodic theorems. R. SINE (1970) has shown that ■■■ converges in the strong operator topology iff the set of fixed points of T separates the fixed points of the adjoint operator T*, T being a contraction operator on any Banach space. In section 4 we prove a generalization in which Vn(T) = ■■■ satisfying the condition (E1) replaces ||T||≦1. In section 3 it will be proved that Theorem 1 (ANZAI 1977) is still valid in the normed linear space if we replace the condition (E) by the condition (E1).
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00300009-0085

慶應義塾大学学術情報リポジトリ(KOARA)に掲載されているコンテンツの著作権は、それぞれの著作者、学会または出版社/発行者に帰属し、その権利は著作権法によって保護されています。引用にあたっては、著作権法を遵守してご利用ください。

The copyrights of content available on the KeiO Associated Repository of Academic resources (KOARA) belong to the respective authors, academic societies, or publishers/issuers, and these rights are protected by the Japanese Copyright Act. When quoting the content, please follow the Japanese copyright act.

KEIO ENGINEERING REPORTS
VOL. 30, NO. 9, pp. 85-92, 1977

ON THE WEAK CONDITIONS FOR MEAN ERGODIC THEOREMS

KAZUO ANZAI

Dept. of Mathematics Keio University, Yokohama 223, Japan

(Received Mar. 3, 1977)

ABSTRACT

In this paper, we shall deal with the weak conditions for a mean ergodic theorems. R. SINE (1970) has shown that $\frac{1}{n}\sum_{i=0}^{n-1} T^i$ converges in the strong operator topology iff the set of fixed points of T separates the fixed points of the adjoint operator T^*, T being a contraction operator on any Banach space. In section 4 we prove a generalization in which $V_n(T)=\sum_{i=1}^{\infty} a_{ni}T^i$ satisfying the condition (E_1) replaces $||T||\leqq 1$. In section 3 it will be proved that Theorem 1 (ANZAI 1977) is still valid in the normed linear space if we replace the condition (E) by the condition (E_1).

1. Preliminaries

Let X be a locally convex space and T a continuous linear mapping of X to X. Let $V_n(T)=\sum_{i=0}^{\infty} a_{ni}T^i$, where (a_{ni}) is a matrix that $V_n(T)$ is well defined as a continuous linear mapping of X to X. Then we call $V_n(T)$ satisfying the condition (E_1) if

(i) $\lim_{n\to\infty} \sum_{i=0}^{\infty} a_{ni}=1$,

(ii) $\lim_{n\to\infty} (I-T)V_n(T)x=0$ for $x\in X$,

(iii) $\{V_n(T): n\geqq 1\}$ is equi-continuous.

A mapping T is said a satisfying the condition (P) if, for every $x\in X$, there exist a sequence $V_n(T)=\sum_{i=0}^{\infty} a_{ni}T^i$ satisfying the condition (E_1) and an $x_0 \in X$ depending on x such that a subsequence of $V_n(T)x$ converges weakly to x_0.

We recall that a matrix (a_{ni}) is said satisfying the condition (E) if the matrix satisfies the following properties

(i) $\lim_{n\to\infty} a_{ni}=0$ $(i=0, 1, 2\cdots)$,

(ii) $\lim_{n\to\infty} \sum_{i=0}^{\infty} a_{ni}=1$,

(iii) $\sum_{i=0}^{\infty} |a_{ni}| \leqq k$ $(n=1, 2, 3, \cdots)$,

(iv) $\lim_{k\to\infty} \sum_{i=k}^{\infty} |a_{ni+1}-a_{ni}|=0$ uniformly in n.

Remark 1. If T is a linear mapping on a complete locally convex space X such that the family of mappings $\{T^n : n \geqq 1\}$ is equi-continuous, and let a matrix (a_{ni}) satisfy the condition (E), then $V_n(T)=\sum_{i=0}^{\infty} a_{ni}T^i$ satisfies the condition (E_1).

The following result is well known.

Theorem A. Let T be a linear mapping on a locally convex space X and let $V_n(T)=\sum_{i=0}^{\infty} a_{ni}T^i$ satisfy the condition (E_1). If, for given $x \in X$, there exists a subsequence $V_{n'}(T)x$ of $V_n(T)x$ which converges weakly to an $x_0 \in X$, then the sequence $V_n(T)\ x$ converges to $x_0 \in X$ and $Tx_0=x_0$.

Let T be a continuous linear mapping on locally convex space X and let T^* the adjoint mapping of T. Throughout this paper we denote by F_T and I_T the set of fixed points of mappings T and T^* respectively. $\frac{1}{n}\sum_{i=1}^{n-1} T^i$ is denoted by $M_n(T)$.

2. Main Theorem

Lemma. Let X be a normed space and let T a bounded linear operator on X such that there exists a sequence $V_n(T)=\sum_{i=0}^{\infty} a_{ni}T^i$ satisfying the condion (E_1). Let ϕ be an element of X^*. Then there exist a subsequence $V_{n'}(T)^*\phi$ of $V_n(T)^*\phi$ and a $\phi_0 \in X^*$ such that $V_{n'}(T)^*\phi$ converges to ϕ_0 in w^*-topology and $\phi_0 \in I_T$. If a subsequence $V_{n''}(T)^*\phi$ of $V_n(T)^*\phi$ converges to a $\phi_0 \in X^*$ in w^*-topology then ϕ_0 is a fixed point of T^*.

Proof. Since the set $\{V_n(T) : n \geqq 1\}$ is bounded in uniform operator topology, the set $\{V_n(T)^*\phi : n \geqq 1\}$ is a bounded subset of X^*. Thus there exist a subsequence $V_{n'}(T)^*\phi$ of $V_n(T)^*\phi$ and a $\phi_0 \in X^*$ such that the subsequence $V_{n'}(T)^*\phi$ converges to ϕ_0 in w^*-topology. Then we have that, for every $x \in X$,

$$
\begin{aligned}
|\langle x, \phi_0 - T^*\phi_0\rangle| \leqq & |\langle x, \phi_0 - V_{n'}(T)^*\phi\rangle| \\
& + |\langle x, V_{n'}(T)^*\phi - T^* V_{n'}(T)^*\phi\rangle| + |\langle x, T^* V_{n'}(T)^*\phi - T^*\phi_0\rangle| \\
\leqq & |\langle x, \phi_0 - V_{n'}(T)^*\phi\rangle| + |\langle x, T^* V_{n'}(T)^*\phi - T^*\phi_0\rangle| \\
& + \|V_{n'}(T)(I-T)x\| \, \|\phi\|.
\end{aligned}
$$

By the condition $(E_1)-$(ii), ϕ_0 is a fixed point of the operator T^*. The lemma is proved.

Theorem 1. Let X be a normed linear space. If a operator T satisfies the condition (P), then, for every $U_n(T)=\sum_{i=0}^{\infty} b_{ni}T^i$ satisfying the condition (E_1) and every $x \in X$, there exists an $x_0 \in X$ depending on x such that $x_0 \in F_T$ and $x_0 = \lim_{n\to\infty} U_n(T)x$.

Proof. We shall show that, for every $\phi \in X^*$, there exists a $\phi_0 \in X^*$ such that the sequence $U_n(T)^*\phi$ converges to ϕ_0 in w^*-topology. By Lemma, there exist a subsequence $U_{n'}(T)^*\phi$ of $U_n(T)^*\phi$ and a $\phi_0 \in I_T$ such that the subsequence $U_{n'}(T)^*\phi$ converges to ϕ_0 in w^*-topology. If there exist a $\psi_0 \in X^*$ and the other subsequence $U_{n''}(T)^*\phi$ such that $U_{n''}(T)^*\phi$ converges to ψ_0 in w^*-topology, by Lemma, ψ_0 is a fixed point of T^*. For given any $y \in X$, by the hypothesis and Theorem A, there exist a $y_0 \in X$ and a sequence $V_n(T)=\sum_{i=0}^{\infty} a_{ni}T^i$ depending on y such that $V_n(T)$ satisfies the condition (E_1) and

$$y_0 = \lim_{n\to\infty} V_n(T)y, \qquad y_0 \in F_T.$$

According to the condition $(E_1)-$(i), we obtain

$$
\begin{aligned}
\langle y_0, \phi_0\rangle &= \lim_{n'\to\infty} \langle y_0, U_{n'}(T)^*\phi\rangle \\
&= \lim_{n'\to\infty} \langle U_{n'}(T)y_0, \phi\rangle = \langle y_0, \phi\rangle.
\end{aligned}
$$

Similary, it follows that $\langle y_0, \psi_0\rangle = \langle y_0, \phi\rangle$, and so, $\langle y_0, \phi_0\rangle = \langle y_0, \psi_0\rangle$. Since $\langle y_0, \phi_0\rangle = \langle y, \phi_0\rangle$ and $\langle y_0, \psi_0\rangle = \langle y, \psi_0\rangle$, we have

$$\langle y, \phi_0\rangle = \langle y, \psi_0\rangle.$$

Hence, the sequence $U_n(T)^*\phi$ converges to ϕ_0 in w^*-topology. Also, by hypothesis, there exist an $x_0 \in X$ and a sequence $W_n(T)=\sum_{i=0}^{\infty} c_{ni}T^i$ depending on x such that $W_n(T)$ satisfies the condition (E_1) and $x_0 = w-\lim_{n\to\infty} W_n(T)x$. Then we can take an element η_0 of X^* satisfying that $\eta_0 = w^*-\lim_{n\to\infty} W_n(T)^*\phi$. Also we have

$$\langle x_0, \phi_0\rangle = \lim_{n\to\infty} \langle W_n(T)x, \phi_0\rangle = \langle x, \phi_0\rangle,$$

and

$$\langle x_0, \phi_0\rangle = \lim_{n\to\infty} \langle x_0, U_n(T)^*\phi\rangle = \langle x_0, \phi\rangle.$$

Thus it follows that $\langle x_0, \phi\rangle = \langle x, \phi_0\rangle$.

On the other hand, we have

$$\langle x, \eta_0\rangle = \lim_{n\to\infty} \langle x, W_n(T)^*\phi\rangle$$
$$= \lim_{n\to\infty} \langle W_n(T)x, \phi\rangle = \langle x_0, \phi\rangle.$$

and so

$$\langle x_0, \phi\rangle = \lim_{n\to\infty} \langle W_n(T)x, \phi\rangle$$
$$= \langle x, \eta_0\rangle = \langle x, \phi_0\rangle$$
$$= \lim_{n\to\infty} \langle U_n(T)x, \phi\rangle.$$

This shows that

$$x_0 = w - \lim_{n\to\infty} W_n(T)x = w - \lim_{n\to\infty} U_n(T)x.$$

Theorem A implies that

$$x_0 = \lim_{n\to\infty} U_n(T)x \quad \text{and} \quad x_0 \in F_T.$$

Thus we get the theorem.

As a consequence of Theorem 1 and Theorem 2 (Anzai 1977), we have following.

Corollary 1. Let X be a normed linear space. Consider a finite number of commuting bounded linear operators $T_j (1 \leqq j \leqq J)$ on X such that

(i) T_j satisfies the condition (P) $(1 \leqq j \leqq J)$,

(ii) for every $T_j (1 \leqq j \leqq J)$, $\|T_j^n\| \leqq K$ $(n \geqq 1)$.

Let T be a convex combination $\sum_{j=1}^{J} \alpha_j T_j$ of linear operators T_j $(1 \leqq j \leqq J)$, where $0 < \alpha_j < 1$ $(1 \leqq j \leqq J)$ and $\sum_{j=1}^{J} \alpha_j = 1$. Then, for every $x \in X$ and $\phi \in X^*$, there exist an $x_0 \in X$ and a $\phi_0 \in X^*$ respectively such that

(1) $x_0 = \lim_{n\to\infty} M_n(T)x = \lim_{n\to\infty} M_n(T_1) \cdots M_n(T_J)x$,

(2) $x_0 \in F_T = \bigcap_{j=1}^{J} F_{T_j}$,

(3) $\phi_0 = w^* - \lim_{n\to\infty} M_n(T)^*\phi$
$= w^* - \lim_{n\to\infty} M_n(T_1)^* \cdots M_n(T_J)^*\phi$,

(4) $\phi_0 \in I_T = \bigcap_{j=1}^{J} I_{T_j}$.

Let S be a compact Hausdorff space and $C_{\boldsymbol{R}}(S)$ a Banach space of real valued continuous functions on S. Then a Markov operator T is a positive ($Tf \geqq 0$ whenever $f \geqq 0$) linear with $T1 = 1$. We call any Markov operator T uniformly mean stable (u.m.s.) if the sequence $M_n(T)f$ converges uniformly for every f in $C_{\boldsymbol{R}}(S)$. Also we recall that a measure μ on S is said invariant measure with respect to T if

$$\int Tf d\mu = \int f d\mu \quad \text{for all} \quad f \in C_{\mathbf{R}}(S).$$

Then we have that μ is an element of I_T if and only if μ is invariant measure.

From Corollary, we have the following remark [c.f. (SINE 1975)].

Remark 2. Let $\{T_j : 1 \leq j \leq J\}$ be a set of commuting Markov operators on $C_{\mathbf{R}}(S)$ whose satisfy the condition (P), and let $T = \sum_{j=1}^{J} \alpha_j T_j$ a convex combination of $T_j (1 \leq j \leq J)$, where $0 < \alpha_j < 1$ and $\sum_{j=1}^{J} \alpha_j = 1$. Then the set of invariant measures with respect to T coincides to the intersection of the sets of invariant measures with respect to $T_j (1 \leq j \leq J)$ and T is u.m.s.

In the case of the noncommuting operators, we have the following result; essentially the same idea has been used by M. EDELSTEIN (1966).

Remark 3. Suppose X is a uniformly convex Banach space, and $T_j (1 \leq j \leq J)$ are contraction linear operators on X whose are not necessary commuting. Let T be a convex combination $\sum_{j=1}^{J} \alpha_j T_j$ of linear operators $T_j (1 \leq j \leq J)$, where $0 < \alpha_j < 1$ and $\sum_{j=1}^{J} \alpha_j = 1$. Then it is easy to show that, for every $x \in X$, there exists an $x_0 \in X$ such that

$$x_0 = \lim_{n \to \infty} M_n(T) x \quad \text{and} \quad x_0 \in F_T = \bigcap_{j=1}^{J} F_{T_j}.$$

3. Equivalence

Theorem 2. Let X be a Banach space and T a bounded linear operator on X satisfying that there exists a sequence $V_n(T) = \sum_{i=0}^{\infty} a_{ni} T^i$ satisfies the condition (E_1). Then the following properties are equivalent.

(1) T satisfies the condition (P).

(2) If $U_n(T) = \sum_{i=0}^{\infty} b_{ni} T^i$ satisfies the condition (E_1), then, for any $x \in X$, there exists an element $x_0 \in X$ depending on x such that $x_0 \in F_T$ and

$$x_0 = \lim_{n \to \infty} V_n(T) x = \lim_{n \to \infty} U_n(T) x.$$

(3) F_T separates the points of I_T.

Proof. By Theorem 1, it follows that the properties (1) and (2) are equivalent.
(2)$\to$(3) By the proof of Theorem 1, we obtain that F_T separates the points of I_T.

(3)→(1) Let $x \in X$ be given. By assumption, Lemma implies that, for every $\phi \in X^*$, $V_n(T)^*\phi$ converges to a $\phi_0 \in I_T$ in w^*-topology. Thus we can define the function h_0 on X^* depending on x as follow

$$h_0(\phi)=\lim_{n\to\infty}\langle x, V_n(T)^*\phi\rangle \qquad \text{for} \quad \phi \in X^*.$$

Then, it is evident that h_0 is a linear function. We shall show that h_0 is continuous in the w^*-topology.

If a net $\{\phi^\alpha\}_{\alpha\in\Lambda}$ converges to ϕ in w^*-topology, then, by Banach-Steinhause's theorem, the set $\{\phi^\alpha : \alpha \in \Lambda\}$ is bounded. Also, by assumption and Lemma, there exist $\phi_0^\alpha(\alpha\in\Lambda)$ and ϕ_0 depending on $\phi^\alpha(\alpha\in\Lambda)$ and ϕ respectively such that $\phi_0^\alpha(\alpha\in\Lambda)$, $\phi_0 \in I_T$ and

$$\phi_0^\alpha = w^*-\lim_{n\to\infty} V_n(T)^*\phi^\alpha(\alpha\in\Lambda), \qquad \phi_0 = w^*-\lim_{n\to\infty} V_n(T)^*\phi.$$

Thus it follows that $h_0(\phi^\alpha)=h_0(\phi_0^\alpha)$ and $h_0(\phi)=h_0(\phi_0)$. Also, for any $y\in F_T$, the net $\{\langle y, \phi_0^\alpha\rangle\}_{\alpha\in\Lambda}$ converges to $\langle y, \phi_0\rangle$ because the net $\{\langle y, \phi^\alpha\rangle\}_{\alpha\in\Lambda}$ converges to $\langle y, \phi\rangle$ and $\langle y, \phi^\alpha\rangle = \langle y, \phi_0^\alpha\rangle(\alpha\in\Lambda)$ and $\langle y, \phi\rangle = \langle y, \phi_0\rangle$. Since F_T separates the points of I_T, by Banach-Steinhause's theorem and Alaoglu's theorem, a net $\{\phi^\alpha\}_{\alpha\in\Lambda}$ converges to ϕ in w^*-topology if and only if, for every $z\in F_T$, the net $\{\langle z, \phi^\alpha\rangle\}_{\alpha\in\Lambda}$ converges to $\langle z, \phi\rangle$. Thus the net $\{\langle x, \phi_0^\alpha\rangle\}_{\alpha\in\Lambda}$ converges to $\langle x, \phi_0\rangle$. Therefore, h_0 is continuous in w^*-topology. Hence, Banach's theorem implies that there exists an $x_0\in X$ such that $\langle x_0, \phi\rangle = h_0(\phi)$. This completes the proof.

Then we have following. The equivalence of the following properties (2) and (3) is a generalization of the (SINE 1975).

Corollary 2. Let X be a Banach space and let T a bounded linear operator on X with $\|T^n\| \leqq K$ $(n\geqq 1)$. Then the following properties are equivalent.

(1) T satisfies the condition (P).

(2) Let $U_n(T)=\sum_{i=1}^{\infty} b_{ni}T^i$ satisfy the condition (E_1). Then, for any $x\in X$, there exists an $x_0\in X$ depending on x such that $x_0\in F_T$ and $x_0=\lim_{n\to\infty} M_n(T)x = \lim_{n\to\infty} U_n(T)x$.

(3) F_T separates the points of I_T.

4. Function Space

Let S be a compact Hausdorff space and let M a subspace of $C(S)$, where $C(S)$ is a Banach space of complex valued continuous functions on S. Then M is called a function space if

(1) M separates the points of S,

(2) M contains the constants.

We denote by $\partial_M S$ the Choquet boundary (PHELPS 1966).

Theorem 4. Let M be a function space on a compact Hausdorff space S and let T a bounded linear operator on M. If, for every $f\in M$, there exist a sequence $V_n(T)=\sum_{i=0}^{\infty} a_{ni}T^i$ satisfying the condition (E_1) and $f_0\in M$ such that a subsequence of $V_n(T)f(x)$ converges to $f_0(x)$ for all $x\in\partial_M S$, then, $f_0\in F_T$ and, for every $U_n(T)=\sum_{i=1}^{\infty} b_{ni}T^i$ satisfying the condition (E_1),

$$f_0=\lim_{n\to\infty} V_n(T)f=\lim_{n\to\infty} U_n(T)f.$$

Proof. By Theorem A and Theorem 1, we have only to show that there exists a subsequence $V_{n'}(T)f$ of $V_n(T)f$ such that

$$\langle f_0, \phi\rangle=\lim_{n'\to\infty}\langle V_{n'}(T)f, \phi\rangle \qquad \text{for} \quad \phi\in M^*.$$

By assumption, there exists a subsequence $V_{n'}(T)f$ of $V_n(T)f$ such that

$$f_0(x)=\lim_{n'\to\infty} V_{n'}(T)f(x) \qquad \text{for} \quad x\in\partial_M S.$$

Let $\phi\in M^*$ be given. By Choquet-Bishop-Deleew's theorem, there exists a measure μ on S such that $\langle g, \phi\rangle=\int_S g d\mu$ for all $g\in M$ and $\mu(E)=0$ for any Baire set E in S which is distinct from $\partial_M S$. Therefore, by Lebesque dominated convergence theorem, we have

$$\lim_{n'\to\infty}\langle V_{n'}(T)f, \phi\rangle=\lim_{n'\to\infty}\int_S V_{n'}(T)f d\mu$$
$$=\int_S f_0 d\mu=\langle f_0, \phi\rangle.$$

Thus the proof is complete.

Corollary 3. Let X be a Banach space, and let T be a bounded linear operator on X. If, for every $x\in X$, we can take an $x_0\in X$ and a sequence $V_n(T)$ satisfying the condition (E_1) and satisfying that there exists a subsequence $V_{n'}(T)x$ of $V_n(T)x$ such that

$$\langle x_0, \phi\rangle=\lim_{n'\to\infty}\langle V_{n'}(T)x, \phi\rangle$$

whenever ϕ is an extreme point of the unit ball of X^*, then, $x_0\in F_T$ and, for every sequence $U_n(T)$ satisfying the condition (E_1),

$$x_0=\lim_{n\to\infty} V_n(T)x=\lim_{n\to\infty} U_n(T)x.$$

Proof. Let K be a unit ball of the dual space X^*. Then K is a compact Hausdorff space in the w^*-topology. We denote by $A(K)$ the space of complex valued continuous affine functions on K. Then $A(K)$ is a Banach space with supremum norm and $\partial_{A(K)}K=exK$, where exK is the set of extreme points of K. Let Φ denote the natural embedding of X into $A(K)$:

$$\Phi(x)(\phi)=\langle x, \phi\rangle \qquad \text{for} \quad \phi\in K.$$

Then Φ is a isometrically isomorphism of X to a subspace of $A(K)$. Also it is clear that

$$\|\Phi(x)\| = \sup_{\phi \in K} |\langle x, \phi \rangle| = \sup_{\phi \in exK} |\langle x, \phi \rangle|.$$

Hence it follows from Theorem 4 that

$$x_0 = w - \lim_{n \to \infty} V_{n'}(T)x.$$

This completes the proof.

Acknowledgement

The author would like to express their thanks to Prof. S. KOIZUMI and Prof. O. TAKENOUCHI for their helpful suggestions.

References

ANZAI, K. (1977): The mean convergence for ergodic theorems, *Proc. J. Academy*, **53**, 34-37.

ANZAI, K. (1977): On a mean ergodic theorem for several operators, *Keio Engineering Reports*, **30**, 59-68.

COHEN, L. W. (1940): On the mean ergodic theorem, *Ann. of Math.*, **41**, 505-509.

DUNFORD, N. and SCHWARTZ, J. (1957): Linear operator, Vol. 1, *Interscience*.

EBERLEIN, W. F. (1949): Abstract ergodic theorems and weak almost periodic functions, *Trans. Amer. Math.*, **67**, 217-240.

EDELSTEIN, M. (1966): A remark on a theorem of M. A. Krasnoselski, *Amer. Math. Monthly*, **73**, 509-510.

PHELPS, R. R. (1966): Lectures on Choquets theorem, *Van Nostrand*.

SINE, R. (1970): A mean ergodic theorem, *Proc. Amer. Math. Soc.*, **24**, 438-439.

SINE, R. (1975): Convex combination of uniformly mean stable Markov operators, *Proc. Amer. Math. Soc.*, **51**, 123-126.

YOSHIDA, K. (1938): Mean ergodic theorem in Banach Space, *Proc. Imp. Acad. Sci.*, **14**, 292-294.