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A NOTE ON WEAK CONSISTENCY OF SIMPLE LEAST SQUARES ESTIMATORS IN A POLYNOMIAL REGRESSION WITH A NONSTATIONARY ERROR PROCESS

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ABSTRACT

Sufficient conditions of weak consistency of Simple Least Squares Estimators for regression parameters and parameters contained in the modulation function in a polynomial regression with a nonstationary error process are given. These conditions are represented in terms of the covariance function of the stationary part and the modulation function of the error stochastic process.

1. Introduction

We often tend to explain the mean function by explanatory variables when the original time series can not be assumed to have mean zero. In this case, that is, when $Y(t)=m(t)+X(t)$, where $m(t)$ is a deterministic function and $X(t)$ is a weakly stationary stochastic process with mean zero, the original time series may be reduced to stationary form by subtracting the time dependent mean. U. GRENANDER and M. ROSENBLATT (1957) gave a necessary and sufficient condition of convergence in mean of the linear estimators to the regression parameters. And F. EICKER (1963) gave a necessary and sufficient condition of weak consistency of Simple Least Squares Estimators for regression parameters in the case where $X(t)$ is an independent random variable.

On the other hand, in treating the field data, we sometimes face on the case where we can not obtain the stationary time series even after removing the trending part.

In this paper, we shall deal with the situation where $m(t)$ is a polynomial of

t of given order and where the residual is a nonstationary stochastic process. We call this nonstationary process *uniformly modulated process*. (see M. B. Priestley (1965)). We seek for a sufficient condition of weak consistency of Simple Least Squares Estimators for the parameters contained in this model under a unified version. Under this model, first, we shall give a sufficient condition of weak consistency of Simple Least Squares Estimators for regression parameters. The condition obtained here can be considered as one of the robust conditions of weak consistency of Simple Least Squares Estimators in this model. Second, by assuming that the distribution of the stationary part of uniformly modulated process is Gaussian and by parametrizing the modulation function, we shall give a sufficient condition of weak consistency of Simple Least Squares Estimators for the modulation parameters given by the estimated residuals for the uniformly modulated process.

2. Weak consistency of Simple Least Squares Estimators for regression parameters

Let us consider a model

$$Y(t)=\sum_{j=0}^{p}\beta_j t^j+c(t)X(t), \tag{2. 1}$$

where the time parameter t is discrete, $Y(t)$ a real-valued stochastic process, $c(t)$ a deterministic function of t and $X(t)$ a weakly stationary stochastic process such that

(i) $EX(t)=0, \quad -\infty<t<\infty$

(ii) $EX(t)X(s)=R(t-s), \quad -\infty<t, \quad s<\infty.$

Then $X(t)$ itself has the integral representation

$$X(t)=\int_{-\pi}^{\pi} e^{it\lambda}dZ(\lambda), \tag{2. 2}$$

where $Z(\lambda)$ is a stochastic process with orthogonal increments.

M. B. Priestley (1965) and V. Mandrekar (1972) defined a class of nonstationary stochastic processes as

$$W(t)=\int_{-\pi}^{\pi} e^{it\lambda}c(t, \lambda)dZ(\lambda), \tag{2. 3}$$

where (i) $Z(\lambda)$ is a stochastic process with orthogonal increments, (ii) for each t, $\int_{-\pi}^{\pi}|c(t, \lambda)|^2dF(\lambda)<\infty$, where $F(\lambda)=E|Z(\lambda)|^2$. They called $W(t)$ *oscillatory processes*.

The error process in the model (2. 1) means $c(t, \lambda)=c(t)$ for all λ, which is a typical oscillatory process and is called *uniformly modulated process* by M. B. Priestley (1965).

After observing $Y(t)$ at $t=1, 2, \cdots, T$, we construct Simple Least Squares Estimators $\{\hat{\beta}_j;\ j=0, 1, \cdots, p\}$ for the regression coefficients $\{\beta_j;\ j=0, 1, \cdots, p\}$. For these estimators we can obtain the following.

Theorem 1. If $Y(t)$ satisfies the following two conditions

1. $\sum_{h=0}^{T}|R(h)|=O(T^{\delta})$ as $T\rightarrow\infty$,

2. $\sum_{t=1}^{T}c(t)^2=o(T^{2-\delta})$ as $T\rightarrow\infty$,

for all $0\leqq\delta\leqq 2$, then $\{\hat{\beta}_j,\ j=0, 1, \cdots, p\}$ are weakly consistent estimators.

Proof. The T observational equations of realizations of the model (2.1) can be written in matrix form

$$\boldsymbol{y}=X\boldsymbol{\beta}+\boldsymbol{u}, \tag{2.4}$$

where $\boldsymbol{y}=[y(1), y(2), \cdots, y(T)]'$ is a vector of T observations of $Y(t)$ at $t=1, 2, \cdots, T$, X design matrix whose (t, m) element is t^{m-1} for $t=1, 2, \cdots, T$ and $m=1, 2, \cdots, p+1$, $\boldsymbol{\beta}=[\beta_0, \beta_1, \cdots, \beta_p]'$ a vector of unknown parameters and $\boldsymbol{u}=[c(1)x(1), c(2)x(2), \cdots, c(T)x(T)]'$ a vector of T realizations of error stochastic process $c(t)X(t)$.

Since $\hat{\boldsymbol{\beta}}=[\hat{\beta}_0, \hat{\beta}_1, \cdots, \hat{\beta}_p]'$ is an unbiased estimator for $\boldsymbol{\beta}$, we need only to prove

$$\operatorname{cov}(\hat{\boldsymbol{\beta}})=E(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'=(X'X)^{-1}X'\textstyle\sum X(X'X)^{-1}\rightarrow 0 \quad\text{as}\quad T\rightarrow\infty,$$

where $\sum$ is the variance-covariance matrix of $\boldsymbol{u}$.

First, we shall evaluate the highest order in T of each element in the matrix $(X'X)^{-1}$. If we denote the (i, j) element of the matrix $X'X$ by $(X'X)_{ij}$, we have

$$(X'X)_{ij}=\sum_{t=1}^{T}t^{i+j-2}, \qquad i, j=1, 2, \cdots, p+1.$$

Let the matrix of the coefficients of the highest order in T for each element of $X'X$ be $\widetilde{X'X}$, then

$$(\widetilde{X'X})_{ij}=\frac{1}{i+j-1}, \qquad i, j=1, 2, \cdots, p+1. \tag{2.5}$$

By the nonsingurality of the matrix (2.5) (see R. T. Gregory and D. L. Karney (1969)), the highest order term in T of $\det X'X$ does not vanish and we can exactly evaluate the highest order in T of each element in the matrix

$$(X'X)^{-1}=\frac{\operatorname{adj} X'X}{\det X'X},$$

where $\operatorname{adj} X'X$ denotes the adjugate matrix of $X'X$. Let $(X'X)_{ij}$ be a_{ij} for $i, j=1, 2, \cdots, p+1$.

Since

$$\det X'X=\textstyle\sum \epsilon(\sigma)a_{1i_1}a_{2i_2}, \cdots, a_{p+1i_{p+1}}$$

where

$$a_{ij_i}=O(T^{j_i+(i-1)}) \quad \text{as} \quad T\rightarrow\infty,$$

then

$$\det X'X=O\left(T^{\sum_{i=1}^{p+1} j_i+\sum_{j=1}^{p} j}\right)=O(T^{(p+1)^2}) \text{ as } T\rightarrow\infty. \tag{2.6}$$

And the highest order in T of each element in $\operatorname{adj} X'X$ is

$$(\operatorname{adj} X'X)_{ij}=O(T^{(p+1)^2-(i+j-1)}) \quad \text{as} \quad T\rightarrow\infty$$

$$\text{for} \qquad i,j=1,2,\cdots,p+1. \tag{2.7}$$

From (2.6) and (2.7),

$$[(X'X)^{-1}]_{ij}=O(T^{-(i+j-1)}) \qquad \text{as} \qquad T\rightarrow\infty$$

$$\text{for} \qquad i,j=1,2,\cdots,p+1. \tag{2.8}$$

Second, we shall evaluate the highest order in T of each element in the matrix $X'\sum X$. By using SCHWARZ's inequality,

$$\begin{aligned}
|(X'\textstyle\sum X)_{ij}| &=|\sum_{t=1}^{T}\sum_{s=1}^{T} s^{i-1}c(s)t^{j-1}c(t)R(t-s)| \\
&=|R(0)\sum_{t=1}^{T} t^{i+j-2}c(t)^2+\sum_{\substack{h=-(T-1)\\(h\neq 0)}}^{T-1} R(h)\sum_{s=1}^{T-h} c(s)c(s+h)s^{i-1}(s+h)^{j-1}| \\
&\leqq T^{i+j-2}R(0)\sum_{t=1}^{T} c(t)^2+2T^{i+j-2}\sum_{h=1}^{T-1}|R(h)|\left(\sum_{s=1}^{T-h} c(s)^2\right)^{1/2}\left(\sum_{s=1}^{T-h} c(s+h)^2\right)^{1/2} \\
&\leqq T^{i+j-2}R(0)\sum_{t=1}^{T} c(t)^2+2T^{i+j-2}\sum_{h=1}^{T-1}|R(h)|\sum_{s=1}^{T} c(s)^2 \\
&=o(T^{i+j-\delta})+o(T^{i+j}) \qquad \text{as} \qquad T\rightarrow\infty \\
&=o(T^{i+j}) \qquad \text{as} \qquad T\rightarrow\infty.
\end{aligned} \tag{2.9}$$

Therefore, from (2.8) and (2.9), we have

$$[(X'X)^{-1}X'\textstyle\sum X(X'X)^{-1}]_{ij}=o(T^{-(i+j-2)}) \quad \text{as} \quad T\rightarrow\infty$$

$$\text{for} \qquad i,j=1,2,\cdots,p+1,$$

and this is equivalent to

$$E(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\rightarrow 0 \qquad \text{as} \qquad T\rightarrow\infty.$$

This shows the weak consistency of Simple Least Squares Estimator $\hat{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$.

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Remark. We usually analize the time series with $\delta=0$ in Condition 1 in Theorem 1. In this case, Condition 2 in Theorem 1 is reduced to $\sum_{t=1}^{T} c(t)^2=o(T^2)$ as $T \to \infty$ and this is satisfied, for example, if $c(t)=t^{1/2-\varepsilon}$ $(\varepsilon>0)$.

3. Weak consistency of Simple Least Squares Estimators for modulation function $c(t)$ by using estimated residuals

In this section, supposing the structure of modulation function $c(t)$, that is, parametrizing $c(t)$, we shall define the estimated residuals by making use of Simple Least Squares Estimator $\hat{\boldsymbol{\beta}}$ and estimate the structual parameters of $c(t)$ by the least squares principle by using the information of the above residuals.

First, we suppose that for each t, the distribution of $X(t)$ is Gaussian with mean 0, variance 1 and that as the structure of $c(t)$

$$c(t)=\exp\left(\sum_{i=1}^{q} c_i\varphi_i(t)\right), \tag{3.1}$$

where $\varphi_i(t)$ is a bounded function of t for each $i=1,2,\cdots,q$. Then the model (2.1) can be written as

$$Y(t)=\sum_{j=0}^{p} \beta_j t^j+\exp\left(\sum_{i=1}^{q} c_i\varphi_i(t)\right)X(t). \tag{3.2}$$

Having the samples of $Y(t)$ at $t=1,2,\cdots,T$, we define the estimated residuals $Z(t)$ by

$$Z(t)=Y(t)-\sum_{j=0}^{p} \hat{\beta}_j t^j, \tag{3.3}$$

where $\{\hat{\beta}_j;\ j=0,1,\cdots,p\}$ are Simple Least Squares Estimators defined in Section 2.

Next, we shall consider the following statistical linear model

$$\log|Z(t)|-E\log|X(t)|=\sum_{i=1}^{q} c_i\varphi_i(t)+\log|X(t)|-E\log|X(t)|, \tag{3.4}$$

$$t=1,2,\cdots,T$$

and construct Simple Least Squares Estimators $\{\hat{c}_i;\ i=1,2,\cdots,q\}$ for $\{c_i;\ i=1,2,\cdots,q\}$, which are obtained by minimizing

$$\sum_{t=1}^{T}\{\log|Z(t)|-E\log|X(t)|-\sum_{i=1}^{q} c_i\varphi_i(t)\}^2. \tag{3.5}$$

Theorem 2. Suppose that the following four conditions hold:

1. $\sum_{h=0}^{\infty}|R(h)|<\infty$

2. $\varphi_i(t)$ is a bounded function of t for $i=1, 2, \cdots, q$.

3. $\sum_{t=1}^{T} \varphi_i(t)^2 = O(T)$ as $T \to \infty$ for $i=1, 2, \cdots, q$.

4. For any $i, j=1, 2, \cdots, q$, the limit

$$\lim_{T\to\infty} \frac{\sum_{t=1}^{T} \varphi_i(t)\varphi_j(t)}{||\varphi_i(t)||_T \, ||\varphi_j(t)||_T}$$

exists and we shall put this limit as r_{ij} where $||\varphi_i(t)||_T = \left(\sum_{t=1}^{T} \varphi_i(t)^2\right)^{1/2}$. The matrix $\{r_{ij};\ i, j=1, 2, \cdots, q\}$ is nonsingular.

Then, Simple Least Squares Estimators $\{\hat{c}_i;\ i=1, 2, \cdots, q\}$ are weakly consistent estimators for the structual parameters $\{c_i;\ i=1, 2, \cdots, q\}$ of $c(t)$.

Proof. Let the design matrix obtained in the model (3.4) be X. The (t, j) element of X is given by

$$(X)_{tj} = \varphi_j(t), \qquad t=1, 2, \cdots, T;\ j=1, 2, \cdots, q.$$

Let Simple Least Squares Estimator of unknown parameter $\boldsymbol{c} = [c_1, c_2, \cdots, c_q]$ be $\hat{\boldsymbol{c}} = [\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_q]$. Since $\hat{\boldsymbol{c}}$ is unbiased, it is enough to show that

$$\operatorname{cov}(\hat{\boldsymbol{c}}) = E(\hat{\boldsymbol{c}} - \boldsymbol{c})(\hat{\boldsymbol{c}} - \boldsymbol{c})' = (X'X)^{-1} X' \sum X (X'X)^{-1} \to 0 \quad \text{as} \quad T \to \infty,$$

where $\sum$ is the variance-covariance matrix of $\log|X(t)| - E\log|X(t)|$, $t=1, 2, \cdots, T$.

First, we shall evaluate the highest order in T of each element of the matrix $(X'X)^{-1}$. From the definition of X,

$$(X'X)_{ij} = \sum_{t=1}^{T} \varphi_i(t)\varphi_j(t) \tag{3.6}$$

for $i, j=1, 2, \cdots, q$.
Let D_T be

$$D_T = \begin{bmatrix} ||\varphi_1||_T & & & 0 \\ & ||\varphi_2||_T & & \\ & & \ddots & \\ 0 & & & ||\varphi_q||_T \end{bmatrix}$$

then $X'X = D_T(D_T^{-1} X' X D_T^{-1}) D_T$.
The conditions 3 and 4 give

$$(D_T^{-1} X' X D_T^{-1})_{ij} = O(1) \qquad \text{as} \qquad T \to \infty$$

for $i, j=1, 2, \cdots, q$.
Then

$$\det X'X = \det D_T(D_T^{-1}X'XD_T^{-1})D_T$$
$$= (\det D_T)^2 \det (D_T^{-1}X'XD_T^{-1})$$
$$= K \prod_{i=1}^{q} ||\varphi_i||_T^2$$
$$= O(T^q) \qquad \text{as} \quad T \to \infty, \tag{3.7}$$

where $K = \det (D_T^{-1}X'XD_T^{-1})$.
Similarly,

$$\text{adj}\, X'X = \text{adj}\, D_T(D_T^{-1}X'XD_T^{-1})D_T$$
$$= O(T^{q-1}) \qquad \text{as} \quad T \to \infty. \tag{3.8}$$

Therefore from (3.7) and (3.8),

$$[(X'X)^{-1}]_{ij} = O(T^{-1}) \qquad \text{as} \quad T \to \infty \qquad \text{for } i, j = 1, 2, \cdots, q. \tag{3.9}$$

Second, we shall evaluate the highest order of the each element in T in the matrix $X' \sum X$. From the model (3.4),

$$(X' \sum X)_{ij} = \sum_{s=1}^{T} \sum_{t=1}^{T} \varphi_i(s)\varphi_j(t)E(\log |X(t)| - \alpha)(\log |X(s)| - \alpha)$$
$$= \sum_{s=1}^{T} \sum_{t=1}^{T} \varphi_i(s)\varphi_j(t)(E \log |X(t)| \log |X(s)| - \alpha^2) \qquad \text{for} \quad i, j = 1, 2, \cdots, q, \tag{3.10}$$

where $\alpha = E \log |X(t)|$. Let the covariance function of $X(t)$ and $X(s)$ be $\rho(t-s)$. We put $F(\rho(t-s)) = E \log |X(t)| \log |X(s)|$ and expand it in the neighbourhood of 0. Then,

$$F(\rho(t-s)) = F(0) + F'(\theta\rho(t-s))\rho(t-s)$$
$$= \alpha^2 + F'(\theta\rho(t-s))\rho(t-s) \qquad (0 < \theta < 1). \tag{3.11}$$

From (3.11), (3.10) is

$$(X' \sum X)_{ij} = \sum_{s=1}^{T} \sum_{t=1}^{T} \varphi_i(s)\varphi_j(t)F'(\theta\rho(t-s))\rho(t-s) \qquad 0 < \theta < 1 \qquad \text{for} \quad i, j = 1, 2, \cdots, q. \tag{3.12}$$

Condition 2 in Theorem 2 says

$$|(X' \sum X)_{ij}| \leqq M \sum_{s=1}^{T} \sum_{t=1}^{T} |F'(\theta\rho(t-s))\rho(t-s)|, \tag{3.13}$$

where M is an absolute constant.

Now we shall evaluate the highest order in T of

$$\sum_{s=1}^{T}\sum_{t=1}^{T}|F'(\theta\rho(t-s))\rho(t-s)|.$$

Let the joint probability density function of $x=X(t)$ and $y=X(s)$ be

$$f(x, y;\ \rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right\}. \tag{3.14}$$

Then

$$f'(x, y;\ \rho)=\frac{\partial f(x, y;\ \rho)}{\partial\rho}=\left\{\frac{\rho}{1-\rho^2}+\frac{(\rho x-y)(\rho y-x)}{(1-\rho^2)^2}\right\}f(x, y;\ \rho). \tag{3.15}$$

Since $E(\log|X(t)|)^2<\infty$, $E(X(t)\log|X(t)|)^2<\infty$ and $E(X(t)^2\log|X(t)|)^2<\infty$,

$$\begin{aligned} F'(\theta\rho)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\log|x|\log|y|\left\{\frac{\rho\theta}{1-\theta^2\rho^2}+\frac{(\theta\rho x-y)(\theta\rho y-x)}{(1-\theta^2\rho^2)^2}\right\}f(x, y;\ \theta\rho)dxdy \\ &=\frac{\rho\theta}{1-\theta^2\rho^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\log|x|\log|y|f(x, y;\ \theta\rho)dxdy \\ &\quad+\frac{1}{(1-\theta^2\rho^2)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\log|x|\log|y|(\theta\rho x-y)(\theta\rho y-x)f(x, y;\ \theta\rho)dxdy \\ &=I_1+I_2, \end{aligned} \tag{3.16}$$

has a definite value.

The order in T of

$$\sum_{s=1}^{T}\sum_{t=1}^{T}|F'(\theta\rho(t-s))\rho(t-s)|=\sum_{s=1}^{T}\sum_{t=1}^{T}|(I_1+I_2)\rho(t-s)|$$

depends on

$$\sum_{s=1}^{T}\sum_{t=1}^{T}|I_2\rho(t-s)|.$$

Since

$$I_2=\frac{1}{(1-\theta^2\rho^2)^{2+1/2}}\{(\theta^2\rho^2-1)N_1-2\rho\theta N_2\},$$

where

$$N_1=(1-\theta^2\rho^2)^{1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\log|x|\log|y|xyf(x, y;\ \theta\rho)dxdy$$

$$N_2=(1-\theta^2\rho^2)^{1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\log|x|\log|y|x^2f(x, y;\ \theta\rho)dxdy,$$

then the highest order in T of $\sum_{s=1}^{T}\sum_{t=1}^{T}|I_2\rho(t-s)|$ depends on

$$\sum_{s=1}^{T}\sum_{t=1}^{T}\frac{\rho(t-s)^2\theta}{(1-\theta^2\rho(t-s)^2)^{2+1/2}} \qquad 0<\theta<1. \tag{3.17}$$

Now,

$$\begin{aligned}
&\sum_{s=1}^{T}\sum_{t=1}^{T}\frac{\rho(t-s)^2\theta}{(1-\theta^2\rho(t-s)^2)^{2+1/2}}\\
&\leqq\sum_{s=1}^{T}\sum_{t=1}^{T}\frac{|\rho(t-s)|}{(1-\theta^2\rho(t-s)^2)^{2+1/2}}\\
&=\frac{T}{(1-\theta^2)^{2+1/2}}+2\sum_{h=1}^{T-1}\sum_{s=1}^{T-h}\frac{|\rho(h)|}{(1-\theta^2\rho(h)^2)^{2+1/2}}\\
&=\frac{T}{(1-\theta^2)^{2+1/2}}+2\sum_{h=1}^{T-1}(T-h)\frac{|\rho(h)|}{(1-\theta^2\rho(h)^2)^{2+1/2}}\\
&\leqq\frac{T}{(1-\theta^2)^{2+1/2}}+4T\sum_{h=1}^{T-1}\frac{|\rho(h)|}{(1-\theta^2\rho(h)^2)^{2+1/2}}
\end{aligned} \tag{3.18}$$

By using Condition 1 in Theorem 2, $\sum_{h=1}^{\infty}|\rho(h)|<\infty$ shows that

$$\sum_{h=1}^{\infty}\frac{|\rho(h)|}{(1-\theta^2\rho(h)^2)^{2+1/2}}<\infty.$$

Then the right hand side of (3.18) is $O(T)$ as $T\to\infty$.
Therefore

$$(X'\textstyle\sum X)_{ij}=O(T) \qquad \text{as} \qquad T\to\infty$$

$$\text{for} \qquad i,j=1,2,\cdots,q. \tag{3.19}$$

Finally, from (3.9) and (3.19), we obtain the result that each element of $\operatorname{cov}(\hat{\boldsymbol{c}})=(X'X)^{-1}X'\sum X(X'X)^{-1}$ is $O(T^{-1})$ as $T\to\infty$. This fact shows that $\hat{\boldsymbol{c}}$ is a weakly consistent estimator of $\boldsymbol{c}$.

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REFERENCES

ANDERSON, T. W. (1971): The Statistical Analysis of Time Series, John Wiley & Sons, Inc., New York.

DUBMAN, M. R. (1965): The Spectral Characterization and Comparison of Nonstationary Processes, Rocketdyne, Research Rep. 25–27.

EICKER, F. (1963): Asymptotic normality and consistency of the least squares estimators for families of linear regression, Ann. Math. Statist. **34**, 447–456.

GREGORY, R. T. and KARNEY, D. L. (1969): A Collection of Matrices for Testing Computational Algorithms, John Wiley & Sons, Inc., New York.

GRENANDER, U. and ROSENBLATT, M. (1957): Statistical Analysis of Stationary Time Series, John Wiley & Sons, Inc., New York.

MANDREKAR, V. (1972): A Characterization of Oscillatory Processes and their Prediction, Pro. Amer. Math. Soc. **32**, 280–284.

PRIESTLEY, M. B. (1965): Evolutionary Spectra and Non-stationary Processes, J. R. statist. Soc. B, **27**, 204–237.

ROZANOV, YU. A. (1967): Stationary Random Processes (English Transl.), Holden-Day, San Francisco.