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FIXED POINTS AND ITERATION OF A KANNAN'S MAPPING IN A BANACH SPACE

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ABSTRACT

Let T be a mapping from a closed convex subset D of a Banach space into a compact subset of D which satisfies

$$\|Tx-Ty\| \leqq \frac{1}{2}a(\|x-Tx\|+\|y-Ty\|)+b\|x-y\| \quad \text{for any } x,\ y \in D,$$

where $a \geqq 0$, $b \geqq 0$ and $a+b \leqq 1$, then the sequence $\{x_n\}$ defined by $x_{n+1}=1/2\ (x_n+Tx_n)$ converges to a fixed point of T for any $x_1 \in D$. As a matter of fact, a theorem which includes this result is proved.

1. Introduction

In [3], the author showed that, if T is a mapping from a closed convex subset D of a Banach space into a compact subset of D and nonexpansive i.e.

$$(1) \qquad \|Tx-Ty\| \leqq \|x-y\| \qquad \text{for any } x,\ y \in D,$$

then the sequence $\{F_{1/2}^n x\}_{n=1}^{\infty}$ converges to a fixed point of T for any $x \in D$, where $F_{1/2}$ is defined by $F_{1/2}x=\frac{1}{2}(x+Tx)$ for all $x \in D$. This result was proved for strictly convex spaces by EDELSTEIN (1966).

Recently, a mapping T from a subset D of a Banach space X into X which satisfies

$$(2) \qquad \|Tx-Ty\| \leqq \frac{1}{2}a(\|x-Tx\|+\|y-Ty\|)+b\|x-y\| \qquad \text{for any } x,\ y \in D,$$

where $a \geqq 0$, $b \geqq 0$ and $a+b \leqq 1$, has been introduced and studied by KANNAN (1969) and subsequently by SOARDI (1971), WONG (1974) and others. Note that condition (2) is more general than condition (1).

In this paper we consider some fixed points theorems and iteration methods for a mapping T from a closed subset D of a Banach space X into a compact subset of X which satisfies

$$(3) \qquad \|Tx-Ty\| < \frac{1}{2}a(\|x-Tx\|+\|y-Ty\|)+b\max\{\|x-y\|, \|x-Ty\|, \|y-Tx\|\}$$

for any $x, y \in D$, where $a>0$, $b \geqq 0$ and $a+b \leqq 1$.

2. Main results

Our main result is the following;

THEOREM 1. Let D be a closed subset of a Banach space X and let T be a mapping from D into a compact subset of X which satisfies condition (3). If there exists a number $t \in (0,1)$ such that $F_t D \subset D$, where F_t is defined by $F_t x=(1-t)x+tTx$ for all $x \in D$, then T has a unique fixed point u in D and the sequence $\{F_t^n x\}_{n=1}^{\infty}$ converges to u for any $x \in D$.

Theorem 1 will follow immediately as a corollary of Theorem 2.

It is clear that if D is convex and T is a self-mapping on D, then $F_t D \subset D$ for any $t \in (0, 1)$. Hence as an immediate consequence of theorem 1, we have the following corollary.

COROLLARY 1. Let D be a closed convex subset of a Banach space X and let T be a mapping from D into a compact subset of D which satisfies condition (3). Then T has a unique fixed point u in D and $\{F_t^n x\}_{n=1}^{\infty}$ converges to u for any $t \in (0, 1)$ and any $x \in D$.

COROLLARY 2. Let D be a closed convex subset of a Banach space and let T be a mapping from D into a compact subset of D which satisfies condition (2). Then T has a fixed point in D and $\{F_{1/2}^n x\}_{n=1}^{\infty}$ converges to a fixed point of T for any $x \in D$.

If $a=0$, this result was proved in [3]. Otherwise it is a special case of Corollary 1.

THEOREM 2. Let D be a closed subset in a Banach space X. Let T be a mapping from D into a compact subset of X which satisfies condition (3). If there exist $t \in (0, 1)$ and $x_1 \in D$ such that $x_n \in D$ for all positive integers n, where x_n is defined iteratively for each positive integer n by $x_{n+1}=(1-t)x_n+tTx_n$, then T has a unique fixed point u in D and the sequence $\{x_n\}_{n=1}^{\infty}$ converges to u.

The following lemma is used to prove Theorem 2.

LEMMA. (a) If $\|Tx_m-Tx_n\| \leqq (1+\varepsilon)R$ and $\|x_{m+1}-Tx_n\| \geqq (1-c\varepsilon)R$, then

$$\|x_m-Tx_n\| \geqq \left(1-\frac{c+t}{1-t}\varepsilon\right)R.$$

(b) If $\|x_m-Tx_n\| \leqq (1+\varepsilon)R$ and $\|x_{m+1}-Tx_n\| \geqq (1-c\varepsilon)R$, then

$$||Tx_m-Tx_n||\geqq(1-(c+1-t)t^{-1}\varepsilon)R.$$

Proof. Since $x_{m+1}-Tx_n=(1-t)(x_m-Tx_n)+t(Tx_m-Tx_n)$, we get that $||x_m-Tx_n|| \geqq(1-t)^{-1}(||x_{m+1}-Tx_n||-t||Tx_m-Tx_n||)\geqq(1-t)^{-1}(1-c\varepsilon-t(1+\varepsilon))R=(1-(c+t)(1-t)^{-1}\varepsilon)R.$
In the same way, the second part of Lemma immediately follows.

Proof of Theorem 2. Let D_0 denote the closure of $T(D)$ and let D_1, denote the closure of the convex hull of the union of D_0 and the point x_1. Then D_0 is compact and a well-known theorem of MAZUR implies that D_1 is also compact. $\{Tx_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ clearly belong to D_0 and D_1 respectively. Then $||x_n-Tx_n|| \leqq d(D_1)<\infty$ for each positive integer n, where $d(D_1)=\sup\{||x-y||\,;x,y\in D\}$, so $\limsup_{n\to\infty} ||x_n-Tx_n||$ exists. We now show that this limit must be zero.

Suppose $\limsup_{n\to\infty}||x_n-Tx_n||=R>0$. Since D_0 is compact, we can take an integer N such that there exists no set $\{y_i\in D_0\,;i=1,2,\cdots,N-1\}$ such that $||y_i-y_j||\geqq\left(\frac{1}{2}\right)R$ for all i and j such that $0\leqq i<j\leqq N-1$. Since we see from condition (3) that $1-t(1-b)\leqq1-ta<1$, it is clear that for any $\varepsilon>0$ there exist integers k_0 and k such that

(4) $(1-t+tb)^{k_0}\,d(D_1)\leqq2^{-1}\varepsilon R,$

(5) $||x_n-Tx_n||\leqq(1+2^{-1}\varepsilon)R$ for any integer n such that $n\geqq k_0,$

(6) $||x_{k+N}-Tx_{k+N}||\geqq(1-\varepsilon)R$ and $k\geqq2k_0.$

We have from (3) and (5) that for all integers m and n such that $m,n>k$,

(7)
$$\begin{aligned}||x_m-x_n||&=||(1-t)x_{m-1}+tTx_{m-1}-(1-t)x_{n-1}-tTx_{n-1}||\\&\leqq(1-t)||x_{m-1}+x_{n-1}||+t\Big[\frac{1}{2}a(||x_{m-1}-Tx_{m-1}||+||x_{n-1}-Tx_{n-1}||)\\&\quad+b\max\{||x_{m-1}-x_{n-1}||,\ ||x_{m-1}-Tx_{n-1}||,\ ||x_{n-1}Tx_{m-1}||\}\Big]\\&\leqq at(1+2^{-1}\varepsilon)R+(1-t+bt)\max\{||x_{m-1}-x_{n-1}||,\ ||x_{m-1}-Tx_{n-1}||,\ ||x_{n-1}-Tx_{m-1}||\}\end{aligned}$$

and

(8)
$$\begin{aligned}||x_m-Tx_n||&=||(1-t)x_{m-1}+tTx_{m-1}-Tx_n||\\&\leqq(1-t)||x_{m-1}-Tx_n||+t||Tx_{m-1}-Tx_n||\\&\leqq(1-t)||x_{m-1}-Tx_n||+t\Big[\frac{1}{2}a(||x_{m-1}-Tx_{m-1}||+||x_n-Tx_n||)\\&\quad+b\max\{||x_{m-1}-x_n||,\ ||x_{m-1}-Tx_n||,\ ||x_n-Tx_{m-1}||\}\Big]\\&\leqq at(1+2^{-1}\varepsilon)R+(1-t+bt)\max\{||x_{m-1}-x_n||,\ ||x_{m-1}-Tx_n||,\ ||x_n-Tx_{m-1}||\}\end{aligned}$$

Performing the calculation according to $(8)+(1-t+bt)\max\{(7)$ with $m-1$ for m, (8) with $m-1$ for m, (8) with n and $m-1$ for m and n respectively$\}$ side by side and eliminating common terms on both sides of the resulting inequality, we have, for any m and n such that $m,n\geqq k$,

$$||x_m - Tx_n|| \leqq at(1+2^{-1}\varepsilon)R(1+(1-t+bt)) + (1-t+bt)^2 \max \{||x_{m-2}-x_{n-1}||, ||x_{m-2}-Tx_{n-1}||, ||x_{n-1}-Tx_{m-2}||, ||x_{m-2}-x_n||, ||x_{m-2}-Tx_n||, ||x_n-Tx_{m-2}||, ||x_{m-1}-x_{n-1}||, ||x_{m-1}-Tx_{n-1}||, ||x_{n-1}-Tx_{m-1}||\},$$

Similarly using (7) and (8) repeatedly, we see, for any m and n such that $m, n \geq k$,

$$\begin{aligned} ||x_m - Tx_n|| &\leqq at(1+2^{-1}\varepsilon)R\{1+(1-t+bt)+(1-t+bt)^2+\cdots+(1-t+bt)^{k_0-1}\} \\ &\quad + (1-t+bt)^{k_0} \max \{||x_{m-i}-x_{n-j}||, ||x_{m-i}-Tx_{n-j}||, ||x_{n-i}-Tx_{m-j}||;\ i, j=1, 2, \cdots, k_0\} \\ &\leqq at(1+2^{-1}\varepsilon)R(1-(1-t+bt)^{k_0})(1-(1-t+bt))^{-1} + (1-t+bt)^{k_0} d(D_1) \\ &\leqq a(1-b)^{-1}(1+2^{-1}\varepsilon)R + (1-t+bt)^{k_0} d(D_1). \end{aligned}$$

Since $a(1-b)^{-1} \leqq 1$, we get from (4) that

(9) $||x_m - Tx_n|| \leqq (1+\varepsilon)R$ for all integers m and n such that $m, n \geqq k$.

In the same way, we can get

(10) $||x_m - x_n|| \leqq (1+\varepsilon)R$ for all integers m and n such that $m, n \geqq k$.

From (3), (5), (9) and (10), we see that for any $m, n \geqq k$,

(11)
$$\begin{aligned} &||Tx_m - Tx_n|| \\ &\leqq \frac{1}{2} a(||x_m - Tx_m|| + ||x_n - Tx_n||) + b \max \{||x_m - x_n||, ||x_m - Tx_n||, ||x_n - Tx_m||\} \\ &\leqq \frac{1}{2} a\{(1+\varepsilon)R + (1+\varepsilon)R\} + b(1+\varepsilon)R \\ &\leqq (1+\varepsilon)R. \end{aligned}$$

Lemma (a) implies from (11) with $k+N-1$ and $k+N$ for m and n respectively and (6) that

$$||x_{k+N-1} - Tx_{k+N}|| \geqq \left(1 - \frac{1+t}{1-t}\varepsilon\right)R,$$

from which and (11) with $k+N-2$ and $k+N$ for m and n respectively, we see by Lemma (a) that

$$||x_{k+N-2} - Tx_{k+N}|| \geqq \left(1 - \frac{2-(1-t)^2}{(1-t)^2}\varepsilon\right)R.$$

Using Lemma (a) and (11) repeatedly, we can obtain in a similar way that for any integer j such that $1 \leqq j \leqq N-1$,

$$||x_{k+j+1} - Tx_{k+N}|| \geqq \left(1 - \frac{2-(1-t)^{N-j-1}}{(1-t)^{N-j-1}}\varepsilon\right)R.$$

From this and (9) with $k+j$ and $k+N$ for m and n respectively, we see by Lemma

(b) that for any j such that $1 \leqq j \leqq N-1$,

$$||Tx_{k+j}-Tx_{k+N}|| \geqq \left\{1-t^{-1}\left(\frac{2-(1-t)^{N-j-1}}{(1-t)^{N-j-1}}+1-t\right)\varepsilon\right\}R \tag{12}$$

$$\geqq \left\{1-\left(\frac{2}{t(1-t)^{N}}-1\right)\varepsilon\right\}R.$$

It follows from (3), (9) and (10) that

$$||Tx_{k+j}-Tx_{k+N}|| \leqq \frac{1}{2}a(||x_{k+j}-Tx_{k+j}||+||x_{k+N}-Tx_{k+N}||)$$
$$+b\max\{||x_{k+j}-x_{k+N}||, ||x_{k+j}-Tx_{k+N}||, ||x_{k+N}-Tx_{k+j}||\}$$
$$\leqq \frac{1}{2}a||x_{k+j}-Tx_{k+j}||+\left(\frac{1}{2}a+b\right)(1+\varepsilon)R,$$

which implies from (12) that for any j such that $1 \leqq j \leqq N-1$,

$$||x_{k+j}-Tx_{k+j}|| \geqq 2a^{-1}\left\{||Tx_{k+j}-Tx_{k+N}||-\left(\frac{1}{2}a+b\right)(1+\varepsilon)R\right\}$$
$$\geqq 2a^{-1}\left\{\left(1-\frac{1}{2}a-b\right)-\left(\frac{2}{t(1-t)^{N}}-1+\frac{1}{2}a+b\right)\varepsilon\right\}R$$
$$\geqq \left\{1-\left(\frac{4}{at(1-t)^{N}}-1\right)\varepsilon\right\}R.$$

From this and (11) with $k+j-1$ and $k+j$ for m and n respectively, we get by Lemma (a) that

$$||x_{k+j-1}-Tx_{k+j}|| \geqq \left(1-\frac{c+t}{1-t}\varepsilon\right)R,$$

where $c=4\{at(1-t)^{N}\}^{-1}-1$. Using Lemma ($a$) and (11) repeatedly, we have in the same way that

$$||x_{k+i+1}-Tx_{k+j}|| \geqq \left\{1-\frac{c+1-(1-t)^{j-i-1}}{(1-t)^{j-i-1}}\varepsilon\right\}R$$

for any integer i such that $0 \leqq i < j$.

By Lemma (b), it follows from this and (9) with $k+i$ and $k+i$ for m and n respectively that

$$||Tx_{k+i}-Tx_{k+j}|| \geqq \left\{1-t^{-1}\left(\frac{c+1-(1-t)^{j-i-1}}{(1-t)^{j-i-1}}+1-t\right)\varepsilon\right\}R$$
$$=\left\{1-\left(\frac{c+1}{t(1-t)^{j-i-1}}-1\right)\varepsilon\right\}R$$
$$\geqq \left(1-\frac{4}{at^{2}(1-t)^{2N}}\right)R$$

for any integer i and j such that $0 \leqq i < j \leqq N-1$.

Setting $\varepsilon=8^{-1}at^{2}(1-t)^{2N}$, this implies that $||Tx_{k+i}-Tx_{k+j}|| \geqq 2^{-1}R$ for any integer

i and j such that $0 \leqq i < j \leqq N-1$. Since Tx_{k+j} is the point in D_0 for each nonnegative integer i, this is incompatible with the definition of N. Hence we obtain that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$.

Since $\{x_n\}_{n=1}^{\infty}$ is a sequence in the intersection of the compact set D_1 and the closed set D, there is a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ that converges to a certain point u of D. Let ε be any positive number. Then there is an n_{i_0} such that $||x_{n_{i_0}} - u|| < \varepsilon$ and $||x_{n_{i_0}} - Tx_{n_{i_0}}|| < \varepsilon$. Hence we have from (3) that

$$
\begin{aligned}
&||u - Tu|| \leqq ||u - x_{n_{i_0}}|| + ||x_{n_{i_0}} - Tx_{n_{i_0}}|| + ||Tx_{n_{i_0}} - Tu|| \\
&\leqq 2\varepsilon + \frac{1}{2} a(||u - Tu|| + ||x_{n_{i_0}} - Tx_{n_{i_0}}||) \\
&\quad + b \max\{||u - x_{n_{i_0}}||, ||u - Tx_{n_{i_0}}||, ||x_{n_{i_0}} - Tu||\} \\
&\leqq 2\varepsilon + \frac{1}{2} a(||u - Tu|| + ||x_{n_{i_0}} - Tx_{n_{i_0}}|| \\
&\quad + b \max\{||u - x_{n_{i_0}}||, ||u - x_{n_{i_0}}|| + ||x_{n_{i_0}} - Tx_{n_{i_0}}||, ||x_{n_{i_0}} - u|| + ||u - Tu||\} \\
&\leqq 2\varepsilon + \frac{1}{2} a(||u - Tu|| + \varepsilon) + b \max\{2\varepsilon, \varepsilon + ||u - Tu||\}.
\end{aligned}
$$

Since ε is arbitrary positive number, this implies that $||u - Tu|| \leqq \left(\frac{1}{2} a + b\right) ||u - Tu||$, which in turn implies from $\frac{1}{2} a + b < 1$ that u is a fixed point of T.

If $\limsup_{n\to\infty} ||x_n - u|| > 0$, then there is a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ that converges to u_1, such that $u_1 \neq u$. By the above argument, u_1, must be a fixed point of T. Since we see that $||u - v|| = ||Tu - Tv|| \leqq \frac{1}{2} a(||u - Tu|| + ||v - Tv||) + b \max\{||u - v||, ||u - Tv||, ||v - Tu||\} = b||u - v||$ for any v in the set of fixed points of T, it follows from $b < 1$ that T has a unique fixed point u, so $u = u_1$. This contradiction implies that $\{x_n\}_{n=1}^{\infty}$ converges to u, which is a unique fixed point of T. This completes the proof.

REFERENCES

[1] EDELSTEIN, M. (1966): A remark on a theorem of M.A. KRASNOSELSKI, Amer. Math. Monthly. vol. 73, 509-510.

[2] GOEBEL, K., KIRK, W.A., and SHIMI, T. N. (1973): Boll. Un. Mat. Ital. vol. 7, 67-75.

[3] ISHIKAWA, S. (1976): Fixed point and iteration of a nonexpansive mapping in a Banach space, Proc. Amer. Mach. Soc. vol. 58, 65-71.

[4] KANNAN, R. (1969): Some results on fixed points—II, Amer. Math. Monthly, vol. 76, 405-408.

[5] KRASNOSELSKI, M.A. (1955): Two remarks on the method of successive approximations, Uspehi, Math. Nauk. vol 10, 123-127.

[6] SOARDI, S. (1971): Su un problema di punto unito di S. Reich, Boll. Un. Mat. Ital. vol. 4, 481-845.

[7] WONG, C.S. (1974): Fixed pointa and characterizations of certain maps. Pacific J. Math. vol. 54, 305-312.