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ON MOTION OF AN IDEAL FLUID WHICH IS FILLED UP IN A ROTATING VESSEL

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ABSTRACT

Let us consider a closed vessel or chamber, which is rotating in prescribed manner, and inside which an ideal fluid is filled up. In this ideal fluid, flow will be set up, which is induced by motion of rotating vessel. In this report, the author has made some study about this flow, especially about effect of this fluid flow upon motion of vessel or chamber body. Thus, effect of fluid flow upon polar moment of inertia of this rotating vessel is studied. Next, taking up the case in which the vessel is made in form of fan-shaped closet, and considering the case of two-dimensional motion of ideal fluid, some detailed discussion is made, giving numerical examples. It is pointed out that effect of flow of the fluid can appear as phenomenon of virtual mass with regard to accelerated rotational motion of the vessel body.

1. General Consideration about Motion of an Ideal Fluid contained in a Rotating Chamber

1-1. Fundamental Equation

In order to study the motion of an ideal fluid, which is filled up in a rotating chamber or vessel, let us use method of moving coordinate axes, which is already given in text-books of hydrodynamics. Refferring to a system of rectangular coordinate axes $OXYZ$, which is moving, at time t, with translational velocity u_a, v_a, w_a and rotational (angular) velocity p_a, q_a, r_a, the fundamental equation of motion for an ideal (incompressible, non-viscous) fluid is given as follows;

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}-r_a v+q_a w+\frac{\partial u}{\partial x}\frac{Dx}{Dt}+\frac{\partial u}{\partial y}\frac{Dy}{Dt}+\frac{\partial u}{\partial z}\frac{Dz}{Dt}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}\\
&\frac{\partial v}{\partial t}-p_a w+r_a u+\frac{\partial v}{\partial x}\frac{Dx}{Dt}+\frac{\partial v}{\partial y}\frac{Dy}{Dt}+\frac{\partial v}{\partial z}\frac{Dz}{Dt}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}\\
&\frac{\partial w}{\partial t}-q_a u+p_a v+\frac{\partial w}{\partial x}\frac{Dx}{Dt}+\frac{\partial w}{\partial y}\frac{Dy}{Dt}+\frac{\partial w}{\partial z}\frac{Dz}{Dt}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned}\right\} \qquad (1)$$

In this system of equations, u, v, w denotes components of absolute velocity of flow of the fluid, p the pressure. These are considered to be functions of x, y, z and t. The position x, y, z of any point P in the fluid is taken to refer to instantaneous position of moving frame $OXYZ$. Values of Dx/Dt, Dy/Dt, Dz/Dt which is the rate of change of coordinates of a fluid particle, relative to the moving frame, are given by

$$\left.\begin{aligned}
\frac{Dx}{Dt}&=u-u_a+r_a y-q_a z\\
\frac{Dy}{Dt}&=v-v_a+p_a z-r_a x\\
\frac{Dz}{Dt}&=w-w_a+q_a x-p_a y
\end{aligned}\right\} \qquad (2)$$

The equation of continuity, for our case of incompressible, non-viscous fluid, is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (3)$$

In eq. (1), X, Y, Z are components of extraneous forces, which act upon mass of the fluid.

1-2. Case of Two-Dimensional Flow.

Now, we shall confine ourselves to simple case of two-dimensional motion, in which we have $w=0$ and u, v are functions of x and y (and t). Also, we shall put

$$X=\partial U/\partial x, \quad Y=\partial U/\partial y.$$

As to motion of our frame of axes $OXYZ$, we shall take up the simple case in which

$$u_a=0, \quad v_a=0, \quad w_a=0; \quad p_a=0, \quad q_a=0, \quad r_a=\omega$$

Thus we are considering the case in which our frame of axes $OXYZ$ is rotating about OZ axis, with angular velocity ω, which is a given function of t.

Under these circumstances, equations (1) and (3) become simplified into the following form;

$$\left.\begin{aligned}\frac{\partial u}{\partial t}-\omega v+(u+\omega y)\frac{\partial u}{\partial x}+(v-\omega x)\frac{\partial u}{\partial y}&=\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x},\\ \frac{\partial v}{\partial t}+\omega u+(u+\omega y)\frac{\partial v}{\partial x}+(v-\omega x)\frac{\partial v}{\partial y}&=\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y}\end{aligned}\right\} \tag{4}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{5}$$

The equation of continuity (5) is satisfied by putting

$$u=\frac{\partial \psi}{\partial y}, \qquad v=-\frac{\partial \psi}{\partial x} \tag{6}$$

where ψ, the stream function, is a function of x and y, and also of t.

Differentiating the first eq. (4) by y and the second by x, and taking differences of both sides, we obtain (after some rearrangements) following equation

$$\left[\frac{\partial}{\partial t}+(u+\omega y)\frac{\partial}{\partial x}+(v-\omega x)\frac{\partial}{\partial y}\right](\Delta\psi)=0 \tag{7}$$

where we write

$$\Delta\psi\equiv\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}$$

If we had at $t=0$, $\Delta\psi=0$, throughout the fluid region, then we shall have $\Delta\psi=0$ for subsequent time at which $0<t$. We shall take, as a special case to satisfy this eq. (7),

$$\Delta\psi=0 \qquad \text{or} \qquad \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0 \tag{8}$$

throughout the fluid region. This eq. (8) will be satisfied by putting

$$u=\frac{\partial \phi}{\partial x}, \qquad v=\frac{\partial \phi}{\partial y} \tag{9}$$

where ϕ is the velocity potential, being a function of x, y and t. Also ϕ must satisfy the Laplace equation

$$\Delta\phi=0 \tag{10}$$

Putting these values of (9) into previous eq. (4), we find that

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2+\left(\frac{\partial \phi}{\partial y}\right)^2\right]+\omega\left[y\frac{\partial \phi}{\partial x}-x\frac{\partial \phi}{\partial y}\right]+C=U-\frac{1}{\rho}p \tag{11}$$

where C is a constant with regard to variables x and y, but may be a function of t.

1-3. Case of Steady Flow

In the case in which the angular velocity ω of rotation of vessel body is constant with respect to time t, the value of velocity potential ϕ is independent of time t, and are to be found by solving following boundary value problem of partial differential equation;

(A) $\Delta\phi=0$ throughout inside space of the vessel, in which the fluid is filled up.

(B) At the boundary wall surface of the vessel, normal component of velocity of flow must be equal to normal component of velocity of rotation of wall itself.

Let us denote by ϕ_s the solution of this boundary value problem, for the case of $\omega=1$, ϕ_s being thus a function of x and y. Then the solution of our problem may be written as

$$\phi(x, y)=\omega\phi_s(x, y) \tag{12}$$

The fluid pressure p_c may also be written, after eq. (11),

$$-\frac{1}{\rho}p_c=\frac{\omega^2}{2}\left[\left(\frac{\partial\phi_s}{\partial x}\right)^2+\left(\frac{\partial\phi_s}{\partial y}\right)^2\right]+\omega^2\left[y\frac{\partial\phi_s}{\partial x}-x\frac{\partial\phi_s}{\partial y}\right]-U+C \tag{13}$$

Resultant force caused by action of this fluid pressure p_c normal to wall surface will be given by

$$F_c=\iint p_c\cos(\nu, \beta)dS \tag{14}$$

where ν is normal direction of surface element dS, and β is direction to which we take the resultant force F_c.

1-4. Case of Non-stationary Motion of the Vessel

Suppose that the vessel is rotating with angular velocity ω, which is a function of time t. In that case we may use eq. (12), wherein ω is taken to be the function of t. The fluid pressure p is no longer given by eq. (13). But instead, we may derive from the general equation (11) in following form;

$$-\frac{1}{\rho}p=\phi_s\frac{d\omega}{dt}+\frac{\omega^2}{2}\left[\left(\frac{\partial\phi_s}{\partial x}\right)^2+\left(\frac{\partial\phi_s}{\partial y}\right)^2\right]+\omega^2\left[y\frac{\partial\phi_s}{\partial x}-x\frac{\partial\phi_s}{\partial y}\right]-U+C \tag{15}$$

The resultant force in direction β, due to action of this fluid pressure upon the wall surface of the vessel, will be given by

$$F_\beta=\iint p\cos(\nu, \beta)dS \tag{16}$$

and it consists of two parts, namely

$$F_{\beta1}=-\rho\frac{d\omega}{dt}\iint\phi_s\cos(\nu, \beta)dS \tag{17}$$

$$F_{\beta 2}=-\rho\omega^2\iint\left[\frac{1}{2}\left(\frac{\partial\phi_s}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\phi_s}{\partial y}\right)^2+\left(y\frac{\partial\phi_s}{\partial x}-x\frac{\partial\phi_s}{\partial y}\right)\right]\cos(\nu,\beta)dS \tag{18}$$

(effect of U is omitted here)
In order to keep the motion of vessel body in state of angular acceleration $d\omega/dt$, it will be required to apply a force to counteract inertia force of amount

$$F_{\beta m}=-I_{\beta m}\cdot\frac{d\omega}{dt} \tag{19}$$

The total inertia force excerted by vessel body and fluid contained in it, will be

$$F_{\beta n}=-\left[I_{\beta m}+\rho\iint\phi_s\cos(\nu,\beta)dS\right]\frac{d\omega}{dt} \tag{20}$$

and thus the ratio

$$k=\frac{\rho}{I_{\beta m}}\iint\phi_s\cos(\nu,\beta)dS \tag{21}$$

will be coefficient of virtual mass, representing the effect of motion of the fluid.

Similar inference can also be made, regarding the moment about Z-axis of rotation of the vessel.

1-5. Case of two-dimensional Flow which is expressed in Polar Coordinates

If we use a system of polar coordinates (r, θ) instead of rectangular coordinates (x, y), we can obtain expressions as given below. The rate of change of coordinates of a fluid particle at $P(s, \theta)$, relative to the moving frame is given by

$$\frac{Dr}{Dt}=v_r, \qquad \frac{D\theta}{Dt}=\frac{v_\theta}{r}-\omega \tag{22}$$

where v_r, v_θ are components of absolute velocity of flow at $P(r, \theta)$. The equation of motion is given by

$$\left.\begin{aligned}\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+(v_\theta-\omega r)\frac{\partial v_r}{r\partial\theta}-\omega v_\theta-\frac{v_\theta^2}{r}&=-\frac{1}{\rho}\frac{\partial p}{\partial r}+X_r\\ \frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+(v_\theta-\omega r)\frac{\partial v_\theta}{r\partial\theta}+\omega v_r+\frac{v_r v_\theta}{r}&=-\frac{1}{\rho}\frac{\partial p}{r\partial\theta}+X_\theta\end{aligned}\right\} \tag{23}$$

the equation of continuity being

$$\frac{1}{r}\frac{\partial(v_r r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}=0 \tag{24}$$

Due to this eq. (24), we can express v_r and v_θ in terms of stream function, as follows

$$v_r = \frac{\partial \psi}{r\partial \theta}, \qquad v_\theta = -\frac{\partial \psi}{\partial r} \tag{25}$$

Putting these values of eq. (25) into eq. (23), and making an equation of form of

$$\frac{\partial}{\partial \theta}[\text{first eq}\cdot(23)] - \frac{\partial}{\partial r}[\text{second eq}\cdot(23)\times r] = 0$$

we have, after some rearrangement

$$\frac{\partial}{\partial t}(r\Delta\psi) - \omega r\frac{\partial}{\partial \theta}(\Delta\psi) + \frac{\partial \psi}{\partial \theta}\frac{\partial}{\partial r}(\Delta\psi) - \frac{\partial \psi}{\partial r}\frac{\partial}{\partial \theta}(\Delta\psi) = 0 \tag{26}$$

where we have put

$$\Delta\psi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \tag{27}$$

If we had $\Delta\psi=0$ throughout the fluid region at time $t=0$, then we shall have $\Delta\psi=0$ in subsequent time $0<t$. In that case we may write

$$v_r = \frac{\partial \phi}{\partial r}, \qquad v_\theta = \frac{\partial \phi}{r\partial \theta} \tag{28}$$

where ϕ is velocity potential of the fluid flow. Putting values of (28) into eq. (23), we obtain

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial r}\right)^2 + \left(\frac{\partial \phi}{r\partial \theta}\right)^2\right] - \omega\frac{\partial \phi}{\partial \theta} + \omega\psi + C(t) = -\frac{1}{\rho}p + U \tag{29}$$

where $C(t)$ is a constant which may be a function of t. Also, in the present instance, formulae (16), (17) may be used. Instead of eq. (18) we have now

$$F_{\beta 2} = -\rho\omega^2 \iint \left[\frac{1}{2}\left(\frac{\partial \phi_s}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi_s}{r\partial \theta} - r\right)^2 + \psi_s\right]\cos(\nu, \beta)dS \tag{18a}$$

And the same inference can be made with respect to concept of virtual mass caused by fluid flow, upon the force and moment in accelerated motion of body of the vessel.

2. Stationary Flow of an Ideal Fluid which is filled up in a Rotating Vessel of Fan-like Form

Concepts of this Chapter 2 has been distributed among post-graduate students, etc., in autographed copies more than twenty years ago. But was never published.

2-1. Expressions for the Steady Flow

Let us consider a fan-shaped region as sketched in Fig. 1. The wall of vessel is composed of two circular arcs of radii r_1 and r_2, the subtended angle being 2α. Other two walls are radial lines $r_2 \leq r \leq r_1$, forming thus a closed figure consisting of two circular arcs and two straight radial lines.

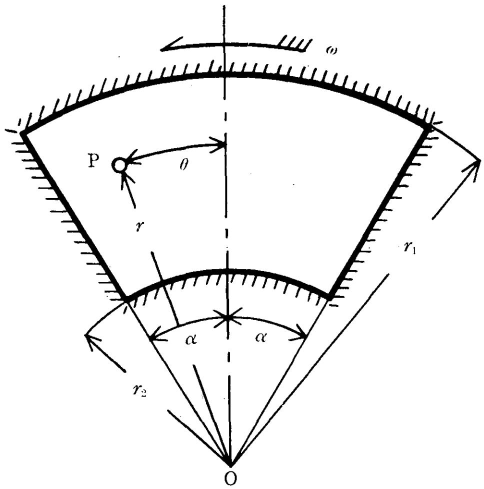

Fig. 1. Fan-shaped vessel, which is notating with angular velocity ω.

The depth of the vessel may be taken to be unity. It is assumed to be making rotational motion with angular velocity ω, about the center of circular arcs. The angular velocity ω is, at first, taken to be of constant value, but it may subsequently be taken to be a function of t.

The steady flow of an ideal fluid, which takes place inside this rotating vessel, can be expressed by means of velocity potential ϕ, as follows,

$$V_r = \frac{\partial\phi}{\partial r}, \qquad V_\theta = \frac{\partial\phi}{r\partial\theta} \tag{30}$$

Boundary conditions to be satisfied by ϕ are

(a) at $r=r_1$ and r_2

$$\frac{\partial\phi}{\partial r} = 0$$

(b) at $\theta = \pm\alpha$

$$\frac{\partial\phi}{\partial\theta} = \omega r^2$$

Also, this function ϕ must be so chosen that it satisfies the Laplace equation

$$\frac{\partial}{r\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{31}$$

For that purpose we take

$$\phi=\phi_1+\phi_2 \tag{32}$$

where we put

$$\phi_1=\frac{\omega}{2\cos 2\alpha}r^2\sin 2\theta \tag{33}$$

$$\phi_2=\sum_{s=1}^{\infty}\left[A_s r^{\lambda s}+\frac{B_s}{r^{\lambda s}}\right]\sin\lambda s\theta \tag{34}$$

According to the above boundary conditions we must have $\lambda=\pi/(2\alpha)$ and s must be odd integers. Also, we must have

$$\frac{\omega r_i}{\cos 2\alpha}b_s+\frac{\lambda_s}{r_i}\left[A_s r_i^{\lambda s}-\frac{B_s}{r_i^{\lambda s}}\right]=0 \quad \text{for} \quad r_i=r_1 \quad \text{and} \quad r_2. \tag{35}$$

In this eq. (35), constants b_s are to be so chosen such that

$$\sin 2\theta=\sum_{s=1}^{\infty}b_s\sin\lambda s\theta \tag{36}$$

s being odd integers 1, 3, 5,$\cdots$ Putting $x=[\pi/(2\alpha)]\theta$ into eq. (36) we have

$$\sin\left(\frac{4\alpha}{\pi}\right)=\sum_s b_s\sin sx \tag{37}$$

for $s=1, 3, 5.\cdots$

This equation (37) means that b_s must be Fourier coefficients for the function

$$f(x)=\sin[(4\alpha/\pi)x] \quad \text{for} \quad -\pi/2\leq x\leq+\pi/2.$$

Regarding eq. (37) as half-range sine series for the given function $f(x)$, as sketched in Fig. 2, we find

$$b_s=\cos 2\alpha\sin[(\pi/2)s]\frac{4\alpha}{4\alpha^2-[(\pi/2)s]^2} \tag{38}$$

Thus we have, by (35)

$$\left.\begin{aligned}
A_s&=\frac{4\omega\alpha}{4\alpha^2-[(\pi/2)s]^2}\frac{\sin[(\pi/2)s]}{\lambda s}\frac{R(\lambda s+2)}{R(2\lambda s)}\\
B_s&=\frac{-4\omega\alpha}{4\alpha^2-[(\pi/2)s]^2}\frac{\sin[(\pi/2)s]}{\lambda s}\frac{R(\lambda s-2)}{R(2\lambda s)}(r_1r_2)^{\lambda s+2}\\
R(\nu)&=r_1^{\nu}-r_2^{\nu}
\end{aligned}\right\} \tag{39}$$

which is established, so long as we have $O<\alpha<\pi/4$. Fig. 2. shows a case of any given function F(x), which can be expanded into a half range sine series similar to r, h, s, expression of eq. (37) Corresponding values of absolute velocity of flow are given as follows;

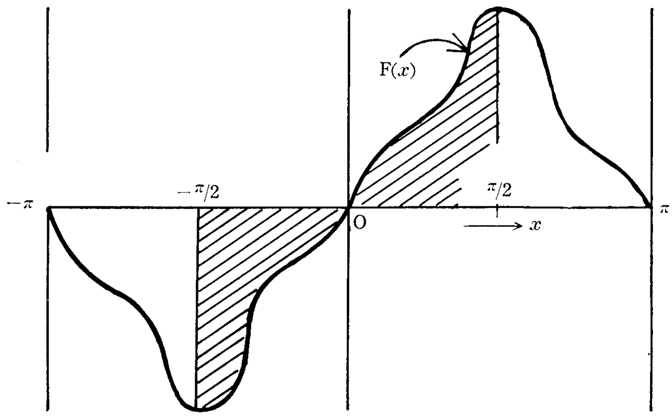

Fig. 2. Illustrating Half-range Sine Series

$$\left.\begin{aligned} V_r &= \frac{\omega r}{\cos 2\alpha}\sin 2\theta + \sum_{s=1}^{\infty}\frac{\lambda s}{r}\left(A_s r^{\lambda s} - \frac{B_s}{r^{\lambda s}}\right)\sin \lambda s\theta \\ V_\theta &= \frac{\omega r}{\cos 2\alpha}\cos 2\theta + \sum_{s=1}^{\infty}\frac{\lambda s}{r}\left(A_s r^{\lambda s} + \frac{B_s}{r^{\lambda s}}\right)\cos \lambda s\theta \end{aligned}\right\} \quad (40)$$

s being odd integers, and A_s, B_s being given by eq. (39).

2-2. Special case in which we have $2\alpha=\pi/2$

In expressions (33), (34) for ϕ_1, ϕ_2, the values become indeterminate if $2\alpha=\pi/2$. But their values can be deduced by putting $2\alpha=(\pi/2)(1-\varepsilon)$ and taking limiting value for $\varepsilon\to 0$. Thus we obtain

$$\begin{aligned} \phi=\phi_1+\phi_2 &= -\omega r_1^2 \frac{\eta^4}{(1-\eta^4)y^2}\log \eta\cdot\sin 2\theta \\ &+\omega r_1^2\frac{1}{12\pi}\left[\frac{H(8)}{H(12)}y^6-\frac{H(4)}{H(12)}\frac{\eta^8}{y^6}\right]\sin 6\theta \\ &-\omega r_1^2\frac{1}{60\pi}\left[\frac{H(12)}{H(20)}y^{10}-\frac{H(8)}{H(20)}\frac{\eta^{12}}{y^{10}}\right]\sin 10\theta+\cdots \end{aligned} \quad (41)$$

where we have put, for shortness

$$H(\nu)=1-\eta^\nu, \qquad \eta=r_2/r_1 \qquad y=r/r_1.$$

3. Effect of Virtual Mass of Fluid Flow, for the Case of Accelerated Rotational Motion of Fan-shaped Vessel

3-1. Formulae for F_1 and M_1

As was already remarked, solutions for flow, given in previous Chapter 2, may be used in the case of non-stationary motion, when we regard the angular velocity of rotation ω to be given function of time t. For that case, formulae (16) for values of force (or moment) caused by nonstationary flow of fluid can be applied. Let us now obtain actual values of this force (moment) for the case of fan-shaped vessel which we discussed in Chap. 3. Thus, using expressions (32) (33) and (34), we have (taking $\theta=\alpha$)

$$-\frac{1}{\rho}F=-\frac{1}{\rho}\int_{r_2}^{r_1}p\,dr=F_1\frac{d\omega}{dt}+\omega^2F_2 \tag{42}$$

where we put

$$F_1=\int_{r_2}^{r_1}\phi_s\,dr=\int_{r_2}^{r_1}[\phi_1+\phi_2](\omega=1)dr \tag{43}$$

As sketched in Fig. 3, the force F_2 has equal values but in opposite directions at two walls $\theta=+\alpha$ and $\theta=-\alpha$, producing no resultant force, as a whole body.

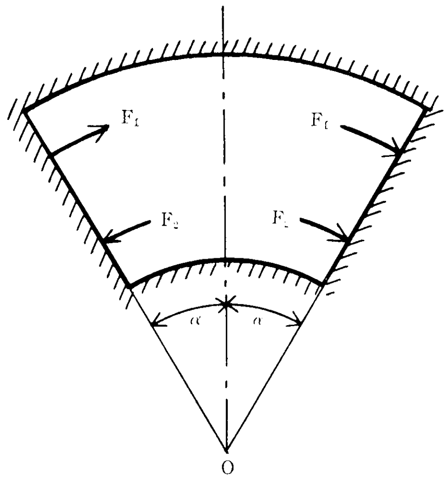

Fig. 3. Sense of Action of Forces F_1 and F_2

On the other hand, force F_1 acts in the same directions at two walls $\theta=+\alpha$ and $\theta=-\alpha$, with equal amounts. So that their resultant force is equal to twice the value of eq. (43). More explicit expressions for force and moment are as follows. First, we put

$$\phi=\omega r_1^2\Phi$$

where we have

$$\Phi=\tan 2\alpha\cdot\xi^2+\sum_{s=1}^{\infty}\left[\frac{4\alpha}{4\alpha^2-[(\pi/2)s]^2}\right]\left[\frac{H(\lambda s+2)}{(\lambda s)H(2\lambda s)}\xi^{\lambda s}-\frac{H(\lambda s-2)}{(\lambda s)H(2\lambda s)}\frac{\eta^{\lambda s+2}}{\xi^{\lambda s}}\right] \tag{44}$$

with the notation

$$\lambda=\pi/(2\alpha),\qquad \xi=r/r_1,\qquad \eta=r_2/r_1,\qquad H(\mu)=1-\eta^{\mu}$$

$$H(-\mu)=-\frac{1}{\eta^{\mu}}H(\mu)$$

Hence, we have

$$\frac{\partial\phi}{\partial r}=\omega r_1^2\frac{\partial\Phi}{\partial r}=\omega r_1\Phi'(\xi)$$

where

$$\Phi'(\xi)=2\tan 2\alpha\cdot\xi+\sum_{s=1}^{\infty}\left[\frac{4\alpha}{4\alpha^2-[(\pi/2)s]^2}\right]\left[\frac{H(\lambda s+2)}{H(2\lambda s)}\xi^{\lambda s-1}+\frac{H(\lambda s-2)}{H(2\lambda s)}\frac{\eta^{\lambda s+2}}{\xi^{\lambda s+1}}\right] \tag{46}$$

Thus we obtain

$$F_1=\int_{r2}^{r1}\frac{\partial\Phi}{\partial t}dr=r_1^3\frac{d\omega}{dt}\int_{\eta}^{1}\Phi(\xi)d\xi=r_1^3\frac{d\omega}{dt}K, \tag{47}$$

where we have put

$$K=\frac{1}{3}\tan 2\alpha\cdot H(3)+\sum_{s=1}^{\infty}\left[\frac{4\alpha}{4\alpha^2-[(\pi/2)s]^2}\right]\left[\frac{H(\lambda s+2)H(\lambda s+1)}{(\lambda s)(\lambda s+1)(H(2\lambda s)}-\frac{H(\lambda s-2)}{(\lambda s)H(2\lambda s)}\frac{\eta^{\lambda s+2}H(-\lambda s+1)}{(-\lambda s+1)}\right] \tag{48}$$

$$M_1=\int_{r2}^{r1}\frac{\partial\phi}{\partial t}rdr=r_1^4\frac{d\omega}{dt}\int_{\eta}^{1}\Phi(\xi)\xi d\xi=r_1^4\frac{d\omega}{dt}\cdot S \tag{49}$$

where we have put

$$S=\frac{1}{4}\tan 2\alpha\cdot H(4)+\sum_{s=1}^{\infty}\left[\frac{4\alpha}{4\alpha^2-[(\pi/2)s]^2}\right]\left[\frac{H(\lambda s+2)H(\lambda s+2)}{(\lambda s)(\lambda s+2)H(2\lambda s)}-\frac{H(\lambda s-2)}{(\lambda s)H(2\lambda s)}\cdot\frac{\eta^{\lambda s+2}H(-\lambda s+2)}{(-\lambda s+2)}\right] \tag{50}$$

3-2. Numerical Example

In order to illustrate the above analytical treatment by some numerical example, let us take up the case of $r_2/r_1=1/2$ ($\eta=0.50$) in which the angle α of fan-shape is varied in three ways as follows;

(1) $\alpha=30°=\pi/6$ $(\lambda=3)$

(2) $\alpha=22.5°=\pi/8$ $(\lambda=4)$

(3) $\alpha=15°=\pi/12$ $(\lambda=6)$

For these three cases, the author has obtained, by formulae (48) and (50) (taking $s=1, 3, 5, 7$ and omitting terms for $s=9, 11, \cdots$) numerical values which give us effect of virtual mass of flowing water, upon accelerated angular motion ($d\omega/dt$) of vessel body. The results of this numerical calculation is shown in Table 1. In this Table 1, we have also shown values of ratios

$(2K)\div$(area of fan-shaped figure of the vessel)

$(2S)\div$(secondary moment about origin, of fan-shaped figure of the vessel)

Table 1

α	(Area)$/r_1^2$	(Secondary Moment)$/r_1^3$	K	S
30	0.39269908	0.30543231	0.38351019 (1.95320137)	0.33485923 (2.19269029)
22.5	0.29452431	0.22907444	0.24209016 (1.64394009)	0.20337043 (1.77558378)
15	0.19634954	0.15271629	0.14882717 (1.51594111)	0.12631900 (1.65429634)

From this Table 1, we observe that values of coefficients of virtual mass varies with the fan-angle α, and that they lie between 1.0 to 2.0.

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