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ON THE KUZMIN'S THEOREM FOR THE CONPLEX CONTINUED FRACTIONS

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ABSTRACT

We define a class of transformations which includes the complex continued fraction transformations. For this class we shall prove KUZMIN's formula which gives a convergence rate on the strong mixing condition and weak Bernoulli property. In the case of the complex continued fraction transformations the rate of strong mixing are exponential.

Introduction

Main purpose of this paper is to prove the KUZMIN's theorem in the case of the class of transformations which includes the complex continued fraction transformations.

Recently SHIOKAWA, KANEIWA and TAMAURA [2] defined a complex continued fraction over $Q(\sqrt{-3})$ and showed its numerical properties. Moreover SHIOKAWA [6], [7] obtained some ergodic properties of the transformation induced by this algorithm. The other hand, SCHWEIGER and WATERMAN [3], [4], [5], [8], [9], have also showed some results about metrical properties on the class of transformations including the PERRON algorithm, but not SHIOKAWA's transformation.

The auther investigates the class of transformations which generalizes WATERMAN's class, including SHIOKAWA's one and HURWITZ's one. In this class, we can see the transformations are weak Bernoulli.

1. Definitions and fundamental properties

In this section we define a class of transformations as a generalization of those induced by the complex continued fractions. Let X be a convex measurable subset of R^n which has finite positive Lebesgue measure, $\mathfrak{B}$ be the σ-field of all Borel subsets of X and $\lambda(\cdot)$ be a normalized Lebesgue measure on X.

We consider the countable partition $\{X_a;\ a\in I,$ *each* X_a *is measurable and connected*$\}$ of X satisfying the condition (a):

(a) *For any* $a\in I$ *there exists an* 1-1, *continuous map* T_a *of* X_a *into* X *such that the components of it have continuous first order partial derivative and det* $DT_a\neq 0$, *where* DT_a *is the Jacobian matrix of* T_a.

We define inductively

$$X_{a_1a_2\cdots a_m}=T_{a_1}^{-1}X_{a_2\cdots a_m},\qquad T_{a_1a_2\cdots a_m}=T_{a_m}\cdots T_{a_2}T_{a_1} \tag{1}$$

where $a_i\in I$, $1\leqq i\leqq m$. Here we note that by definition $X_{a_1a_2\cdots a_m}$ may be empty for some $a_1a_2\cdots a_m$. We thus obtain for any $m\geqq 1$ a partition $\{X_{a_1a_2\cdots a_m}\}$ of X with a family of mappings $T_{a_1a_2\cdots a_m}$ of $X_{a_1a_2\cdots a_m}$ into X.

Now we further require the following assumptions (b), (c), (d) and (e):

(b) *There exist finite number, say* N, *of subsets* $U_1, U_2,\cdots, U_N$ *of* X *with positive measure such that for any* $a_1, a_2,\cdots, a_m\in I$, $(U_0=X)$,

$$T_{a_1a_2\cdots a_m}X_{a_1a_2\cdots a_m}\begin{cases}=U_j, & \text{for some } j,\ 0\leq j\leqq N,\\ =\phi, & \text{if } X_{a_1a_2\cdots a_m}=\phi.\end{cases}$$

(c) *There exists a constant* $C>1$ *such that*

$$\sup_{x\in U_j}|\det D\psi_{a_1a_2\cdots a_m}(x)|<C\cdot\inf_{x\in U_j}|\det D\psi_{a_1a_2\cdots a_m}(x)| \tag{2}$$

uniformly in $a_1a_2\cdots a_m$ *and* j, *where* $\psi_{a_1a_2\cdots a_m}$ *is the mapping of* $T_{a_1a_2\cdots a_m}X_{a_1a_2\cdots a_m}=U_j$ *onto* $X_{a_1a_2\cdots a_m}$ *defined as the inverse of* $T_{a_1a_2\cdots a_m}$.

(d) $\qquad \sup_{a_1a_2\cdots a_m}\operatorname{diam}\ (X_{a_1a_2\cdots a_m})=\sigma(m)\to 0$ as $m\to\infty$.

(e) *For any* j, $1\leqq j\leqq N$, *there exists* $X_{a_1a_2\cdots a_s}$ *such that* $T_{a_1a_2\cdots a_s}X_{a_1a_2\cdots a_s}=U_0(=X)$ and

$$X_{a_1a_2\cdots a_s}\subset U_j,$$

where the length s is independent of j.

We can define, under these assumptions, the transformation T,

$$T=T_a \text{ on } X_a \text{ for } a\in I.$$

Thus $T^m=T_{a_1a_2\cdots a_m}$ on $X_{a_1a_2\cdots a_m}$.

Since

$$\lambda(X_{a_1a_2\cdots a_m})=\int_{U_j}|\det D\psi_{a_1a_2\cdots a_m}(x)|dx,\quad T^mX_{a_1a_2\cdots a_m}=U_j, \tag{3}$$

we have by (2)

$$\inf|\det D\psi_{a_1a_2\cdots a_m}(x)|\cdot\lambda(U_j)<\lambda(X_{a_1a_2\cdots a_m})<C\cdot\inf|\det D\psi_{a_1\cdots a_m}(x)\cdot\lambda(U_j). \tag{4}$$

By the definition of (1), we can get

$$T^{m+k}X_{a_1a_2\cdots a_ma_{m+1}\cdots a_{m+k}}=T^m(X_{a_{m+1}\cdots a_{m+k}}\cap U_j). \tag{5}$$

Remark. Assumption (d) implies $\bigvee_{m=1}^{\infty} T^{-m}X=\xi$.

In this section all positive constants Q_1, Q_2, Q_3 depend at most on C and $\min_j \lambda(U_i)$.

We define the sets of m-tuple of indices $A(m), A^0(m)$:

$$A(m)=\{(a_1a_2\cdots a_m);\ T^m X_{a_1a_2\cdots a_m}\neq\phi\}$$

and

$$A^0(m)=\{(a_1a_2\cdots a_m);\ T^m X_{a_1a_2\cdots a_m}=X\}.$$

Lemma 1. *Let $(a_1a_2\cdots a_k)$ be a k-tuple of indices given arbitrarily. Then for any m ($\geqq s$ of (e)), there exist Q_1 and Q_2 such that*

$$\sum_{(b_1b_2\cdots b_m);(a_1\cdots a_kb_1\cdots b_m)\in A^0(k+m)} \lambda(X_{a_1\cdots a_kb_1\cdots b_m})\geqq Q_1\cdot\lambda(X_{a_1a_2\cdots a_k}), \tag{6}$$

and so for any $m\geqq 1$

$$\sum_{(a_1a_2\cdots a_m)\in A^0(m)} \lambda(X_{a_1a_2\cdots a_m})\geqq Q_2. \tag{7}$$

Proof. If there exists Q_1 of (6), then

$$\sum_{(a_1a_2\cdots a_m)\in A^0(m)} \lambda(X_{a_1a_2\cdots a_m})\geqq Q_1$$

provided $m\geqq s+1$. So we may choose in (7)

$$Q_2=\min\left\{\min_{1\leqq m\leqq s}\ \sum_{A^0(m)} \lambda(a_1a_2\cdots a_m),\quad Q_1\right\}.$$

To prove (6), we first suppose that $m=s$. By (5) and the assumptions (b), (c), we have $\{(b_1b_2\cdots b_s);\ (a_1a_2\cdots a_kb_1\cdots b_s)\in A^0(k+s)\}\neq\phi$ if $X_{a_1a_2\cdots a_k}\neq\phi$. Hence

$$\begin{aligned}&\sum_{(b_1b_2\cdots b_s);\ (a_1\cdots a_kb_1\cdots b_s)\in A^0(k+s)} \lambda(X_{a_1a_2\cdots a_kb_1\cdots b_s})\\ &\geqq \sum_{(b_1\cdots b_s)} \inf|\det D\psi_{a_1\cdots a_k}(x)|\cdot\lambda(X_{b_1\cdots b_s})\\ &\geqq C^{-1}\cdot\lambda(X_{a_1\cdots a_k})\sum_{(b_1\cdots b_s)}\lambda(X_{b_1\cdots b_s}).\end{aligned}$$

Since by (5), $\{(b_1b_2\cdots b_s);\ (a_1a_2\cdots a_kb_1\cdots b_s)\in A^0(k+s)\}=\{(b_1\cdots b_s)\in A^0(s);\ X_{b_1b_2\cdots b_s}\subset U_j,\ T^kX_{a_1\cdots a_k}=U_j\}$, and we have

$$\begin{aligned}&\sum_{(b_1b_2\cdots b_s);\ (a_1\cdots a_kb_1\cdots b_s)\in A^0(k+s)} \lambda(X_{a_1a_2\cdots a_kb_1\cdots b_s})\\ &\geqq C^{-1}\lambda(X_{a_1a_2\cdots a_k})\sum_{(b_1\cdots b_s)\in A^0(s);\ X_{b_1\cdots b_s}\subset U_j}\lambda(X_{b_1\cdots b_s}).\end{aligned}$$

So we choose Q_1 as

$$Q_1=C^{-1}\min_{0\leqq j\leqq N}\sum_{\substack{(b_1\cdots b_s)\in A^0(s);\\ X_{b_1\cdots b_s}\subset U_j}}\lambda(X_{b_1\cdots b_s}).$$

If $m>s$, then

$$\sum_{(b_1b_2\cdots b_m)} \lambda(X_{a_1a_2\cdots a_kb_1\cdots b_m})$$
$$= \sum_{(a_1\cdots a_kb_1\cdots b_{m-s})\in A(k+m-s)} \sum_{\substack{(b_{m-s+1}\cdots b_m)\in A^0(s);\\ (a_1\cdots b_{m-s})\in A(m-s)}} \lambda(X_{a_1\cdots a_kb_1\cdots b_{m-s}})$$
$$\times \frac{\lambda(X_{a_1\cdots b_m})}{\lambda(X_{a_1\cdots b_{m-s}})}$$
$$\geqq \sum_{\substack{(b_1\cdots b_{m-s});\\ (a_1\cdots a_kb_1\cdots b_{m-s})\in A(k+m-s)}} \lambda(X_{a_1\cdots a_mb_1\cdots b_{m-s}})\cdot Q_1$$
$$=\lambda(X_{a_1a_2\cdots a_k})\cdot Q_1.$$

Remark. If we assume (b′) instead of (b),

(b′) $$\inf_{m,\ A(m)} \lambda(T^m X_{a_1a_1\cdots a_m})>0,$$

then we need (6) as the condition (e). And (b′) is an extension of (L) in WATERMAN [8].

Theorem 1. *The transformation T is irreducible; i.e. if $T^{-1}E=E\in\mathfrak{B}$, then $\lambda(E)=0$ or* 1.

Proof. Assume that $T^{-1}E=E$ and $\lambda(E)=0$. For any $X_{a_1a_2\cdots a_m}$, $(a_1a_2\cdots a_m)\in A(m)$, we have

$$\lambda(E\cap X_{a_1a_2\cdots a_m})$$
$$\geqq \sum_{\substack{(b_1b_2\cdots b_s);\\ (a_1\cdots a_mb_1\cdots b_s)\in A^0(m+s)}} \int_X I_E(x)\cdot|\det D\psi_{a_1\cdots a_mb_1\cdots b_s}(x)|dx$$
$$\geqq C^{-1}\sum_{(b_1\cdots b_s)} \lambda(X_{a_1\cdots a_mb_1\cdots b_s})\cdot\lambda(E)$$
$$\geqq C^{-1}\cdot Q_1\cdot\lambda(X_{a_1a_2\cdots a_m})\cdot\lambda(E),$$

where I_E is the indicator function of E. So

$$\lambda(E\cap F)\geqq C^{-1}Q_1\lambda(F)\lambda(E)$$

for any $F\in\mathfrak{B}$, since the family of all $\{X_{a_1\cdots a_m};\ (a_1\cdots a_m)\in A(m)\}$ generates $\mathfrak{B}$. Hence, putting $F=E^C$, it must be $\lambda(F)=0$ and $\lambda(E)=1$.

Theorem 2. *There exists an unique T-invariant probability measure μ equivalent to Lebesgue measure λ such that*

$$Q_3^{-1}\lambda(E)\leqq\mu(E)\leqq Q_3\cdot\lambda(E),\ E\in\mathfrak{B}. \tag{8}$$

Proof. If there exists T-invariant measure, then it is unique by **Theorem 1.** To prove the existence of the invariant measure μ satisfying the inequality (8), it is enough to show that for any $k\geqq0$

$$Q_3^{-1}\cdot\lambda(E)\leqq\lambda(T^{-k}E)\leqq Q_3\lambda(E),\ E\in\mathfrak{B}. \tag{9}$$

By (4) and (7) we find

$$\lambda(T^{-k}E)=\sum_{(a_1a_2\cdots a_k)\in A(k)}\int_{U_j\cdot E}|\det D\psi_{a_1\cdots a_k}(x)|dx$$
$$\geqq\sum_{A^0(k)}\int_E|\det D\psi_{a_1\cdots a_k}(x)|dx$$
$$\geqq C^{-1}\cdot Q_2\cdot\lambda(E).$$

On the other hand

$$\lambda(T^{-k}E)\leqq\sum_{A(k)}C\cdot inf|\det D\psi_{a_1a_2\cdots a_k}(x)|\cdot\lambda(E)$$
$$\leqq[\min_j\lambda(U_j)]^{-1}C\lambda(E).$$

Thus putting

$$Q_3=max\{C\cdot Q_2^{-1},\ C\cdot\min_j\lambda(U_j)^{-1}\},$$

we obtain (9).

Theorem 3. *T is an exact endomorphism with invariant measure μ.*

Proof. Similar to that of Theorem 5.3 in [8].

2. KUZMIN's theorem

We need next two additional conditions (f) and (g) in order to show the so-called KUZMIN's theorem:

(f) *There exists a constant K such that*

$$||\det D\psi_{a_1a_2\cdots a_m}(x)|-|\det D\psi_{a_1a_2\cdots a_m}(y)||$$
$$\leqq K\cdot\lambda(X_{a_1a_2\cdots a_m})\cdot\lambda(_{a_1\cdots a_m})\cdot||x-y||,\ x,y\in T^mX_{a_1a_2\cdots a_m}, \tag{10}$$

uniformly in x,y and $(a_1a_2\cdots a_m)\in A(m)$.

We define the partition ξ

$$\xi=\bigvee_{m,\ A(m)}\{T^mX_{a_1a_2\cdots a_m},(T^mX_{a_1a_2\cdots a_m})^C\}$$
$$=\bigvee_j\{U_j,U_j^C\}$$

and

$$A_\xi(m)=\{(a_1a_2\cdots a_m);\ X_{a_1a_2\cdots a_m}\not\subset A\ for\ any\ A\in\xi\}.$$

(g) $$\sum_{(a_1a_2\cdots a_m)\in A_\xi(m)}\lambda(X_{a_1a_2\cdots a_m})=\gamma(m)\to 0\quad as\quad m\to\infty.$$

Moreover, there exists a constant M such that

$$|D_{i,j}\psi_{a_1a_2\cdots a_m}(x)|\leqq M$$

uniformly in $i,j(1\leqq i,j\geqq n)$ and $(a_1a_2\cdots a_m)\in A(m)$, where $D_{i,j}\psi_{a_1\cdots a_m}$ is the (i,j)-component of the Jacobian matrix of $\psi_{a_1a_2\cdots a_m}$.

Theorem 4. *Suppose that a real-valued integrable function h_0 on X satisfies the following condition (i) and* $\{h_1, h_2, \cdots\}$ is *a sequence of functions defined reccursively by (ii):*
(i) There exists constants B and L such that

$$B^{-1}<h_0<B \quad on \quad X \tag{11}$$

and

$$|h_0(x)-h_0(y)|<L\|x-y\| \tag{12}$$

when x and y are contained in the same element of ξ.

(*ii*)
$$h_m(x)=\sum_{a;\ TX_a \ni x} h_{m-1}(\psi_a(x))\cdot|\det D\psi_a(x)|. \tag{13}$$

Then

$$h_m(x)=H_0\cdot\rho(x)+O(\sigma(m)+\gamma(m))$$

where

$$H_0=\int_X h_0(x)\,dx \quad and \quad \rho(x)=\frac{d\mu}{dx}.$$

Here and henceforth all the O's and the constants $Q_4, Q_5, \cdots$ depend possibly on C, M, B, L and $\min_j \lambda(U_i)$.

Remark. If we adopt (b′) in place of (b), then it is necessary to assume that the partition ξ is countable.

Lemma 2. *If*

$$h(x)=\sum_{a;\ TX_a \ni x} h(\psi_a(x))\cdot|\det D\psi_a(x)|, \tag{14}$$

then

$$h(x)=H\cdot\rho(x), \qquad H=\int_X h(x)dx.$$

Lemma 3. *For any $m\geqq 1$, we have*

$$h_m(x)=\sum_{\substack{(a_1a_2\cdots a_m);\\ T^m X_{a_1\cdots a_m}\ni x}} h_0(\psi_{a_1\cdots a_m}(x))\cdot|\det D\psi_{a_1\cdots a_m}(x)|. \tag{15}$$

Lemma 4. *For any $m\geqq 1$, we have*

$$\int_X h_m(x)\,dx=\int_X h_0(x)\,dx. \tag{16}$$

The proof of these **Lemmas** are the same that in [5], [9].

Lemma 5. *If x and y are contained in the same element of ξ, then*

$$|h_m(x)-h_m(y)|\leqq Q_4\cdot\|x-y\|.$$

Proof. By the assumption, $x\in T^m X_{a_1a_2\cdots a_m}$ if and only if $y\in T^m X_{a_1a_2\cdots a_m}$. Hence

$$|h_m(x)-h_m(y)|$$
$$\leqq \sum_{\substack{(a_1a_2\cdots a_m):\\ T^mX_{a_1\cdots a_m}\ni x,\, y}} |h_0(\psi_{a_1\cdots a_m}(x))\cdot|\det D\psi_{a_1\cdots a_m}(x)|$$
$$-h_0(\psi_{a_1\cdots a_m}(y))\cdot|\det D\psi_{a_1\cdots a_m}(y)||$$
$$\leqq \sum |h_0(\psi_{a_1\cdots a_m}(x))|\cdot||\det D\psi_{a_1\cdots a_m}(x)|-|\det D\psi_{a_1\cdots a_m}(y)||$$
$$+\sum |\det D\psi_{a_1\cdots a_m}(y)|\cdot|h_0(\psi_{a_1\cdots a_m}(x))-h_0(\psi_{a_1\cdots a_m}(y))|.$$

By (10) and (11)

$$\sum |h_0(\psi_{a_1\cdots a_m}(x))|\cdot||\det D\psi_{a_1\cdots a_m}(x)|-|\det D\psi_{a_1\cdots a_m}(y)||$$
$$\leqq K\cdot B\cdot||x-y||.$$

Observing that if x and y are contained in the same element of ξ, then $\psi_{a_1a_2\cdots a_m}(x)$ and $\psi_{a_1a_2\cdots a_m}(y)$ are also in the same element of ξ even if $X_{a_1a_2\cdots a_m}\in A_\xi(m)$, we obtain

$$\sum |\det D\psi_{a_1\cdots a_m}(y)|\cdot|h_0(\psi_{a_1\cdots a_m}(x))-h_0(\psi_{a_1\cdots a_m}(y))|$$
$$\leqq L\cdot \sum |\det D\psi_{a_1\cdots a_m}(y)|\cdot|\psi_{a_1\cdots a_m}(x)-\psi_{a_1\cdots a_m}(y)|$$
$$\leqq L\cdot n\cdot M\cdot C\cdot||x-y||.$$

So the lemma is proved with $Q_4=max(L\cdot B, L\cdot n\cdot M\cdot C)$.

Proof of **Theorem 4.** It is sufficient to show that

$$h_{m+k}(x)-h_m(x)=O(\sigma(m)+\gamma(m))$$

as $m\to\infty$, uniformly in $k\geqq 1$ and $x\in X$. By (b) and **Lemma 3**, there is a constant Q_5 such that

$$Q_5^{-1}<h_m(x)<Q_5 \quad \text{for any} \quad m\geqq 0,$$

and we get for any m and k

$$g_0h_m(x)<h_{m+k}(x)<G_0h_m(x) \tag{17}$$

where we may choose $g_0^{-1}=Q_5^2=G_0$.

Now define

$$v_m(x)=h_{m+k}(x)-g_0h_m(x).$$

Then from **Lemma 3** and (4),

$$v_m(x)=\sum_{\substack{A(m):\\ T^mX_{a_1\cdots a_m}\ni x}} v_0(\psi_{a_1\cdots a_m}(x))\cdot|\det D\psi_{a_1\cdots a_m}(x)|$$
$$\geqq \sum_{A^0(m)} v_0(\psi_{a_1\cdots a_m}(x))\cdot|\det D\psi_{a_1\cdots a_m}(x)|$$
$$\geqq C^{-1}\sum_{A^0(m)} v_0(\psi_{a_1\cdots a_m}(x))\cdot\lambda(X_{a_1\cdots a_m}).$$

Moreover, using the mean-valued theorem,

$$v_m(x) - C^{-1} \sum \int_{X_{a_1 \cdots a_m}} v_0(x)\, dx$$
$$\geqq C^{-1} \sum |v_0(\psi_{a_1 \cdots a_m}(x)) - v_0(\psi_{a_1 \cdots a_m}(x'))| \cdot \lambda(X_{a_1 \cdots a_m})$$

for some $x' \in T^m X_{a_1 a_2 \cdots a_m}$.
Because of $v_0 = h_k - g_0 h_0$, we have

$$\begin{aligned} &| \sum \{v_0(\psi_{a_1 \cdots a_m}(x)) - v_0(\psi_{a_1 \cdots a_m}(x'))\} \cdot \lambda(X_{a_1 \cdots a_m})| \\ &\quad \leqq \sum |h_k(\psi_{a_1 \cdots a_m}(x)) - h_k(\psi_{a_1 \cdots a_m}(x'))| \cdot \lambda(X_{a_1 \cdots a_m}) \\ &\qquad + g_0 \sum |h_0(\psi_{a_1 \cdots a_m}(x)) - h_0(\psi_{a_1 \cdots a_m}(x'))| \cdot \lambda(X_{a_1 \cdots a_m}) \\ &\quad \leqq Q_6(\sigma(m) + \gamma(m)), \end{aligned}$$

using **Lemma 5.** Hence

$$\begin{aligned} &v_m(x) - C^{-1} \sum \int_{X_{a_1 \cdots a_m}} v_0(x)\, dx \\ &\qquad > -Q_7(\sigma(m) + \gamma(m)). \end{aligned}$$

So

$$\begin{aligned} &h_{m+k}(x) \\ &\quad > h_m \Big\{ g_0 + C^{-1} Q_5^{-1} \sum \int_{X_{a_1 \cdots a_m}} (h_k(x) - g_0 h_0(x)) dx \\ &\qquad - Q_5^{-1} Q_7(\sigma(m) + \gamma(m)) \Big\} \\ &\quad > h_m(x) g_1, \end{aligned}$$

where

$$\begin{aligned} g_1 &= \alpha(m) \cdot g_0 + \beta(m, k); \\ \alpha(m) &= \Big(1 - C^{-1} Q_5^{-1} \sum \int_{X_{a_1 \cdots a_m}} h_0(x)\, dx \Big) \cdot g_0 \\ \beta(m, k) &= C^{-1} Q_5^{-1} \sum \int_{X_{a_1 \cdots a_m}} h_k(x) dx - Q_5^{-1} Q_7(\sigma(m) + \gamma(m)). \end{aligned}$$

Next, if we start, in (17), with

$$V_m(x) = G_0 \cdot h_{m+k}(x) - h_m(x),$$

we obtain in the same manner

$$\begin{aligned} &G_1 \cdot h_{m+k}(x) > h_m(x), \\ &\qquad G_1 = \alpha(m) G_0 + \delta(m, k), \\ &\delta(m, k) = C^{-1} Q_5^{-1} \sum \int_{X_{a_1 \cdots a_m}} h_k(x) dx + Q_5^{-1} Q_8(\sigma(m) + \gamma(m)). \end{aligned}$$

Thus, we can construct two sequences

$$g_r = \alpha(m) g_{r-1} + \beta(m, k),$$
$$G_r = \alpha(m) G_{r-1} + \delta(m, k),$$

which satisfy

$$g_r \cdot h_m(x) < h_{m+k}(x) < G_r \cdot h_m(x)$$

for any $k \geqq 1$ and $m \geqq 1$. Noticing here that

$$\alpha(m) < 1, \beta(m, k) > 0 \quad \text{and} \quad \delta(m, k) > 0$$

for all $m \geqq 1$, we may find

$$\lim_{r\to\infty} g_r = \frac{\beta(m, k)}{1-\alpha(m)} = Q(m, k) + O(\sigma(m) + \gamma(m)),$$

$$\lim G_r = \frac{\delta(m, k)}{1-\alpha(m)} = Q(m, k) + O(\sigma(m) + \gamma(m))$$

where

$$Q(m, k) = \frac{\sum_{A^0(m)} \int_{X_{a_1 \cdots a_m}} h_k dx}{\sum_{A^0(m)} \int_{X_{a_1 \cdots a_m}} h_k dx}$$

Hence

$$h_{m+k}(x) - Q(m, k) \cdot h_m(x) = O(\sigma(m) + \gamma(m)).$$

We integrate this inequality on X, and using **Lemma 4** we get

$$Q(m, k) = 1 + O(\sigma(m) + \gamma(m)).$$

Consequently we have

$$h_{m+k}(x) - h_m(x) = O(\sigma(m) + \gamma(m)),$$

and the proof of the theorem is now complete.

Corollary 1. *For any* $E \in \mathfrak{B}$,

$$|\lambda(T^{-m}E) - \mu(E)| < Q_9 \cdot \lambda(E) \cdot (\sigma(m) + \gamma(m)).$$

Proof. We may put $h_0(x) = 1$.

Corollary 2. *Let* $F = X_{a_1 a_2 \cdots a_k}$ *and* $E \in \mathfrak{B}$, *then*

$$|\mu(T^{-m}E \cap F) - \mu(E) \cdot \mu(F)|$$
$$\leqq \mu(E) \cdot \mu(F) \cdot Q_{10}(\sigma(m-k) + \gamma(m-k)).$$

And so the transformation T *is weak Bernoulli.*

Proof. Put $h_0(x) = \mu(X_{a_1 a_2 \cdots a_k})^{-1} I_{X_{a_1 a_2 \cdots a_k}}(x) \cdot \rho(x)$, then the proof is the same as Theorem 6.3 in [8].

3. The complex continued fractions.

The complex continued fraction transformation induced by SHIOKAWA [6], [7] is an example of the transformation T with $s=2$, $\sigma(m)=O(\theta^m)$ and $\gamma(m)=O(\eta^m)$ where

$$\theta=3^{-\frac{1}{3}},\ \eta=\sqrt{2\left(\frac{\pi^4}{90}-1\right)}.$$

We will show that the complex continued fraction in the case of $Q(\sqrt{-1})$ defined by HURWITZ [1] also satisfies the assumptions (a), (b),$\cdots$,(g) with $s=1$. From now on, $Q_{10}, Q_{11},\cdots$ are absolute constants.

Let

$$X=\{z\,;\ z=u+vi,\ -1/2<u, v<1/2\} \tag{18}$$

and

$$I_{(i)}=\{u+vi\,;\ u \text{ and } v \text{ are integers.}\} \tag{19}$$

For any complex number z, $[z]_i$ is equal to $a\in I_{(i)}$ such that $z-a\in X$; i.e. $[z]_i$ is the nearest point of $I_{(i)}$. We define the partition $\{X_a\}$ and the transformation T on X

$$\begin{gathered} X_a=\left\{z\in X;\left[\frac{1}{z}\right]_i=a\right\} \quad \text{for} \quad a\in I, \\ I=I_{(i)}\backslash\{0,1,-1,\mathrm{i},-\mathrm{i}\} \end{gathered} \tag{20}$$

and

$$Tz=\frac{1}{z}-\left[\frac{1}{z}\right]_i \quad \text{for} \quad z\in X. \tag{21}$$

Also we define

$$a_m(z)=\left[\frac{1}{T^{m-1}z}\right]_i \quad \text{for} \quad z\in X. \tag{22}$$

A complex number $z\in X$ is expanded in

$$z=\frac{1|}{|a_1}+\frac{1|}{|a_2}+\cdots+\frac{1|}{|a_m+T^m z} \qquad (m\geqq 1) \tag{23}$$

provided $T^k z\neq 0$ for all $k\leqq m$. As usual, we put

$$\frac{p_m}{q_m}=\frac{1|}{|a_1}+\frac{1|}{|a_2}+\cdots+\frac{1|}{|a_m} \tag{24}$$

and have the following formulae;

$$p_m=a_m\cdot p_{m-1}+p_{m-2},\ q_m=a_m\cdot q_{m-1}+q_{m-2} \tag{25}$$

$$\frac{q_{m-1}}{q_m}=\frac{1|}{|a_m}+\cdots+\frac{1|}{|a_2}+\frac{1|}{|a_1} \tag{26}$$

$$p_m\cdot q_{m-1}-p_{m-1}\cdot q_m=(-1)^{m-1} \tag{27}$$

where $p_{-1}=1, q_{-1}=0, p_0=0, q_0=1$. Furthermore we get

$$|q_m|<|q_{m+1}| \quad \text{for any } z\in X \text{ and } m\geqq 1, \tag{28}$$

and

$$\lim \frac{p_m}{q_m}=z \quad \text{for} \quad z\in X$$

(see HURWITZ [1]).

It is clear that the transformation T satisfies the assumption (a).

We put

$$\begin{aligned}
&U_0=X,\\
&U_1=\{z\in X;\ |z+i|\geqq 1\},\ U_2=U_1\times i,\ U_3=U_2\times i,\ U_4=U_3\times i,\\
&U_5=\{z\in X;\ |z+1+i|\geqq 1\},\ U_6=U_5\times i,\cdots\cdots,\\
&U_9=\{z\in X;\ |z+i|\geqq 1,\ |z+1|\geqq 1\},\cdots\cdots,\ U_{12}=U_{11}\times i,
\end{aligned}$$

where $U_j\times i=\{z';\ z'=z\times i,\ z\in U_j\}$, then $\{U_j;\ j=0,1,2,\cdots,12\}$ satisfies (b). This is shown by induction (see figure-1).

By (23), (24), (25), we get $\psi_{a_1a_2\cdots a_m}$ (z) in (c) as

$$\psi_{a_1a_2\cdots a_m}(z)=\frac{p_m+p_{m-1}z}{q_m+q_{m-1}z} \tag{29}$$

where $z\in U_j=T^mX_{a_1a_2\cdots a_m}$. From this equation and Cauchy-Riemann equation, it follows that

$$\begin{aligned}
|\det D\psi_{a_1a_2\cdots a_m}| &= \left|\frac{\partial\psi_{a_1a_2\cdots a_m}}{\partial(x,y)}\right|, \quad z=x+iy\\
&=\left|\frac{d\psi_{a_1a_2\cdots a_m}}{dz}\right|^2\\
&=|q_m|^{-4}\left|1+\frac{q_{m-1}}{q_m}z\right|^{-4}
\end{aligned} \tag{30}$$

Hence we have

$$Q_{11}^{-1}|q_m|^{-4}<|\det D\psi_{a_1a_2\cdots a_m}|<Q_{11}|q_m|^{-4} \tag{31}$$

and this implies (c).

Next proposition means the assumption (d).

Proposition 1. *It follows that*

$$|\psi_{a_1a_2\cdots a_m}(z)-\psi_{a_1a_2\cdots a_m}(w)|<Q_{12}\theta^{-m}$$

uniformly $z,w\in T^mX_{a_1a_2\cdots a_m}$ *and* $(a_1a_2\ldots a_m)\in A(m)$, *where* $\theta=3-\sqrt{2}$.

Proof. In general

$$|\psi_{a_1a_2\cdots a_m}(z)-\psi_{a_1a_2\cdots a_m}(w)|=Q_{12}\cdot|q_m|^{-2},$$

so it is sufficient to prove

$$\left|\frac{q_m}{q_{m-2}}\right|>3-\sqrt{2} \tag{32}$$

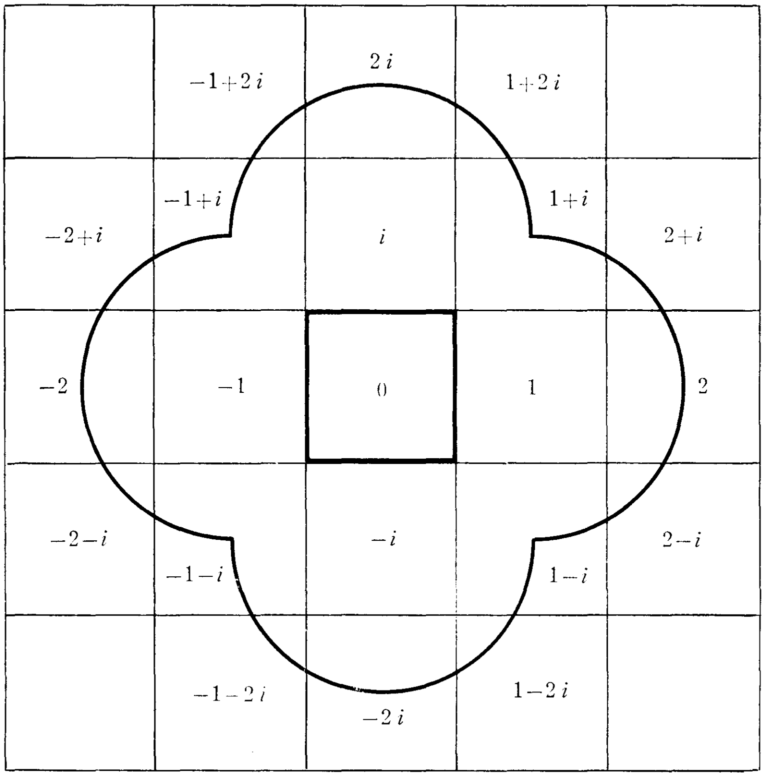

Fig. 1-(a) X and $X^{-1}=\{z'\,;\ z'=1/z,\ z\in X\}$.

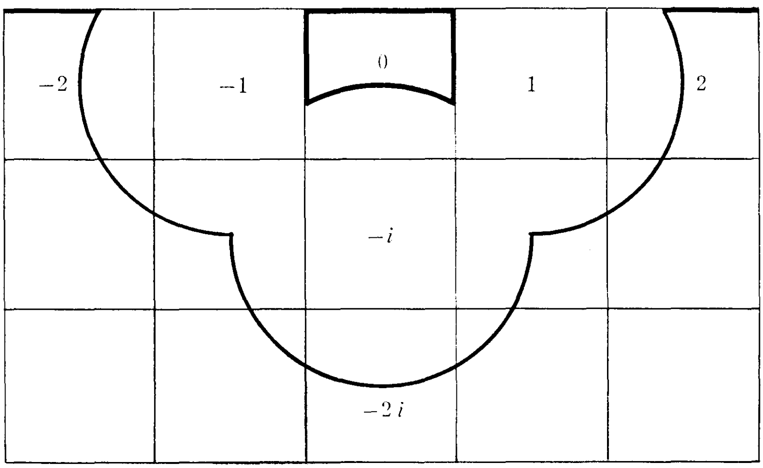

Fig. 1-(b) U_1 and U_1^{-1}.

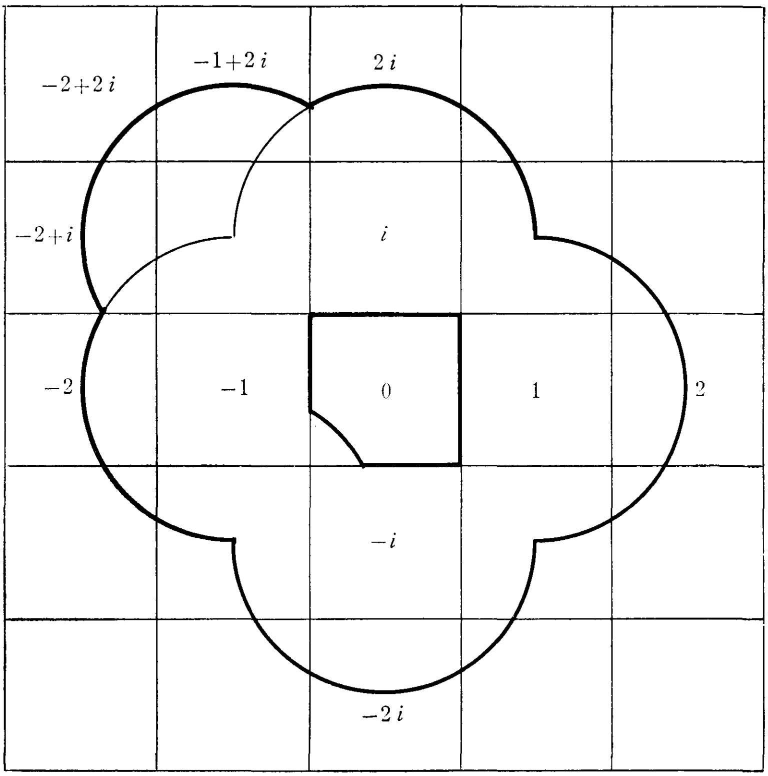

Fig. 1-(c) U_5 and U_5^{-1}.

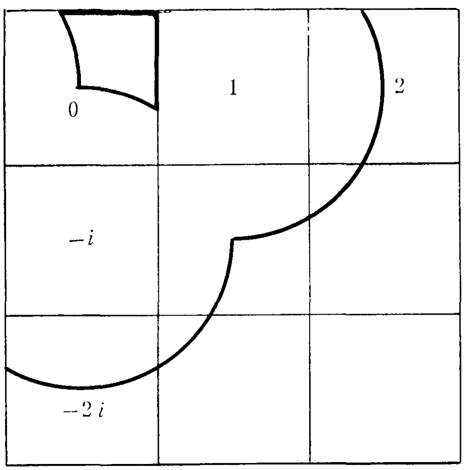

Fig. 1-(d) U_9 and U_9^{-1}.

for $m \geqq 2$. By (25) and (28)

$$\left|\frac{q_m}{q_{m-2}}\right| = \left|\frac{a_m(a_{m-1}q_{m-2}+q_{m-3})+q_{m-2}}{q_{m-2}}\right|$$
$$= \left|a_m a_{m-1}+1+a_m \frac{q_{m-3}}{q_{m-2}}\right|$$
$$> |a_m a_{m-1}+1| - |a_m|.$$

From this inequality and dependency of a_m and a_{m-1}, we get

$$\left|\frac{q_m}{q_{m-2}}\right| \geqq 3-\sqrt{2}\ .$$

The assumption (e) is evident with $s=1$ (see figure-1). Finally we examine the additional conditions (f) and (g). From (30) and (31), we have

$$\left||\det D\psi_{a_1a_2\cdots a_m}(z)| - |\det D\psi_{a_1a_2\cdots a_m}(w)|\right|$$
$$\leqq Q_{13}\cdot|z-w|\cdot|q_m|^{-4}$$

and this implies (f).

Proposition 2. *It follows that*

$$\sum_{A\xi(m)} |q_m|^{-4} \leqq Q_{14}(\sqrt{2}/3)^{4m}.$$

Proof. In this transformation T, the partition ξ decomposes $X_{a_1a_2\cdots a_m}$ on the unit circles with centers $(1+i)$, $(1-i)$, $(-1+i)$, $(-1-i)$ respectively. Since these unit circles are symmetric about real line or imaginary line, it is sufficient to consider the unit circle with center $(1+i)$ (see figure-2). For $m=1$, the unit circle crosses X_{1+2i}, X_{2+2i} and X_{2+i}. Hence we have by induction

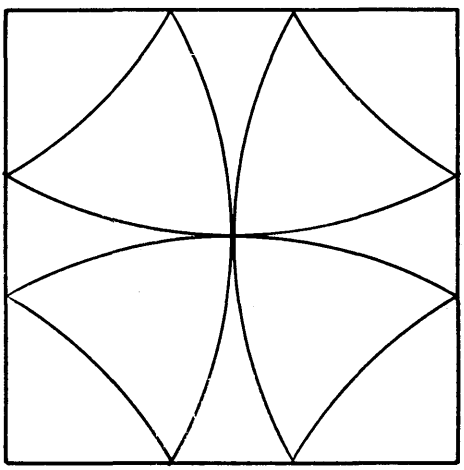

Fig. 2-(a) the partition ξ.

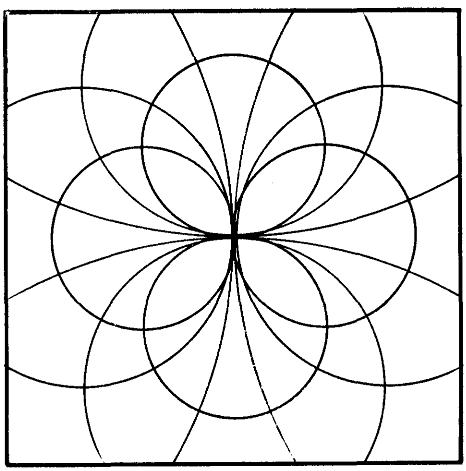

Fig. 2-(b) the partition $\{X_a\}$.

$$A_{\xi}(m)=\{(a_1a_2\cdots a_m);\ a_{2k+1}=2+2i,\ a_{2k}=-2+2i,\quad for\quad 1\leqq 2k,\ 2k+1\leqq m-1,$$
$$a_m=1+2i\ or\ 2+2i\ or\ 2+i,\quad if\ m\ is\ odd,$$
$$a_m=-1+2i\ or\ -2+2i\ or\ -2+2i,\quad if\ m\ is\ even.\}$$

Because of $q_m=q(a_1a_2...a_m)=a_1\cdot q(a_2a_3...a_m)+p(a_2a_3...a_m)$, we get

$$|q_m|=|a_1|\cdot|q(a_2a_3\cdots a_m)|\cdot\left|1+\frac{p(a_2a_3\cdots a_m)}{a_1\cdot q(a_2a_3\cdots a_m)}\right|$$
$$\geqq\sqrt{8}\cdot q(a_2a_3\cdots a_m)\cdot\frac{3}{4}.$$

Thus we have

$$\sum_{A_{\xi}(m)}|q_m|^{-4}<\sum_{A_{\xi}(m)}|q_{m-1}|^{-4}\left(\frac{\sqrt{2}}{3}\right)^4.$$

Since $\lambda(X_{a_1a_2\cdots a_n})\sim|q_m|^{-4}$, **Proposition 2** involves the condition (g).

As stated above, the transformation T satisfies (a), (b),$\cdots$(g). Consequently we have KUZMIN's formula as follows:

$$h_m(x)\leq H_0\cdot\rho(x)+Q_{15}((3-\sqrt{2})^{-m}).$$

Remark. In the case of $Q(\sqrt{-3})$ defined by HURWITZ, it is easy to show that the assumption (a), (b),$\cdots$, (e) is satisfied. So we can calculate the order of $\sigma(m)$ and examine the other conditions in the same way. Thus, three types of the complex continued fraction transformations are weak Bernoulli endomorphisms.

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