KEIO ENGINEERING REPORTS
VOL. 29, NO. 7, pp. 73-86, 1976

SOME ERGODIC PROPERTIES OF A COMPLEX CONTINUED FRACTION ALGORITHM

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(*Received May 7, 1976*)

ABSTRACT

Some ergodic properties of a continued fraction algorithm for complex numbers are given.

Recently R. Kaneiwa, J. Tamura and the author [2] proved, by making use of a certain kind of continued fraction algrithm for complex numbers, a theorem of Perron on complex Diophantine approximations [4]: *For any complex number* θ *not belonging to the imaginary quadratic field* $Q(\sqrt{-3})$ *there exist infinitely many integers* p, q *in* $Q(\sqrt{-3})$ *such that*

$$\left|\theta-\frac{p}{q}\right|<\frac{1}{\sqrt[4]{13}|q|^2}.$$

If $\theta=\frac{1}{2}(\zeta+\sqrt{\zeta^2+4})$, *where* $\zeta=\frac{1}{2}(1+\sqrt{-3})$, *the constant* $\sqrt[4]{13}$ *can not be improved.*

In this paper we shall investigate some ergodic properties of the complex continued fractions defined in [2].

1. Definition of the algorithm

Every complex number z can be uniquely written in the form $z=u\zeta+v\bar{\zeta}$, where u and v are real, and $\bar{w}$ is the complex conjugate of a complex number w. We put

$$[z]=[u]\zeta+[v]\bar{\zeta},$$

where, in the right-hand side, x is the largest rational integer not exceeding a real number x. Note that if z is real then $[z]$ becomes the ordinary Gauss' symbol. Thus we define a continued fraction algorithm as follows;

$$\left.\begin{array}{ll} r_n=r_n(z)=r_{n-1}^{-1}-[r_{n-1}^{-1}]\ (n\geq 1), & r_0=z-[z] \\ a_n=a_n(z)=[r_{n-1}^{-1}]\ (n\geq 1), & a_0=[z] \end{array}\right\} \tag{1}$$

These procedures terminate, i.e. $r_n=0$ for some $n\geq 0$, if and only if z belongs to $Q(\sqrt{-3})$. Hence every complex number z can be expanded in the form

$$z=a_0+\frac{1|}{|a_1}+\cdots+\frac{1|}{|a_n+r_n} \qquad (n\geq 0), \tag{2}$$

provided $r_k\neq 0$ for all $k<n$. We put

$$X=\{u\zeta+v\bar{\zeta};\ 0\leq u,\ v<1\},$$

and define a trnsformation T of X onto itself by

$$Tz=\frac{1}{z}-\left[\frac{1}{z}\right] \quad (z\in X), \tag{3}$$

which is an extension of the well-known '*continued fraction transformation*'

$$Tx=\frac{1}{x}-\left[\frac{1}{x}\right] \quad (x\in[0,1)) \tag{4}$$

(cf. [1]). Thus the remainder $r_n=r_n(z)$ in the algorithm (1) is the nth power of the transformation (3) (i.e. $r_n=T^n z$, for all $z\in X$.)

Now we exhibit some basic properties of the algorithm (1). Let Z_ζ be the ring of all integers in $Q(\sqrt{-3})$ and let N_ζ be the subset of Z_ζ defined by

$$N_\zeta=\{a\zeta+b\bar{\zeta};\ a, b \text{ non-negative integers with } a+b\geq 1\}.$$

We put

$$D=\{u\zeta+v\bar{\zeta};\ u, v\geq 0\},$$

and set

$$Y=D\backslash\{z;\ z^{-1}\in X\}.$$

Thus by the definitions we have

$$\begin{array}{l} \{a_0(z);\ z\in C\}=Z_\zeta, \\ \{a_n(z);\ z\in C\}=N_\zeta=(D\cap Z_\zeta)\backslash\{0\} \quad (n\geq 1), \end{array} \tag{5}$$

where C is the set of all complex numbers; and

$$\max_{z\in Y}|z|=\frac{2}{\sqrt{3}}, \tag{6}$$

$$\min_{z\in D\backslash X} |z| = \frac{\sqrt{3}}{2}, \tag{7}$$

Let $a_1, a_2, \cdots, a_n, \cdots$ be any sequence of complex numbers in $D\backslash\{0\}$ not necessarily integral. Every finite continued fraction

$$\frac{1}{|a_1} + \frac{1}{|a_2} + \cdots + \frac{1}{|a_n}$$

is well-defined, since the fractions $a_n^{-1}, a_{n-1}+a_n^{-1}, \cdots$ are different from zero. If, more precisely, $a_n \in D\backslash X$ for all $n\geq 1$, then $a_n^{-1} \in Y\backslash\{0\}$ and so $a_{n-1}+a_n^{-1} \in D\backslash X$. Repeating this process we get

$$a_1 + \frac{1}{|a_2} + \cdots + \frac{1}{|a_n} \in D\backslash X \tag{8}$$

and

$$\frac{1}{|a_1} + \frac{1}{|a_2} + \cdots + \frac{1}{|a_n} \in Y\backslash\{0\}. \tag{9}$$

Let $a_0 \in Z_\zeta$ and $a_n \in N_\zeta$ $(n\geq 1)$. Every finite continued fraction

$$a_0 + \frac{1}{|a_1} + \cdots + \frac{1}{|a_n}$$

has a canonical representation p_n/q_n $(p_n, q_n \in Z_\zeta)$, called nth approximant, in the form of an ordinary fraction. Especially if the sequence $a_0, a_1, \cdots$ is given by the algorithm (1) we call the fraction p_n/q_n the nth approximant of z. Thus from the general theory of finite continued fractions we have the following formulae: (For the proofs see [5].)

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2} \qquad (n\geq 1), \tag{10}$$

$$\frac{1}{|a_n} + \frac{1}{|a_{n-1}} + \cdots + \frac{1}{|a_1} = \frac{q_{n-1}}{q_n} \qquad (n\geq 1), \tag{11}$$

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \qquad (n\geq 0), \tag{12}$$

where $p_{-1}=1$, $q_{-1}=0$, $p_0=a_0$, $q_0=1$. Further if p_n/q_n is the nth approximant of z, then

$$z - \frac{p_n}{q_n} = (-1)^n \left(a_{n+1} + T^{n+1}z + \frac{q_{n-1}}{q_n} \right)^{-1} q_n^{-2}. \tag{13}$$

Lemma 1. (R. Kaneiwa, I. Shiokawa, and J. Tamura [2]) *Let $a_0 \in Z_\zeta$ and $a_n \in N_\zeta$ $(n\geq 1)$. Then we have*

$$|q_n| \to \infty \quad as \quad n \to \infty.$$

where q_n is the denominator of the nth approximant.

For completeness we prove this lemma.

Proof. Suppose, on the contrary, that $q_n \not\to \infty$ as $n \to \infty$. So we can choose an infinite subsequence $\{q_{n_j}\}_{j=1}^{\infty}$ such that $|q_{n_j}| < M$ for all $n \geq 1$, where M is a constant independent of j. But from (6) and (9) we have

$$\left|\frac{p_n}{q_n}\right| < \left|a_0 + \frac{2}{\sqrt{3}}\right|$$

and so

$$|p_{n_j}| < \left(|a_0| + \frac{2}{\sqrt{3}}\right)M,$$

where the right-hand side is also independent of j. It follows from these inequalities that $p_{n_j}/q_{n_j} = p_{n_k}/q_{n_k}$ for some j and k with $j<k$, since the ring of all integers in $Q(\sqrt{-3})$ is discrete. Hence we have

$$\frac{1}{|a_{n_j+1}} + \frac{1}{|a_{n_j+2}} + \cdots + \frac{1}{|a_{n_k}} = 0,$$

which contradicts (9).

Lemma 2. (ibid.) *Let z be any complex number not belonging to $Q(\sqrt{-3})$ and let p_n/q_n be its nth approximent. Then we have*

$$z = \lim_{n\to\infty} \frac{p_n}{q_n}.$$

Proof. By (13) as well as (7), (8), (11) we have

$$\left|z - \frac{p_n}{q_n}\right| < \frac{2}{\sqrt{3}} |q_n|^{-2},$$

which tend to zero as $n \to \infty$.

Lemma 3. (ibid) *With the same notations as in Lemma 1, the nth approximant p_n/q_n converges to some complex number which belongs to $a_0 + Y$.*

Proof. Similar to that of Lemma 2.

By means of Lemma 2, every complex number z can be expressed as an infinite regular continued fraction whose partial denominators $a_n(z)$ are integers in $Q(\sqrt{-3})$;

$$z = a_0(z) + \frac{1}{|a_1(z)} + \frac{1}{|a_2(z)} + \cdots.$$

This continued fraction expansion is a natural extension of the ordinary real one, since both algorithms coincide when z is real. (For further properties of the algorithm see [2] and [3].)

2. Admissible sequences and fundamental cells

We put

$$A^{(n)}=\{a_1(z)\cdots a_n(z);\ z\in X\}\qquad (1\leq n\leq\infty)$$

Sequences belonging to $A^{(n)}$ $(1\leq n\leq\infty)$ will be called admissible. (Note that Lemma 3 suggests the existence of non-admissible sequences.) For any $a_1\cdots a_n\in A^{(n)}$ we define

$$X_{a_1\cdots a_n}=\{z\in X;\ a_k(z)=a_k,\ 1\leq k\leq n\},$$

which will be called a fundamental cell of rank n. Thus we have

$$X=\bigcup_{a_1\cdots a_n\in A^{(n)}} X_{a_1\cdots a_n}$$

where $X_{a_1\cdots a_n}\cap X_{b_1\cdots b_n}=\phi$ if $a_k\neq b_k$ for some k with $1\leq k\leq n$; i.e. the set of all fundamental cells of rank n forms a partition of X. Besides, for any fixed infinite admissible sequence $a_1a_2\cdots$ we find

$$X\supset X_{a_1}\supset\cdots\supset X_{a_1\cdots a_{n-1}}\supset X_{a_1\cdots a_n}$$

and (by Lemma 2)

$$\text{diam}\,(X_{a_1\cdots a_n})\to 0\qquad \text{as } n\to\infty.$$

Hence every Lebesgue measurable subset of X may be approximated with any accuracy by finite unions of mutually disjoint fundamental cells.

For any given $a_1\cdots a_n\in A^{(n)}$ we define a function of z by

$$\psi_{a_1\cdots a_n}(z)=\frac{1\ |}{|\ a_1}+\cdots+\frac{1\ |}{|\ a_{n-1}}+\frac{1\ |}{|\ a_n+z}$$

or equivalently

$$=\frac{p_n+p_{n-1}z}{q_n+q_{n-1}z}\qquad (z\in X).\tag{14}$$

Because of the formula (12) the linear transformation $\psi_{a_1\cdots a_n}$ has the inverse

$$(\psi_{a_1\cdots a_n})^{-1}(z)=\frac{p_n-q_nz}{-p_{n-1}+q_{n-1}z}\qquad (z\in\psi_{a_1\cdots a_n}(X)).\tag{15}$$

But the equality (2) can be rewritten in the form

$$z=\psi_{a_1\cdots a_n}(T^nz)\qquad (z\in X).$$

Hence for each $a_1\cdots a_n\in A^{(n)}$ the nth power of T restricted on the cell $X_{a_1\cdots a_n}$ is identical with the inverse of $\psi_{a_1\cdots a_n}$; i.e.

$$T^nz=(\psi_{a_1\cdots a_n})^{-1}(z)\qquad (z\in X_{a_1\cdots a_n}).\tag{16}$$

Especially we have

$$X_{a_1\cdots a_n}=\psi_{a_1\cdots a_n}(T^n X_{a_1\cdots a_n}). \tag{17}$$

Now we need some notations: Put

$$U_1=\left\{z\in X;\ \left|z+\frac{\sqrt{-3}}{3}\right|>\frac{\sqrt{3}}{3}\right\},$$

$$U_2=\{z\in X;\ \mathrm{Im}(z)>0\},$$

$$U_3=\{z\in X;\ \bar{z}\in U_1,\ \mathrm{Im}(z)>0\},$$

and define

$$U_{-j}=\{\bar{z};\ z\in U_j\}\qquad (j=1,2,3).$$

Further we set $U_0=X$ for notational convenience. Considering the reciprocals $U_j^{-1}=\{z;\ z^{-1}\in U_j\}$, we obtain (see Fig. 1)

$$X=\psi_\zeta(U_1)\cup\psi_{\bar\zeta}(U_{-1})\cup\Big(\bigcup_{\substack{a\in N_\zeta\\ a\neq\zeta,\bar\zeta}}\psi_a(X)\Big), \tag{18.0}$$

$$U_1=\psi_\zeta(U_{-3})\cup\psi_{\bar\zeta}(U_{-1})\cup\Big(\bigcup_{k=1}^{\infty}\psi_{\zeta+k}(U_{-2})\Big)\cup\Big(\bigcup_{\substack{a\in N_\zeta,\,a\neq\bar\zeta\\ \mathrm{Im}(a)\leq 0}}\psi_a(X)\Big), \tag{18.1}$$

$$U_2=\psi_{\bar\zeta}(U_{-1})\cup\Big(\bigcup_{k=1}^{\infty}\psi_k(U_{-2})\Big)\cup\Big(\bigcup_{\substack{a\in N_\zeta,\,a\neq\zeta\\ \mathrm{Im}(a)<0}}\psi_a(X)\Big), \tag{18.2}$$

and

$$U_3=\psi_{\bar\zeta}(U_3)\cup\bigcup_{k=1}^{\infty}(\psi_k(U_{-2})\cup\psi_{\bar\zeta+k}(U_2)). \tag{18.3}$$

Taking the complex conjugate of (18.1)-(18.3) we have also the same relations for U_{-1}, U_{-2}, and U_{-3} to which we assign (18.-1), (18.-2), and (18.-3) resp.

In any case U_j can be written in the form

$$U_j=\bigcup_{a\in M_j}\psi_a(U_k) \tag{19}$$

where M_j is a subset of N_ζ and $k(-3\leq k\leq 3)$ are chosen uniquely according as j and a. In addition, we note that

$$\psi_a(X)\cap\psi_b(X)=\phi \tag{20}$$

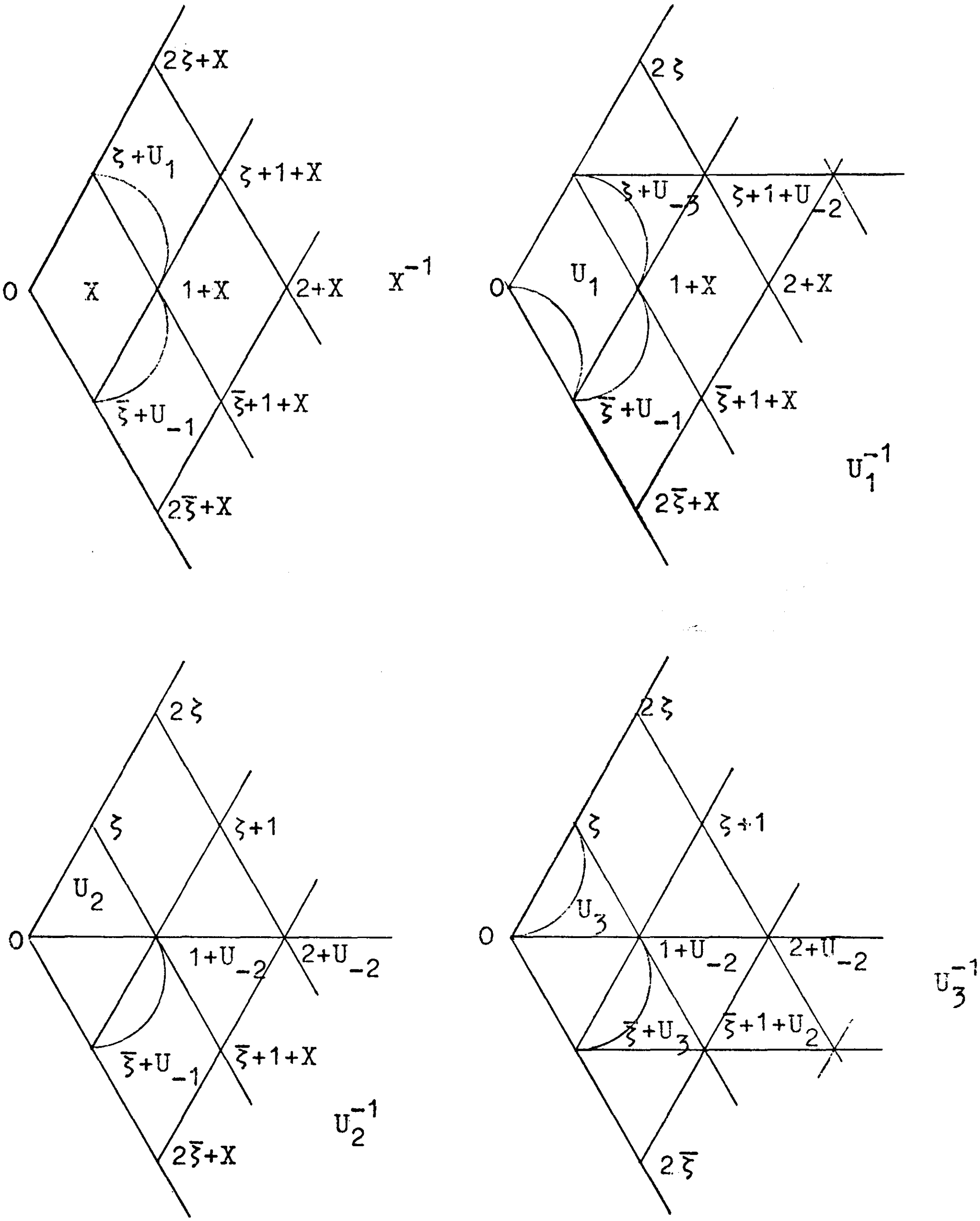
whenever $a\neq b(a, b\in N_\zeta)$.

LEMMA 4. *Let $n\geq 1$ and let $a_1\cdots a_n\in A^{(n)}$. Then we have*

$$X_{a_1\cdots a_n}=\psi_{a_1\cdots a_n}(U_j), \tag{21}$$

and so

$$T^n X_{a_1\cdots a_n}=U_j \tag{22}$$



2ζ+X
ζ+U1
ζ+1+X
0
X
1+X
2+X
X-1
ζ̄+U-1
ζ̄+1+X
2ζ̄+X
2ζ
ζ+U-3
ζ+1+U-2
U1
1+X
2+X
ζ̄+U-1
ζ̄+1+X
2ζ̄+X
U1-1
2ζ
ζ
ζ+1
U2
1+U-2
2+U-2
ζ̄+U-1
ζ̄+1+X
2ζ̄+X
U2-1
2ζ
ζ
ζ+1
U3
1+U-2
2+U-2
ζ̄+U3
ζ̄+1+U2
2ζ̄
U3-1

Fig. 1

for some j $(-3\leq j\leq 3)$.

Proof. By induction on n. First we prove (21). If $n=1$ (21) follows from (18.0). Suppose that (21) holds for all $a_1\cdots a_n\in A^{(n)}$. Then we have

$$\begin{aligned}X_{a_1\cdots a_{n+1}}&=\{z\in X_{a_1\cdots a_n};\ a_{n+1}(z)=a_1(T^n z)=a_{n+1}\}\\&=\{\psi_{a_1\cdots a_n}(w);\ w\in U_j,\ a_1(w)=a_{n+1}\}\\&=\psi_{a_1\cdots a_n}(\psi_{a_{n+1}}(U_k)),\ (\text{by }(14),(15))\\&=\psi_{a_1\cdots a_{n+1}}(U_k),\end{aligned}$$

for $a_1\cdots a_{a+1}\in A^{(n+1)}$, where j is defined by $U_j=T^nX_{a_1\cdots a_n}$ and k chosen uniquely in (19). (22) follows from (17) and (21).

Let E be any subset of X. Then by Lemma 4 we have

$$\begin{aligned}T^{-n}E&=\{z\in X;\ T^n z\in E\}\\&=\bigcup_{a_1\cdots a_n\in A^{(n)}}\{z\in X_{a_1\cdots a_n};\ T^n z\in E\cap T^n X_{a_1\cdots a_n}\}\\&=\bigcup_{a_1\cdots a_n\in A^{(n)}}\psi_{a_1\cdots a_n}(E\cap T^n X_{a_1\cdots a_n}).\end{aligned}\tag{23}$$

3. Estimates of the Lebesgue measure

Let m be the Lebesgue measure on the complex plane and let $\mathfrak{B}$ be the σ-field of all measurable subsets of X. Then we have for any $a_1\cdots a_n\in A^{(n)}$ and any $E\in\mathfrak{B}$

$$m(\psi_{a_1\cdots a_n}(E))=\iint_E|\psi'_{a_1\cdots a_n}(z)|^2dxdy,\qquad z=x+iy.\tag{24}$$

But using (12) we find

$$|\psi'_{a_1\cdots a_n}(z)|^2=|q_n|^{-4}\left|1+\frac{q_{n-1}}{q_n}z\right|^{-4}.\tag{25}$$

Hence

$$3^{-4}<|q_n|^4|\psi'_{a_1\cdots a_n}(z)|^2<3^4\tag{26}$$

and

$$3^{-4}<|q_n|^{-4}|(\psi^{-1}_{a_1\cdots a_n})'(z)|^2<3^4,\tag{27}$$

because (from (6), (7), (9), (11))

$$3^{-1}<\frac{\sqrt{3}}{2}\leq\left|1+\frac{q_{n-1}}{q_n}z\right|\leq 1+\frac{2}{\sqrt{3}}<3.\tag{28}$$

(23) and (25) give the estimates

$$3^{-4}m(E)<|q_n|^4m(\psi_{a_1\cdots a_n}(E))<3^4m(E)\qquad(E\in\mathfrak{B})\tag{29}$$

Eepecially, taking account of the fact that $3^{-2}<m(U_j)<1$ $(-3\leq j\leq 3)$, we have

$$3^{-6}<|q_n|^4 m(X_{a_1\cdots a_n})<3^4 \qquad (a_1\cdots a_n\in A^{(n)}). \tag{30}$$

We write

$$S(n)=\sum_{a_1\cdots a_n\in A^{(n)}}|q_n|^{-4}.$$

Then for any $n\geq 1$ we have

$$3^{-5}<S(n)<3^6. \tag{31}$$

Indeed it follows from (30) that

$$3^4 S(n)>\sum_{A^{(n)}} m(X_{a_1\cdots a_n})=m(X)>3^{-1}$$

and

$$3^{-6}S(n)<m(X)<1.$$

By means of Lemma 4 the set $A^{(n)}$ of all admissible sequences can naturally be divided into seven subsets; we put

$$A_j^{(n)}=\{a_1\cdots a_n\in A(n);\ T^n X_{a_1\cdots a_n}=U_j\} \qquad (-3\leq j\leq 3),$$

then

$$A^{(n)}=\bigcup_{j=-3}^{3} A_j^{(n)}.$$

By (18.j) $(-3\leq j\leq 3)$ we have the following relations for $n>1$;

$$\begin{aligned} A_0^{(n)}=\{a_1\cdots a_n\in A^{(n)};\ & a_1\cdots a_{n-1}\in A_0^{(n-1)},\ a_n\neq\zeta,\bar{\zeta};\\ \text{or } & a_1\cdots a_{n-1}\in A_1^{(n-1)},\ a_n\neq\bar{\zeta},\ \mathrm{Im}\,(a_n)\leq 0;\\ \text{or } & a_1\cdots a_{n-1}\in A_{-1}^{(n-1)},\ a_n\neq\zeta,\ \mathrm{Im}\,(a_n)\geq 0;\\ \text{or } & a_1\cdots a_{n-1}\in A_2^{(n-1)},\ a_n\neq\bar{\zeta},\ \mathrm{Im}\,(a_n)<0;\\ \text{or } & a_1\cdots a_{n-1}\in A_{-2}^{(n-1)},\ a_n\neq\zeta,\ \mathrm{Im}\,(a_n)>0\}, \end{aligned} \tag{32.0}$$

$$A_1^{(n)}=\{a_1\cdots a_n\in A^{(n)};\ a_1\cdots a_{n-1}\in A_0^{(n-1)}\cup A_{-1}^{(n-1)}\cup A_{-2}^{(n-1)},\ a_n=\zeta\}, \tag{32.1}$$

$$\begin{aligned} A_2^{(n)}=\{a_1\cdots a_n\in A^{(n)};\ & a_1\cdots a_{n-1}\in A_{-1}^{(n-1)}\cup A_3^{(n-1)},\ a_n-\zeta\in N;\\ \text{or } & a_1\cdots a_{n-1}\in A_{-2}^{(n-1)}\cup A_{-3}^{(n-1)},\ a_n\in N\}, \end{aligned} \tag{32.2}$$

$$A_3^{(n)}=\{a_1\cdots a_n\in A^{(n)};\ a_1\cdots a_{n-1}\in A_{-1}^{(n-1)}\cup A_{33}^{(n-1)},\ a_n=\bar{\zeta}\}, \tag{32.3}$$

and

$$A_{-j}^{(n)}=\{\bar{a}_1\cdots\bar{a}_n;\ a_1\cdots a_n\in A_j^{(n)}\} \qquad (j=1,2,3), \tag{32.-j}$$

where N is the set of all positive integers.

We write

$$S_j(n) = \sum_{a_1 \cdots a_n \in A_j^{(n)}} |q_n|^{-4} \qquad (-3 \leq j \leq 3).$$

Thus we have

$$S_j(n) = S_{-j}(n) \qquad (-3 \leq j \leq 3), \tag{33}$$

and

$$S(n) = \sum_{j=-3}^{3} S_j(n). \tag{34}$$

LEMMA 5. *For any j $(-3 \leq j \leq 3)$ we have*

$$S_j(n) > 3^{-12} \qquad (n \geq 3). \tag{35}$$

Proof. Let $a_1 \cdots a_n \in A^{(n)}$ and let p_n/q_n the nth approximant. Then it follows from (10) and (28) that

$$3^{-1}|a_n q_{n-1}| < |q_n| < 3|a_n q_{n-1}|. \tag{36}$$

By (32.0), (33) and (36) we have

$$\begin{aligned} 3S_0(n) > & \sum_{a \in N_\zeta \setminus \{\zeta, \bar{\zeta}\}} |a|^{-4} \sum_{a_1 \cdots a_{n-1} \in A_0^{(n-1)}} |q_{n-1}|^{-4} \\ & + 2 \sum_{\substack{a \in N_\zeta \setminus \{\bar{\zeta}\} \\ \operatorname{Im}(a) \leq 0}} |a|^{-4} \sum_{a_1 \cdots a_{n-1} \in A_1^{(n-1)}} |q_{n-1}|^{-4} \\ & + 2 \sum_{\substack{a \in N_\zeta \setminus \{\bar{\zeta}\} \\ \operatorname{Im}(a) < 0}} |a|^{-4} \sum_{a_1 \cdots a_{n-1} \in A_2^{(n-1)}} |q_{n-1}|^{-4} \\ & > S_0(n-1) + 2S_1(n-1) + 2|\zeta + 1|^{-4} S_2(n-1) \\ & > 3^{-2}(S_0(n-1) + S_1(n-1) + S_2(n-1)). \end{aligned}$$

Hence

$$S_0(n) > 3^{-3}(S_0(n-1) + S_1(n-1) + S_2(n-1)).$$

In the same way we get

$$S_1(n) > 3^{-1}(S_0(n-1) + S_1(n-1) + S_2(n-1)),$$

$$S_2(n) > 3^{-1}(S_1(n-1) + S_2(n-1) + S_3(n-1)),$$

and

$$S_3(n) > 3^{-1}(S_1(n-1) + S_3(n-1))$$

(using (32.1)-(32.3)). These inequalities as well as (31), (33), and (34) imply that

$$S_0(n) > 3^{-6} \sum_{j=0}^{3} S_j(n-2) > 3^{-7} S(n) > 3^{-12}.$$

Similarly we may obtain (35) for any $j (-3 \leq j \leq 3)$.

In what follows we shall use (35) only with $j = 0$.

4. Invariant measure and ergodicity

THEOREM 1. *Let E be any measurable subset of X such that $T^{-1}E=E$. Then $m(E)=0$ or $m(X)$.*

Proof. We assume that $m(E)>0$. By Lemma 4 and (23) we find for any $a_1\cdots a_n \in A^{(n)}$

$$\begin{aligned} E\cap X_{a_1\cdots a_n} &= T^{-n}E\cap \psi_{a_1\cdots a_n}(T^n X_{a_1\cdots a_n}) \\ &= \psi_{a_1\cdots a_n}(E\cap T^n X_{a_1\cdots a_n}). \end{aligned}$$

From this as well as (29) and (30) we have

$$\begin{aligned} m(E\cap X_{a_1\cdots a_n}) &\geq 3^{-4}|q_n|^{-4} m(E\cap T^n X_{a_1\cdots a_n}) \\ &\geq 3^{-8} m(X_{a_1\cdots a_n}) \min\{m(E\cap U_3), m(E\cap U_{-3})\}. \end{aligned} \tag{37}$$

But (18.3) and (23) implies that

$$E\cap U_3 = T^{-1}E\cap U_3 \supset \psi_{\zeta+1}(E\cap U_2)\cup \psi_1(E\cap U_{-2}).$$

Beside for any measurable subset F of U_2 we have by (24) with (25)

$$m(\psi_1(F)) = \iint_F |1+z|^{-4}dxdy > \iint_F |\zeta+1+z|^{-4}dxdy = m(\psi_{\zeta-1}(F)).$$

Hence

$$\begin{aligned} m(E\cap U_3) &> m(\psi_{\zeta+1}(E\cap U_2)) + m(\psi_{\zeta+1}(E\cap U_{-2})) \\ &= m(\psi_{\zeta+1}(E)) > 3^{-4}|\zeta+1|^{-4} m(E) = 3^{-6} m(E). \end{aligned} \tag{38}$$

(using (29)). Similarly we get

$$m(E\cap U_{-3}) > 3^{-6} m(E). \tag{39}$$

By (37), (38), and (39) the inequality

$$m(E\cap F) \geq 3^{-14} m(E) m(F). \tag{40}$$

hold for all fundamental cell F, and so for any measurable set F in X. Thus, putting $F=X\setminus E$ in (40), we have

$$m(E)m(X\setminus E)=0,$$

which implies $m(E)=m(X)$.

THEOREM 2. *There exists an unique, T-invariant probability measure μ equivalent to Lebesgue measure such that the inequalities*

$$3^{-15}\frac{m(E)}{m(X)} \leq \mu(E) \leq 3^{10}\frac{m(E)}{m(X)} \tag{41}$$

hold for all $E\in\mathfrak{B}$.

Proof. To prove the existence it is enough to show that the inequalities

$$3^{-15}m(E)<m(T^{-n}E)<3^{10}m(E) \qquad (E\in\mathfrak{B}) \tag{42}$$

hold for all $n\geq 0$ (see *F.* Schweiger [5] §6–§7). By (23), (29), and (31) we have

$$\begin{aligned} m(T^{-n}E) &< \sum_{A^{(n)}} m(\psi_{a_1\cdots a_n}(E)) \\ &\leq 3^4 m(E) S(n) \leq 3^{10} m(E) \end{aligned}$$

To prove the left-hand side inequalities in (42), we suppose first that $E\subset U_3$. Then, by (23), (29), and Lemma 5,

$$m(T^{-n}E)\geq \sum_{A_0^{(n)}} m(\psi_{a_1\cdots a_n}(E))\geq 3^{-16}m(E),$$

as required. In the same way, these inequalities hold for any $E\subset U_2\backslash U_3$. The left-hand side of the inequalities (42) is also true for any subset E of U_{-3} or $U_{-2}\backslash U_{-3}$. As a result (42) holds for any subset E of X, since

$$E=(E\cap U_3)\cup(E\cap(U_2\backslash U_3))\cup(E\cap U_{-2})\cup(E\cap(U_{-2}\backslash U_{-3})).$$

By Theorem 1 T-invariant probability measure μ is uniquely given by the limit

$$\mu(E)=\frac{1}{m(X)}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} m(T^{-k}E) \qquad (E\in\mathfrak{B}) \tag{43}$$

(see F. Schweiger [5]). So (41) follows from (42) and (43).

THEOREM 3. *T is ergodic with respect to μ; i.e. for any $f\in L^1(X)$ we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k z)=\int_X f(z)d\mu, \quad a.e.$$

Proof. Follows from Theorem 1, 2 and Birkhoff's individual ergodic theorem.

As an application of Theorem 3, we have

$$\lim_{n\to\infty}(a_1(z)\cdots a_n(z))^{1/n}=e^{\alpha}, \quad a.e.$$

where

$$\alpha=\int_X \log a_1(z)d\mu.$$

(Note that $f(z)=\log a_1(z)\in L^1(X)$, where $-\frac{\pi}{3}\leq \arg f(z)\leq\frac{\pi}{3}$, since the series $\sum_{a\in N_\zeta} a^{-4}\log a$ is convergent.)

5. Exactness

A measure-preserving transformation T on a normalized measure space $(X, \mathfrak{B}, \mu)$

is said to be *exact* if

$$\bigcap_{n=0}^{\infty} T^{-n}\mathfrak{B}=\{\phi, X\},$$

or equivalently, if for every set E of positive measure with the measurable images $TE, T^2E, \cdots$ the relation

$$\lim_{n\to\infty} \mu(T^nE)=1 \tag{44}$$

holds (see V.A. Rohlin [6].)

Theorem 4. *The transformation T is exact.*
The proof requires the following

Lemma 6. *Let $\varepsilon>0$ and let E be any measurable set such that*

$$\mu(U_j\backslash E)<\varepsilon$$

for some j $(-3\leq j\leq 3)$. Then

$$\mu(TE)>1-3^{31}\varepsilon.$$

Proof of Lemma 6. It is clearly enough to consider only the case $j=\pm 3$. Further we may assume $j=3$, since the following arguments are available for the conjugate case $j=-3$. Note first that

$$\psi_1(U_{-2})\cup\psi_{\bar{\zeta}+1}(U_2)\subset U_3, \tag{45}$$

(by (18.3)). By (41) and (27) with $n=1$ and $a_n=1$, we get

$$\begin{aligned}\mu(T(\psi_1(U_{-2})\backslash E))&\leq 3^{10}m(X)^{-1}m(T(\psi_1(U_{-2})\backslash E))\\ &\leq 3^{14}m(X)^{-1}m(\psi_1(U_{-2})\backslash E)\leq 3^{29}\mu(\psi_1(U_{-2})\backslash E).\end{aligned} \tag{46}$$

while, using (27) with $n=1$ and $a_1=\bar{\zeta}+1$,

$$\mu(T(\psi_{\bar{\zeta}+1}(U_2)\backslash E))\leq 3^{31}\mu(\psi_{\bar{\zeta}+1}(U_2)\backslash E)). \tag{47}$$

Combining (45), (46), and (47) we find

$$\begin{aligned}&\mu(T((\psi_1(U_{-2})\cup\psi_{\bar{\zeta}+1}(U_2))\backslash E))\\ &\quad\leq 3^{31}\mu((\psi_1(U_{-2})\cup\psi_{\bar{\zeta}+1}(U_2))\backslash E)\\ &\quad\leq 3^{31}\mu(U_3\backslash E)\leq 3^{31}\varepsilon.\end{aligned} \tag{48}$$

Therefore, by (45), (48), and (18.3), we obtain

$$\begin{aligned}\mu(TE)&\geq\mu(T((\psi_1(U_{-2})\cup\psi_{\bar{\zeta}+1}(U_2))\backslash E))\\ &\geq\mu(T(\psi_1(U_{-2})\cup\psi_{\bar{\zeta}+1}(U_2)))-\mu(T((\psi_1(U_{-2})\cup\psi_{\bar{\zeta}+1}(U_2))\backslash E))\\ &>1-3^{31}\varepsilon.\end{aligned}$$

Proof of Theorem 3. We prove (44). Let $E\in\mathfrak{B}$ given arbitrary. (Note that, by the definition of T, $E\in\mathfrak{B}$ if and only if $TE\in\mathfrak{B}$.) Let $\varepsilon>0$. Then there exists a fundamental interval $F=X_{a_1\cdots a_n}$ such that

$$m(F\backslash E)<3^{-50}\varepsilon m(F). \tag{49}$$

Otherwise, the inequality

$$m(F\backslash E)\geq 3^{-50}\varepsilon m(F)$$

holds for all fundamental interval F, and so it holds also for arbitrary measurable set F. Putting $F=E$ we have $m(F)=0$; a contradiction.

Using Lemma 4, (41), (16), (27), (30), and (49), we get

$$\begin{aligned}\mu(T^nF\backslash T^nE)&\leq\mu(T^n(F\backslash E))\\&\leq 3^{11}m(T^n(F\backslash E))\leq 3^{15}|q_n|^4m(F\backslash E)\\&\leq 3^{19}m(F)^{-1}m(F\backslash E)<3^{-31}\varepsilon.\end{aligned} \tag{50}$$

Noticing that $T^nF=U_j$ for some j by Lemma 4, we have from (50) and Lemma 6

$$\mu(T^{n+1}E)>1-\varepsilon.$$

Since $\mu(E), \mu(TE), \mu(T^2E), \cdots$ is non-decreasing, the relation (44) is proved.

As a general property of exact transformations (see V.A. Rohlin [4]) we have

Corollary. *The transformation T is mixing of all degrees. In particular T is strongly mixing; i.e. for any $E, F\in\mathfrak{B}$ we have*

$$\lim_{n\to\infty}\mu(T^{-n}E\cap F)=\mu(E)\mu(F).$$

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