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SOME ERGODIC PROPERTIES OF A COMPLEX CONTINUED FRACTION ALGORITHM

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ABSTRACT

Some ergodic properties of a continued fraction algorithm for complex numbers are given.

Recently R. Kaneiwa, J. Tamura and the author [2] proved, by making use of a certain kind of continued fraction algrithm for complex numbers, a theorem of Perron on complex Diophantine approximations [4]: For any complex number θ not belonging to the imaginary quadratic field $Q(\sqrt{-3})$ there exist infinitely many integers p, q in $Q(\sqrt{-3})$ such that

$$\left| \theta - rac{p}{q} \right| < rac{1}{\sqrt[4]{13}|q|^2}$$

If $\theta = \frac{1}{2}(\zeta + \sqrt{\zeta^2 + 4})$, where $\zeta = \frac{1}{2}(1 + \sqrt{-3})$, the constant $\sqrt[4]{13}$ can not be improved.

In this paper we shall investigate some ergodic properties of the complex continued fractions defined in [2].

1. Definition of the algorithm

Every complex number z can be uniquely written in the form $z = u\zeta + v\overline{\zeta}$, where u and v are real, and \overline{w} is the complex conjugate of a complex number w. We put

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$$[z] = [u]\zeta + [v]\zeta,$$

where, in the right-hand side, x is the largest rational integer not exceeding a real number x. Note that if z is real then [z] becomes the ordinary Gauss' symbol. Thus we define a continued fraction algorithm as follows;

$$r_{n} = r_{n}(z) = r_{n-1}^{-1} - [r_{n-1}^{-1}] \quad (n \ge 1), \quad r_{0} = z - [z] \\ a_{n} = a_{n}(z) = [r_{n-1}^{-1}] \quad (n \ge 1), \quad a_{0} = [z]$$

$$(1)$$

These procedures terminate, i.e. $r_n=0$ for some $n \ge 0$, if and only if z belongs to $Q(\sqrt{-3})$. Hence every complex number z can be expanded in the form

$$z = a_0 + \frac{1}{|a_1|} + \dots + \frac{1}{|a_n + r_n|} \qquad (n \ge 0),$$
(2)

provided $r_k \neq 0$ for all k < n. We put

$$X = \{ u\zeta + v\zeta; 0 \le u, v < 1 \},\$$

and define a trnsformation T of X onto itself by

$$Tz = \frac{1}{z} - \left[\frac{1}{z}\right] \quad (z \in X), \tag{3}$$

which is an extension of the well-known 'continued fraction transformation'

$$Tx = \frac{1}{x} - \left[\frac{1}{x}\right] \quad (x \in [0, 1)) \tag{4}$$

(cf. [1]). Thus the remainder $r_n = r_n(z)$ in the algorithm (1) is the *n*th power of the transformation (3) (i.e. $r_n = T^n z$, for all $z \in X$.)

Now we exhibit some basic properties of the algorithm (1). Let Z_{ζ} be the ring of all integers in $Q(\sqrt{-3})$ and let N_{ζ} be the subset of Z_{ζ} defined by

$$N_{\zeta} = \{a\zeta + b\overline{\zeta}; a, b \text{ non-negative integers with } a+b \ge 1\}.$$

We put

$$D = \{ u\zeta + v\bar{\zeta}; u, v \ge 0 \},$$

and set

$$Y=D\backslash\{z; z^{-1}\in X\}.$$

Thus by the definitions we have

$$\{a_0(z); z \in C\} = Z_{\zeta},$$

$$\{a_n(z); z \in C\} = N_{\zeta} = (D \cap Z_{\zeta}) \setminus \{0\} \quad (n \ge 1),$$

(5)

where C is the set of all complex numbers; and

$$\max_{z \in Y} |z| = \frac{2}{\sqrt{3}}, \qquad (6)$$

$$\min_{z \in D \setminus X} |z| = \frac{\sqrt{3}}{2}, \tag{7}$$

Let $a_1, a_2, \dots, a_n, \dots$ be any sequence of complex numbers in $D \setminus \{0\}$ not necessarily integral. Every finite continued fraction

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$$

is well-defined, since the fractions $a_n^{-1}, a_{n-1} + a_n^{-1}, \cdots$ are different from zero. If, more precisely, $a_n \in D \setminus X$ for all $n \ge 1$, then $a_n^{-1} \in Y \setminus \{0\}$ and so $a_{n-1} + a_n^{-1} \in D \setminus X$. Repeating this process we get

$$a_1 + \frac{1}{a_2} + \dots + \frac{1}{a_n} \epsilon D \setminus X \tag{8}$$

and

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \in Y \setminus \{0\}.$$
(9)

Let $a_0 \in \mathbb{Z}_{\zeta}$ and $a_n \in \mathbb{N}_{\zeta}$ $(n \ge 1)$. Every finite continued fraction

$$a_0 + \frac{1}{a_1} + \dots + \frac{1}{a_n}$$

has a canonical representation p_n/q_n ($p_n, q_n \in Z_{\zeta}$), called *n*th approximant, in the form of an ordinary fraction. Especially if the sequence a_0, a_1, \cdots is given by the algorithm (1) we call the fraction p_n/q_n the *n*th approximant of *z*. Thus from the general theory of finite continued fractions we have the following formulae: (For the proofs see [5].)

$$p_n = a_n p_{n-1} + p_{n-2}, \ q_n = a_n q_{n-1} + q_{n-2} \qquad (n \ge 1), \tag{10}$$

$$\frac{1}{a_n} + \frac{1}{a_{n-1}} + \dots + \frac{1}{a_1} = \frac{q_{n-1}}{q_n} \qquad (n \ge 1),$$
(11)

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \qquad (n \ge 0), \tag{12}$$

where $p_{-1}=1$, $q_{-1}=0$, $p_0=a_0$, $q_0=1$. Further if p_n/q_n is the *n*th approximant of *z*, then

$$z - \frac{p_n}{q_n} = (-1)^n \left(a_{n+1} + T^{n+1}z + \frac{q_{n-1}}{q_n} \right)^{-1} q_n^{-2}.$$
 (13)

LEMMA 1. (R. Kaneiwa, I. Shiokawa, and J. Tamura [2]) Let $a_0 \in Z_{\zeta}$ and $a_n \in N_{\zeta}$ ($n \ge 1$). Then we have

$$|q_n| \to \infty$$
 as $n \to \infty$.

where q_n is the denominator of the nth approximant.

For completeness we prove this lemma.

Proof. Suppose, on the contrary, that $q_n \to \infty$ as $n \to \infty$. So we can choose an infinite subsequence $\{q_{n_j}\}_{j=1}^{\infty}$ such that $|q_{n_j}| < M$ for all $n \ge 1$, where M is a constant independent of j. But from (6) and (9) we have

$$\left|\frac{p_n}{q_n}\right| < \left|a_0 + \frac{2}{\sqrt{3}}\right|$$

and so

$$|p_{n_j}| < \left(|a_0| + \frac{2}{\sqrt{3}}\right)M,$$

where the right-hand side is also independent of j. It follows from these inequalities that $p_{n_j}|q_{n_j}=p_{n_k}|q_{n_k}$ for some j and k with j < k, since the ring of all integers in $Q(\sqrt{-3})$ is discrete. Hence we have

$$\frac{1}{a_{n_j}+1} + \frac{1}{a_{n_j}+2} + \dots + \frac{1}{a_{n_k}} = 0,$$

which contradicts (9).

LEMMA 2. (ibid.) Let z be any complex number not belonging to $Q(\sqrt{-3})$ and let p_n/q_n be its nth approximent. Then we have

$$z=\lim_{n\to\infty}\frac{p_n}{q_n}.$$

Proof. By (13) as well as (7), (8), (11) we have

$$\left|z-\frac{p_n}{q_n}\right|<\frac{2}{\sqrt{3}}|q_n|^{-2},$$

which tend to zero as $n \to \infty$.

LEMMA 3. (ibid) With the same notations as in Lemma 1, the nth approximant p_n/q_n converges to some complex number which belongs to $a_0 + Y$.

Proof. Similar to that of Lemma 2.

By means of Lemma 2, every complex number z can be expressed as an infinite regular continued fraction whose partial denominators $a_n(z)$ are integers in $Q(\sqrt{-3})$;

$$z=a_0(z)+\frac{1}{|a_1(z)|}+\frac{1}{|a_2(z)|}+\cdots.$$

This continued fraction expansion is a natural extension of the ordinary real one, since both algorithms coincide when z is real. (For further properties of the algorithm see [2] and [3].)

2. Admissible sequences and fundamental cells

We put

$$A^{(n)} = \{a_1(z) \cdots a_n(z); z \in X\} \qquad (1 \le n \le \infty)$$

Sequences belonging to $A^{(n)}$ $(1 \le n \le \infty)$ will be called admissible. (Note that Lemma 3 suggests the existence of non-admissible sequences.) For any $a_1 \cdots a_n \in A^{(n)}$ we define

$$X_{a_1\cdots a_n} = \{z \in X; a_k(z) = a_k, 1 \le k \le n\},\$$

which will be called a fundamental cell of rank n. Thus we have

$$X = \bigcup_{a_1 \cdots a_n \in A^{(n)}} X_{a_1 \cdots a_n}$$

where $X_{a_1\cdots a_n} \cap X_{b_1\cdots b_n} = \phi$ if $a_k \neq b_k$ for some k with $1 \le k \le n$; i.e. the set of all fundamental cells of rank n forms a partition of X. Besides, for any fixed infinite admissible sequence $a_1a_2\cdots$ we find

$$X \supset X_{a_1} \supset \cdots \supset X_{a_1 \cdots a_{n-1}} \supset X_{a_1 \cdots a_n}$$

and (by Lemma 2)

diam
$$(X_{a_1\cdots a_n}) \to 0$$
 as $n \to \infty$.

Hence every Lebesgue measurable subset of X may be approximated with any accuracy by finite unions of mutually disjoint fundamental cells.

For any given $a_1 \cdots a_n \in A^{(n)}$ we define a function of z by

$$\psi_{a_1\cdots a_n}(z) = \frac{1}{|a_1|} + \cdots + \frac{1}{|a_{n-1}|} + \frac{1}{|a_n+z|}$$

or equivalently

$$=\frac{p_n+p_{n-1}z}{q_n+q_{n-1}z} \qquad (z \in X).$$
(14)

Because of the formula (12) the linear transformation $\psi_{a_1\cdots a_n}$ has the inverse

$$(\psi_{a_{1}\cdots a_{n}})^{-1}(z) = \frac{p_{n} - q_{n}z}{-p_{n-1} + q_{n-1}z} \quad (z \in \psi_{a_{1}\cdots a_{n}}(X)).$$
(15)

But the equality (2) can be rewitten in the form

$$z = \phi_{a_1 \cdots a_n}(T^n z) \qquad (z \in X).$$

Hence for each $a_1 \cdots a_n \in A^{(n)}$ the *n*th power of *T* restricted on the cell $X_{a_1 \cdots a_n}$ is identical with the inverse of $\phi_{a_1 \cdots a_n}$; i.e.

$$T^n z = (\phi_{a_1 \cdots a_n})^{-1}(z) \qquad (z \in X_{a_1 \cdots a_n}).$$

$$\tag{16}$$

Especially we have

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$$X_{a_1\cdots a_n} = \phi_{a_1\cdots a_n}(T^n X_{a_1\cdots a_n}).$$
⁽¹⁷⁾

Now we need some notations: Put

$$U_{1} = \left\{ z \in X; \left| z + \frac{\sqrt{-3}}{3} \right| > \frac{\sqrt{3}}{3} \right\},$$
$$U_{2} = \{ z \in X; \text{ Im}(z) > 0 \},$$
$$U_{3} = \{ z \in X; \ \bar{z} \in U_{1}, \text{ Im}(z) > 0 \},$$

and define

$$U_{-j} = \{ \bar{z} ; z \in U_j \}$$
 $(j=1, 2, 3).$

Further we set $U_0 = X$ for notational convenience. Considering the reciprocals $U_j^{-1} = \{z; z^{-1} \in U_j\}$, we obtain (see Fig. 1)

$$X = \psi_{\zeta}(U_1) \cup \psi_{\overline{\zeta}}(U_{-1}) \cup \left(\bigcup_{\substack{a \in N_{\zeta} \\ a \neq \zeta, \overline{\zeta}}} \psi_a(X)\right),$$
(18.0)

$$U_{1} = \psi_{\zeta}(U_{-3}) \cup \psi_{\bar{\zeta}}(U_{-1}) \cup \left(\bigcup_{k=1}^{\infty} \psi_{\zeta+k}(U_{-2})\right) \cup \left(\bigcup_{\substack{a \in N_{\zeta}, a \neq \bar{\zeta} \\ \operatorname{Im}(a) \leq 0}} \psi_{a}(X)\right),$$
(18.1)

$$U_{2} = \psi_{\xi}(U_{-1}) \cup \left(\bigcup_{k=1}^{\infty} \psi_{k}(U_{-2})\right) \cup \left(\bigcup_{\substack{a \in N_{\zeta}, a \neq \zeta \\ \operatorname{Im}(a) < 0}} \psi_{a}(X)\right),$$
(18.2)

and

$$U_{3} = \psi_{\xi}(U_{3}) \cup \bigcup_{k=1}^{\infty} (\psi_{k}(U_{-2}) \cup \psi_{\xi+k}(U_{2})).$$
(18.3)

Taking the complex conjugate of (18.1)-(18.3) we have also the same relations for U_{-1} , U_{-2} , and U_{-3} to which we assign (18.-1), (18.-2), and (18.-3) resp.

In any case U_j can be written in the form

$$U_j = \bigcup_{a \in M_j} \phi_a(U_k) \tag{19}$$

where M_j is a subset of N_{ζ} and $k(-3 \le k \le 3)$ are chosen uniquely according as j and a. In addition, we note that

$$\psi_a(X) \cap \psi_b(X) = \phi \tag{20}$$

whenever $a \neq b(a, b \in N_{\zeta})$.

LEMMA 4. Let $n \ge 1$ and let $a_1 \cdots a_n \in A^{(n)}$. Then we have

$$X_{a_1\cdots a_n} = \phi_{a_1\cdots a_n}(U_j),\tag{21}$$

and so

$$T^n X_{a_1 \cdots a_n} = U_j \tag{22}$$







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Fig. 1

for some j $(-3 \le j \le 3)$.

Proof. By induction on *n*. First we prove (21). If n=1 (21) follows from (18.0). Suppose that (21) holds for all $a_1 \cdots a_n \in A^{(n)}$. Then we have

$$\begin{aligned} X_{a_{1}\cdots a_{n-1}} &= \{ z \in X_{a_{1}\cdots a_{n}}; \ a_{n+1}(z) = a_{1}(T^{n}z) = a_{n+1} \} \\ &= \{ \psi_{a_{1}\cdots a_{n}}(w); \ w \in U_{j}, a_{1}(w) = a_{n+1} \} \\ &= \psi_{a_{1}\cdots a_{n}}(\psi_{a_{n+1}}(U_{k})), \ (\text{by (14), (15)}) \\ &= \psi_{a_{1}\cdots a_{n+1}}(U_{k}), \end{aligned}$$

for $a_1 \cdots a_{a+1} \in A^{(n+1)}$, where *j* is defined by $U_j = T^n X_{a_1 \cdots a_n}$ and *k* chosen uniquely in (19). (22) follows from (17) and (21).

Let E be any subset of X. Then by Lemma 4 we have

$$T^{-n}E = \{z \in X; T^n z \in E\}$$

$$= \bigcup_{a_1 \cdots a_n \in A^{(n)}} \{z \in X_{a_1 \cdots a_n}; T^n z \in E \cap T^n X_{a_1 \cdots a_n}\}$$

$$= \bigcup_{a_1 \cdots a_n \in A^{(n)}} \psi_{a_1 \cdots a_n}(E \cap T^n X_{a_1 \cdots a_n}).$$
(23)

3. Estimates of the Lebesgue measure

Let *m* be the Lebesgue measure on the complex plane and let \mathfrak{B} be the σ -field of all measurable subsets of *X*. Then we have for any $a_1 \cdots a_n \in A^{(n)}$ and any $E \in \mathfrak{B}$

$$m(\phi_{a_1\cdots a_n}(E)) = \iint_E |\phi'_{a_1\cdots a_n}(z)|^2 dx dy, \qquad z = x + iy.$$
⁽²⁴⁾

But using (12) we find

$$|\psi'_{a_{\Gamma}\cdots a_{n}}(z)|^{2} = |q_{n}|^{-4} \left| 1 + \frac{q_{n-1}}{q_{n}} z \right|^{-4}.$$
(25)

Hence

$$3^{-4} < |q_n|^4 |\psi'_{a_1 \cdots a_n}(z)|^2 < 3^4$$
(26)

and

$$3^{-4} < |q_n|^{-4} |(\psi_{a_1\cdots a_n}^{-1})'(z)|^2 < 3^4, \tag{27}$$

because (from (6), (7), (9), (11))

$$3^{-1} < \frac{\sqrt{3}}{2} \le \left| 1 + \frac{q_{n-1}}{q_n} z \right| \le 1 + \frac{2}{\sqrt{3}} < 3.$$
(28)

(23) and (25) give the estimates

$$3^{-4}m(E) < |q_n|^4 m(\phi_{a_1\cdots a_n}(E)) < 3^4 m(E) \qquad (E \in \mathfrak{B})$$
⁽²⁹⁾

Eepecially, taking account of the fact that $3^{-2} < m(U_j) < 1$ $(-3 \le j \le 3)$, we have

$$3^{-6} < |q_n|^4 m(X_{a_1 \cdots a_n}) < 3^4 \qquad (a_1 \cdots a_n \in A^{(n)}).$$
(30)

We write

$$S(n) = \sum_{a_1 \cdots a_n \in A^{(n)}} |q_n|^{-4}.$$

Then for any $n \ge 1$ we have

$$3^{-5} < S(n) < 3^{6}$$
. (31)

Indeed it follows from (30) that

$$3^{4}S(n) > \sum_{A^{(n)}} m(X_{a_{1}\cdots a_{n}}) = m(X) > 3^{-1}$$

and

 $3^{-6}S(n) < m(X) < 1.$

By means of Lemma 4 the set $A^{(n)}$ of all admissible sequences can naturally be divided into seven subsets; we put

$$A_{j}^{(n)} = \{a_{1} \cdots a_{n} \in A(n); \ T^{n} X_{a_{1} \cdots a_{n}} = U_{j}\} \qquad (-3 \le j \le 3),$$

then

$$A^{(n)} = \bigcup_{j=-3}^{3} A_j^{(n)}.$$

By (18.j) $(-3 \le j \le 3)$ we have the following relations for n > 1;

$$A_{0}^{(n)} = \{a_{1} \cdots a_{n} \in A^{(n)}; a_{1} \cdots a_{n-1} \in A_{0}^{(n-1)}, a_{n} \neq \zeta, \zeta; \\ \text{or } a_{1} \cdots a_{n-1} \in A_{1}^{(n-1)}, a_{n} \neq \zeta, \text{ Im } (a_{n}) \leq 0; \\ \text{or } a_{1} \cdots a_{n-1} \in A_{-1}^{(n-1)}, a_{n} \neq \zeta, \text{ Im } (a_{n}) \geq 0; \\ \text{or } a_{1} \cdots a_{n-1} \in A_{2}^{(n-1)}, a_{n} \neq \zeta, \text{ Im } (a_{n}) < 0; \\ \text{or } a_{1} \cdots a_{n-1} \in A_{-2}^{(n-1)}, a_{n} \neq \zeta, \text{ Im } (a_{n}) < 0; \\ \text{or } a_{1} \cdots a_{n-1} \in A_{-2}^{(n-1)}, a_{n} \neq \zeta, \text{ Im } (a_{n}) > 0\},$$
(32.0)

$$A_{1}^{(n)} = \{a_{1} \cdots a_{n} \in A^{(n)}; a_{1} \cdots a_{n-1} \in A_{0}^{(n-1)} \cup A_{-1}^{(n-1)} \cup A_{-2}^{(n-1)}, a_{n} = \zeta\},$$
(32.1)

$$A_{2}^{(n)} = \{a_{1} \cdots a_{n} \in A^{(n)}; a_{1} \cdots a_{n-1} \in A_{-1}^{(n-1)} \cup A_{3}^{(n-1)}, a_{n} - \bar{\zeta} \in N; \\ \text{or } a_{1} \cdots a_{n-1} \in A_{-2}^{(n-1)} \cup A_{-3}^{(n-1)}, a_{n} \in N\},$$
(32.2)

$$A_{3}^{(n)} = \{a_{1} \cdots a_{n} \in A^{(n)}; a_{1} \cdots a_{n-1} \in A_{-1}^{(n-1)} \cup A_{33}^{(n-1)}, a_{n} = \bar{\zeta}\},$$
(32.3)

and

$$A_{-j}^{(n)} = \{ \bar{a}_1 \cdots \bar{a}_n ; a_1 \cdots a_n \in A_j^{(n)} \} \quad (j = 1, 2, 3),$$
(32.-j)

where N is the set of all positive integers.

We write

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$$S_j(n) = \sum_{a_1 \cdots a_n \in A_j(n)} |q_n|^{-4} \quad (-3 \le j \le 3).$$

Thus we have

$$S_j(n) = S_{-j}(n)$$
 (-3 $\leq j \leq 3$), (33)

and

$$S(n) = \sum_{j=-3}^{3} S_j(n).$$
(34)

LEMMA 5. For any $j (-3 \le j \le 3)$ we have

$$S_j(n) > 3^{-12}$$
 ($n \ge 3$). (35)

Proof. Let $a_1 \cdots a_n \in A^{(n)}$ and let p_n/q_n the *n*th approximant. Then it follows from (10) and (28) that

$$3^{-1}|a_nq_{n-1}| < |q_n| < 3|a_nq_{n-1}|.$$
(36)

By (32.0), (33) and (36) we have

$$\begin{split} 3S_{0}(n) &> \sum_{a \in N_{\zeta} \setminus \{\zeta, \zeta\}} |a|^{-4} \sum_{a_{1} \cdots a_{n-1} \in A_{0}^{(n-1)}} |q_{n-1}|^{-4} \\ &+ 2 \sum_{\substack{a \in N_{\zeta} \setminus \{\zeta\} \\ 1m(a) \leq 0}} |a|^{-4} \sum_{a_{1} \cdots a_{n-1} \in A_{1}^{(n-1)}} |q_{n-1}|^{-4} \\ &+ 2 \sum_{\substack{a \in N_{\zeta} \setminus \{\zeta\} \\ 1m(a) < 0}} |a|^{-4} \sum_{a_{1} \cdots a_{n-1} \in A_{2}^{(n-1)}} |q_{n-1}|^{-4} \\ &> S_{0}(n-1) + 2S_{1}(n-1) + 2|\zeta + 1|^{-4}S_{2}(n-1) \\ &> 3^{-2}(S_{0}(n-1) + S_{1}(n-1) + S_{2}(n-1)). \end{split}$$

Hence

$$S_0(n) > 3^{-3}(S_0(n-1) + S_1(n-1) + S_2(n-1)).$$

In the same way we get

$$S_{1}(n) > 3^{-1}(S_{0}(n-1) + S_{1}(n-1) + S_{2}(n-1)),$$

$$S_{2}(n) > 3^{-1}(S_{1}(n-1) + S_{2}(n-1) + S_{3}(n-1)),$$

and

$$S_3(n) > 3^{-1}(S_1(n-1) + S_3(n-1))$$

(using (32.1)-(32.3)). These inequalities as well as (31), (33), and (34) imply that

$$S_0(n) > 3^{-6} \sum_{j=0}^{3} S_j(n-2) > 3^{-7}S(n) > 3^{-12}.$$

Similarly we may obtain (35) for any $j(-3 \le j \le 3)$.

In what follows we shall use (35) only with j=0.

4. Invariant measure and ergodicity

THEOREM 1. Let E be any measurable subset of X such that $T^{-1}E=E$. Then m(E)=0 or m(X).

Proof. We assume that m(E) > 0. By Lemma 4 and (23) we find for any $a_1 \cdots a_n \in A^{(n)}$

$$E \cap X_{a_1 \cdots a_n} = T^{-n} E \cap \psi_{a_1 \cdots a_n} (T^n X_{a_1 \cdots a_n})$$
$$= \psi_{a_1 \cdots a_n} (E \cap T^n X_{a_1 \cdots a_n}).$$

From this as well as (29) and (30) we have

$$m(E \cap X_{a_{1}\cdots a_{n}}) \ge 3^{-4} |q_{n}|^{-4} m(E \cap T^{n} X_{a_{1}\cdots a_{n}})$$

$$\ge 3^{-8} m(X_{a_{1}\cdots a_{n}}) \min \{m(E \cap U_{3}), m(E \cap U_{-3})\}.$$
(37)

But (18.3) and (23) implies that

$$E\cap U_3 = T^{-1}E\cap U_3 \supset \phi_{\zeta+1}(E\cap U_2) \cup \phi_1(E\cap U_{-2}).$$

Beside for any measurable subset F of U_2 we have by (24) with (25)

$$m(\psi_1(F)) = \iint_F |1+z|^{-4} dx dy > \iint_F |\zeta+1+z|^{-4} dx dy = m(\psi_{\zeta-1}(F)).$$

Hence

$$m(E \cap U_3) > m(\phi_{\zeta+1}(E \cap U_2)) + m(\phi_{\zeta+1}(E \cap U_{-2})) = m(\phi_{\zeta+1}(E)) > 3^{-4} |\zeta+1|^{-4} m(E) = 3^{-6} m(E).$$
(38)

(using (29)). Similarly we get

$$m(E \cap U_{-3}) > 3^{-6} m(E).$$
 (39)

By (37), (38), and (39) the inequality

$$m(E \cap F) \ge 3^{-14} m(E) m(F).$$
 (40)

hold for all fundamental cell F, and so for any measurable set F in X. Thus, putting $F = X \setminus E$ in (40), we have

$$m(E)m(X \setminus E) = 0,$$

which implies m(E) = m(X).

THEOREM 2. There exists an unique, T-invariant probability measure μ equivalent to Lebesgue measure such that the inequalities

$$3^{-15} \frac{m(E)}{m(X)} \le \mu(E) \le 3^{10} \frac{m(E)}{m(X)} \tag{41}$$

hold for all $E \in \mathfrak{B}$.

Proof. To prove the existence it is enough to show that the inequalities

$$3^{-15}m(E) < m(T^{-n}E) < 3^{10}m(E)$$
 (*E* \in **B**) (42)

hold for all $n \ge 0$ (see F. Schweiger [5] § 6-§ 7). By (23), (29), and (31) we have

$$\begin{split} m(T^{-n}E) &< \sum_{A^{(n)}} m(\phi_{a_1\cdots a_n}(E)) \\ &\leq 3^4 m(E) S(n) \leq 3^{10} m(E) \end{split}$$

To prove the left-hand side inequalities in (42), we suppose first that $E \subset U_3$. Then, by (23), (29), and Lemma 5,

$$m(T^{-n}E) \ge \sum_{A_0^{(n)}} m(\phi_{a_1\cdots a_n}(E)) \ge 3^{-16}m(E),$$

as required. In the same way, these inequalities hold for any $E \subset U_2 \setminus U_3$. The lefthand side of the inequalities (42) is also true for any subset E of U_{-3} or $U_{-2} \setminus U_{-3}$. As a result (42) holds for any subset E of X, since

$$E = (E \cap U_3) \cup (E \cap (U_2 \setminus U_3)) \cup (E \cap U_{-2}) \cup (E \cap (U_{-2} \setminus U_{-3})).$$

By Theorem 1 T-invariant probability measure μ is uniquely given by the limit

$$\mu(E) = \frac{1}{m(X)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} m(T^{-k}E) \qquad (E \in \mathfrak{B})$$
(43)

(see F. Schweiger [5]). So (41) follows from (42) and (43).

THEOREM 3. T is ergodic with respect to μ ; i.e. for any $f \in L^1(X)$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f(T^kz)=\int_Xf(z)d\mu, \quad a.e.$$

Proof. Follows from Theorem 1, 2 and Birkhoff's individual ergodic theorem. As an application of Theorem 3, we have

$$\lim_{n\to\infty} (a_1(z)\cdots a_n(z))^{1/n} = e^{\alpha}, \quad a.e.$$

where

$$\alpha = \int_X \log \alpha_1(z) d\mu.$$

(Note that $f(z) = \log a_1(z) \in L^1(X)$, where $-\frac{\pi}{3} \le \arg f(z) \le \frac{\pi}{3}$, since the series $\sum_{a \in N_{\zeta}} a^{-4} \log a$ is convergent.)

5. Exactness

A measure-preserving transformation T on a normalized measure space (X, \mathfrak{B}, μ)

is said to be exact if

$$\bigcap_{n=0}^{\infty} T^{-n}\mathfrak{B} = \{\phi, X\},\$$

or equivalently, if for every set E of positive measure with the measurable images TE, T^2E, \cdots the relation

$$\lim_{n \to \infty} \mu(T^n E) = 1 \tag{44}$$

holds (see V.A. Rohlin [6].)

THEOREM 4. The transformation T is exact. The proof requires the following

LEMMA 6. Let $\varepsilon > 0$ and let E be any measurable set such that

 $\mu(U_{J} \setminus E) < \epsilon$

for some j ($-3 \le j \le 3$). Then

 $\mu(TE) > 1 - 3^{31}\varepsilon.$

Proof of Lemma 6. It is clearly enough to consider only the case $j=\pm 3$. Further we may assume j=3, since the following arguments are available for the conjugate case j=-3. Note first that

$$\psi_1(U_{-2}) \cup \psi_{\bar{\zeta}+1}(U_2) \subset U_3, \tag{45}$$

(by (18.3)). By (41) and (27) with n=1 and $a_n=1$, we get

$$\mu(T(\psi_1(U_{-2})\backslash E)) \le 3^{10} m(X)^{-1} m(T(\psi_1(U_{-2})\backslash E)) \le 3^{14} m(X)^{-1} m(\psi_1(U_{-2})\backslash E) \le 3^{29} \mu(\psi_1(U_{-2})\backslash E).$$
(46)

while, using (27) with n=1 and $a_1=\bar{\zeta}+1$,

$$\mu(T(\phi_{\xi+1}(U_2)\backslash E)) \leq 3^{31}\mu(\phi_{\xi+1}(U_2)\backslash E)).$$

$$(47)$$

Combining (45), (46), and (47) we find

$$\mu(T((\psi_1(U_{-2}) \cup \psi_{\xi+1}(U_2)) \setminus E))) \\
\leq 3^{31} \mu((\psi_1(U_{-2}) \cup \psi_{\xi+1}(U_2)) \setminus E)) \\
\leq 3^{31} \mu(U_3 \setminus E) \leq 3^{31} \varepsilon.$$
(48)

Therefore, by (45), (48), and (18.3), we obtain

$$\begin{split} \mu(TE) &\geq \mu(T((\phi_1(U_{-2}) \cup \phi_{\xi+1}(U_2)) \setminus E)) \\ &\geq \mu(T(\phi_1(U_{-2}) \cup \phi_{\xi+1}(U_2))) - \mu(T((\phi_1(U_{-2}) \cup \phi_{\xi+1}(U_2)) \setminus E)) \\ &> 1 - 3^{31}\varepsilon. \end{split}$$

Proof of Theorem 3. We prove (44). Let $E \in \mathfrak{B}$ given arbitrary. (Note that, by the definition of $T, E \in \mathfrak{B}$ if and only if $TE \in \mathfrak{B}$.) Let $\varepsilon > 0$. Then there exists a

fundamental interval $F = X_{a_1 \cdots a_n}$ such that

$$m(F \setminus E) < 3^{-50} \varepsilon m(F). \tag{49}$$

Otherwise, the inequality

 $m(F \setminus E) \ge 3^{-50} \varepsilon m(F)$

holds for all fundamental interval F, and so it holds also for arbitrary measurable set F. Putting F=E we have m(F)=0; a contradiction.

Using Lemma 4, (41), (16), (27), (30), and (49), we get

$$\mu(T^{n}F \setminus T^{n}E) \leq \mu(T^{n}(F \setminus E))$$

$$\leq 3^{11}m(T^{n}(F \setminus E)) \leq 3^{15}|q_{n}|^{4}m(F \setminus E)$$

$$\leq 3^{19}m(F)^{-1}m(F \setminus E) < 3^{-31}\varepsilon.$$
(50)

Noticing that $T^n F = U_j$ for some j by Lemma 4, we have from (50) and Lemma 6

$$\mu(T^{n+1}E) > 1 - \varepsilon.$$

Since $\mu(E), \mu(TE), \mu(T^2E), \cdots$ is non-decreasing, the relation (44) is proved.

As a general property of exact transformations (see V.A. Rohlin [4]) we have

Corollary. The transformation T is mixing of all degrees. In particular T is strongly mixing; i.e. for any $E, F \in \mathfrak{B}$ we have

$$\lim_{n\to\infty}\mu(T^{-n}E\cap F)\!=\!\mu(E)\mu(F).$$

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