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ON THE FLOW OUTSIDE A NONSADDLE COMPACT INVARIANT SET

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ABSTRACT

In this paper, we deal with a nonsaddle compact invariant set M of abstract dynamical systems placing special emphasis on the behaviour of orbits lying in the vicinity of M. Given a neighbourhood U of M, we divide $\bar{U}-M$ into several subsets according to the behaviour of orbits in them. These subsets will be called hyperbolic, parabolic and elliptic regions and our aim is to clarify how these regions are distributed in $\bar{U}-M$.

1. Introduction and Preliminaries

Let (X, π) be an abstract dynamical system with phase space X and phase map π defined by usual axioms. Also, as our standing hypothesis, we assume that X is a *locally compact metric space.*

The following notation will be used throughout the paper.

For any $x \in X$, we denote by:

$C^+(x)$, a *positive half-orbit* from x;
$C^-(x)$, a *negative half-orbit* from x;
$C(x)$, an *orbit* through x, i.e. $C(x)=C^+(x)\cup C^-(x)$;
$L^+(x)$, an *ω-limit set* of x;
$L^-(x)$, an *α-limit set* of x;
$J^+(x)$, a *positive prolongational limit set* of x;
$J^-(x)$, a *negative prolongational limit set* of x.

The object of this paper lies in the study of the behaviour of orbits in the vicinity of a compact invariant set which is the most fundamental part of the local theory of dynamical systems. For that purpose, we introduce following concepts.

Let M be a compact invariant set of (X, π) which is not open, and U be its arbitrary neighbourhood. We divide $\bar{U}-M$ into following subsets:

$$G_U=[x;\ x\in \bar{U}-M,\ C^+(x)\not\subset U,\ C^-(x)\not\subset U],$$

$$N_U^+=[x;\ x\in\bar{U}-M,\ C^+(x)\subset U],$$

$$N_U^-=[x;\ x\in\bar{U}-M,\ C^-(x)\subset U],$$

$$N_U=N_U^+\cap N_U^-,$$

and we call each connected component of:

G_U, a *hyperbolic region*,
$N_U^+-N_U$, a *positive parabolic region*,
$N_U^--N_U$, a *negative parabolic region*,
N_U, an *elliptic region.*

Obviously G_U is open and N_U^+, N_U^- and N_U are closed in $\bar{U}-M$, and

$$\bar{U}-M=G_U\cup N_U^+\cup N_U^-,$$

$$G_U\cap N_U^+=\emptyset,\quad G_U\cap N_U^-=\emptyset.$$

It is also obvious that N_U is an invariant set.

The following definition gives a very important classification of compact invariant sets.

Definition. If there exists a neighbourhood U of M such that $\bar{G}_U\cap M\neq\emptyset$, then M is called a *saddle set*. Otherwise it is called a *nonsaddle set.*

Compared with the saddle set, the nonsaddle set is much easier to treat. This is mainly due to the fact that, if M is a nonsaddle set, we may suppose that $\bar{U}-M$ has no hyerbolic regions as we shall see later.

Saddle property of a compact invariant set will be characterized by following theorems.

Theorem A. *If there exists either*

1) *an* $x\notin M$ *such that*

$$L^+(x)\cap M\neq\emptyset,\qquad J^+(x)\not\subset M,$$

or

2) *an* $x'\notin M$ *such that*

$$L^-(x')\cap M\neq\emptyset,\qquad J^-(x')\not\subset M,$$

then M *is a saddie set.*

Theorem B. (Converse of Theorem A). *If* M *is a saddle set isolated from minimal sets* (*i.e. there exists a neighbourhood* U *of* M *such that* $\bar{U}-M$ *contains no minimal sets*), *then there exist*

1) *an* $x\notin M$ *such that*

$$L^+(x)\cap M\neq\emptyset,\qquad J^+(x)\not\subset M,$$

and

2) *an* $x'\notin M$ *such that*

$$L^-(x')\cap M\neq\emptyset, \qquad J^-(x')\not\subset M.$$

Theorem C. (Contraposition of Theorem A). *If M is a nonsaddle set, then $x\notin M$ and $L^+(x)\cap M\neq\emptyset$ implies $M\supset J^+(x)\supset L^+(x)$, and $x\notin M$ and $L^-(x)\cap M\neq\emptyset$ implies $M\supset J^-(x)\supset L^-(x)$.*

To investigate the behaviour of orbits in some neighbourhood of M, we have to determine all possible configurations of hyperbolic, parabolic and elliptic regions. This is by no means an easy problem and results obtained so far are quite meagre except for some very special dynamical systems. As a first step towards the complete solution of this problem, we shall be concerned with the case when M is a nonsaddle set which is undoubtedly an easier part of the study.

We conclude this section with the statement of one more theorem which will be used in the next section.

Theorem D. *A compact invariant set M is positively (negatively) asymptotically stable if and only if $N_U^-=\emptyset$ ($N_U^+=\emptyset$) for some neibourhood U of M.*

2. The Case when $N_U=\emptyset$

Hereafter M always denotes a nonsaddle compact invariant set which is not open and isolated from minimal sets. U always denotes an open neighbourhood of M such that $\bar{U}-M$ contains no minimal sets. Since our study is entirely local-theoretic, all we need is a sufficiently small neighbourhood of M. So we may always assume that $\bar{U}$ is compact because of the local compactness of X. Also we assume that $\partial U\neq\emptyset$. This means that we exclude the case when U coincides with the whole (or a connected component containing M) of X and therefore is quite a natural assumption.

Since M is a nonsaddle set, $\bar{G}_U\cap M=\emptyset$ for any neighbourhood U of M. So if a neighourhood V of M is chosen sufficently small, we have

$$\bar{G}_U\cap V=\emptyset.$$

Therefore $U\backslash\bar{G}_U\supset V$ which shows that $U\backslash\bar{G}_U=U'$ is also a neighbourhood of M and $G_{U'}=\emptyset$. In other words, every open neighbourhood U of M contains an open neighbourhood U' with $G_{U'}=\emptyset$. Thus we get

Proposition 1. *If M is a nonsaddle set, there exists a fundamental system of neighbourhoods of M such that $G_U=\emptyset$ for every member U of this fundamental system.*

So, from now on, we always assume that $G_U=\emptyset$. This greatly simplifies our argument.

$\bar{U}-M$ generally consists of several connected components. We denote them by $C_\alpha(\bar{U}-M)$, or simply C_α, $\alpha\in I$, where I is the set of indices α (which might even be uncountable). As is well known, C_α is closed in $\bar{U}-M$.

As G_U is supposed to be empty,

$$\bar{U}-M=N_U^- \cup N_U^+.$$

Therefore we have only to consider following four cases:

$$\text{(I)} \quad N_U^+=\bar{U}-M, \quad N_U^-=\emptyset,$$

$$\text{(II)} \quad N_U^-=\bar{U}-M, \quad N_U^+=\emptyset,$$

$$\text{(III)} \quad N_U^+\neq\emptyset, \quad N_U^-\neq\emptyset, \quad N_U=\emptyset,$$

$$\text{(IV)} \quad N_U^+\neq\emptyset, \quad N_U^-\neq\emptyset, \quad N_U\neq\emptyset.$$

By Theorem D stated at the end of the preceding section, we immediately have

Proposition 2. *If the case* (I) *takes place, M is positively asymptotically stable. If the case* (II) *takes place, M is negatively asymptotically stable.*

Next consider the case (III). Since N_U^+ and N_U^- are both closed in $\bar{U}-M$, $N_U^+ \cap C_\alpha$ and $N_U^- \cap C_\alpha$ are both closed in C_α. As N_U is empty, we have

$$C_\alpha=(N_U^+ \cap C_\alpha) \cup (N_U^- \cap C_\alpha),$$

$$(N_U^+ \cap C_\alpha) \cap (N_U^- \cap C_\alpha)=N_U \cap C_\alpha=\emptyset.$$

But as C_α is connected, we have either

$$N_U^+ \cup C_\alpha=C_\alpha, \qquad N_U^- \cap C_\alpha=\emptyset,$$

or

$$N_U^- \cap C_\alpha=C_\alpha, \qquad N_U^+ \cap C_\alpha=\emptyset.$$

Therefore C_α is a parabolic region either positive or negative. If C_α is a positive parabolic region, then for any $x \in C_\alpha$, we have $C^-(x)\not\subset\bar{U}$, $C^+(x)\subset\bar{U}$ and $L^+(x)\subset\bar{U}$. As $\bar{U}$ is supposed to be compact, $L^+(x)$ is a compact invariant set and hence contains a minimal set in it. Since $\bar{U}-M$ contains no minimal sets, such a minimal set naturally lies in M. Hence $L^+(x)\cap M\neq\emptyset$. By Theorem C, this implies $M\supset L^+(x)$.

Analogously if C_α is a negative parabolic region, $x \in C_\alpha$ implies $C^+(x)\not\subset\bar{U}$, $L^-(x)\subset M$. Thus we get

Proposition 3. *If the case* (III) *takes place, each connected component C_α of $\bar{U}-M$ is either a positive parabolic region or a negative parabolic region. In the former case, every orbit in C_α tends to M positively and leaves C_α negatively. In the latter case, every orbit in C_α tends to M negatively and leaves C_α positively.*

3. The Case when $N_U\neq\emptyset$

Next we consider the case (IV). In this case, the situation becomes much more complicated because of the existence of elliptic regions.

Now C_α will be classified into following four types:

(i) $C_\alpha \subset N_U^+ - N_U$,

(ii) $C_\alpha \subset N_U^- - N_U$,

(iii) $C_\alpha \subset N_U$,

(iv) $C_\alpha \cap N_U \neq \emptyset$, $C_\alpha \not\subset N_U$.

When C_α is of the type (i) or (ii), C_α itself is a parabolic region either positive or negative. Therefore the behaviour of orbits is just the same as in the case (III).

When C_α is of the type (iii), C_α itself is an elliptic region and hence every orbit in C_α tends to M both positively and negatively.

It is to be noticed that all the C_α cannot be of the type (iii). Indeed, if that is the case, $\bar{U}-M=N_U$ and hence $\bar{U}$ is a compact invariant set. Then evidently ∂U is also a compact invariant set and hence contains a minimal set. This is however impossible because U is so chosen that $\bar{U}-M$ contains no minimal sets. So some of the C_α must necessarily be of the type (i), (ii) or (iv).

So far, the discussion is quite simple. All the difficulties concentrate on the case when C_α is of the type (iv).

We start with the proof of the following lemma.

Lemma 1. *There exists a neighbourhood W of N_U such that $y \in W$ implies*

$$M \supset L^+(y) \neq \emptyset \qquad \text{and} \qquad M \supset L^-(y) \neq \emptyset.$$

Proof. Suppose that there exists a sequence $\{x_n\} \subset X-M$ such that $x_n \to x \in N_U$ and $L^+(x_n) \not\subset M$ or $L^+(x_n)=\emptyset$.

If $L^+(x_n)=\phi$, the $\pi(x_n, t) \notin \bar{U}$ if t is sufficiently large.

If $L^+(x_n) \neq \emptyset$ and $L^+(x_n) \not\subset M$, then as M is a nonsaddle set, we have $L^+(x_n) \cap M=\emptyset$ by Theorem C. If $L^+(x_n)$ is not compact, then evidently $L^+(x_n) \not\subset \bar{U}$ because $\bar{U}$ is compact. If $L^+(x)$ is compact, it contains a minimal set which does not lie in M. But as $\bar{U}-M$ contains no minimal sets, we must have $L^+(x_n) \not\subset \bar{U}$. So, in either case, we can find $t>T$ such that $\pi(x_n, t) \notin \bar{U}$ for any positive T.

Thus, anyway, there exists a sequence $\{t_n\}$ such that

$$t_n>0, \qquad t_n \to \infty, \qquad \pi(x_n, t_n) \notin \bar{U}.$$

Let $\tilde{X}$ be a one-point compactification of X and $(\tilde{X}, \tilde{\pi})$ be a natural extension of (X, π) onto $\tilde{X}$. Also denote by $\tilde{L}^+(x)$ and $\tilde{J}^+(x)$ the ω-limit set and the positive prolongational limit set of x in $(\tilde{X}, \tilde{\pi})$ respectively.

$\tilde{X}$ being compact, a sequence $\{\pi(x_n, t_n)\}=\{\tilde{\pi}(x_n, t_n)\}$ has a cluster point z in $\tilde{X}$. Since $\pi(x_n, t_n) \notin \bar{U}$, we have

$$z \in \tilde{X}-U \subset \tilde{X}-M.$$

Since $x_n \to x$ and $t_n \to \infty$, $z \in \tilde{J}^+(x)$. Hence

$$\tilde{J}^+(x) \not\subset M.$$

On the other hand, as $x \in N_U$, we necessarily have

$$L^+(x)\cap M\neq\emptyset.$$

As it is obvious that $\tilde{L}^+(x)\supset L^+(x)$, this implies $\tilde{L}^+(x)\cap M\neq\emptyset$.

Consequently, by Theorem A, M is a saddle set of the dynamical system $(\tilde{X},\tilde{\pi})$. Since the saddle property is entirely a local one, this implies that M is also a saddle set of (X,π) contrary to our assumption.

Thus we conclude that, if y is sufficiently close to N_U, we have $M\supset L^+(y)\neq\emptyset$.

In a similar way, one can show that $M\supset L^-(y)\neq\emptyset$ if y is sufficiently close to N_U, and this completes the proof.

Based on this lemma, we prove the following

Lemma 2. *Suppose that $\bar{U}$ is locally connected. Then, if C_α is of the type* (iv), *we have*

$$(N_U^+-N_U)\cap C_\alpha\neq\emptyset,\qquad (N_U^--N_U)\cap C_\alpha\neq\emptyset.$$

Proof. For simplicity, we write

$$N_U^+\cap C_\alpha=N_\alpha^+,\quad N_U^-\cap C_\alpha=N_\alpha^-,\quad N_U\cap C_\alpha=N_\alpha.$$

Then what we have to show is that

$$N_\alpha^+\neq N_\alpha,\qquad N_\alpha^-\neq N_\alpha.$$

Assume that $N_\alpha^+=N_\alpha$ to derive a contradiction. Then $N_\alpha^-\neq N_\alpha$ because $N_\alpha^+=N_\alpha=N_\alpha^-$ implies $C_\alpha\subset N_U$ contrary to the assumption. Thus we have

$$N_\alpha^-\supset N_\alpha=N_\alpha^+,\qquad N_\alpha^--N_\alpha\neq\emptyset.$$

Since (G_U being empty) $N_\alpha^+\cup N_\alpha^-=C_\alpha$, we have

$$C_\alpha=N_\alpha^-=(N_\alpha^--N_\alpha)\cup N_\alpha,$$

$$N_\alpha^--N_\alpha\neq\emptyset,\qquad N_\alpha\neq\emptyset.$$

As N_α is closed in C_α and C_α is connected, N_α is not open in C_α. Therefore we can find a sequence $\{x_n\}\subset N_\alpha^--N_\alpha$ tending to $x\in N_\alpha$. Then, by Lemma 1, we may suppose that $M\supset L^+(x_n)\neq\emptyset$. However, as $x_n\in N_\alpha^--N_\alpha$, $\pi(x_n,t)$ leaves $\bar{U}$ when t increases and re-enter $\bar{U}$ to stay there forever. So there exists $t_n>0$ for each x_n such that

$$\pi(x_n,t)\notin\bar{U},\qquad t_n>t>t_n-\varepsilon,$$

$$y_n=\pi(x_n,t_n)\in\partial U,\qquad C^+(y_n)\subset\bar{U},$$

where ε is a suitably small positive number. Then obviously $y_n\in N_U^+-N_U$. We shall prove that at most a finite number of y_n can lie outside C_α.

Let us assume the contrary. Then there exists an infinite subsequence $\{y_{n_k};\ k=1,2,\cdots\}$ of $\{y_n\}$ which lies outside C_α. Since $y_n\in\partial U$ and ∂U is compact, we may suppose that $y_{n_k}\to y\in\partial U$.

Since $\bar{U}$ is supposed to be locally connected and $\bar{U}-M$ is open in $\bar{U}$, C_α is an open set in $\bar{U}-M$. Hence the set $(\bar{U}-M)-C_\alpha$ is closed in $\bar{U}-M$. As $y_{n_k}\in(\bar{U}-M)-C_\alpha$ and

$$y_{n_k} \to y \in \partial U \subset \bar{U} - M,$$

we have $y \in (\bar{U} - M) - C_\alpha$. Hence $y \notin C_\alpha$.

Since $\pi(x_{n_k}, t_{n_k}) = y_{n_k} \to y \notin C_\alpha$ and $x_{n_k} \to x \in N_\alpha$ as $k \to \infty$ and N_α is an invariant set in C_α, $\{t_{n_k}\}$ cannot be bounded. So we may suppose that $t_{n_k} \to \infty$, and this implies $y \in J^+(x)$. As $y \in \partial U$, this shows that

$$J^+(x) \not\subset M.$$

On the other hand, as $x \in N_\alpha \subset N_U$, we have $L^+(x) \cap M \neq \emptyset$. Thus, by Theorem A, M must be a saddle set contrary to our assumption.

Therefore almost all y_n belong to C_α. Since $y_n \in N_U^+ - N_U$ as we have already remarked, we have

$$C_\alpha \cap (N_U^+ - N_U) = N_\alpha^+ - N_\alpha \neq \emptyset.$$

This contradiction shows that $N_\alpha^+ \neq N_\alpha$.

$N_\alpha^- \neq N_\alpha$ can be proved similarly.

From this lemma, we see that, if C_α is of the type (iv) and $\bar{U}$ is locally connected,

$$C_\alpha \cap (N_U^+ - N_U) \neq \emptyset, \qquad C_\alpha \cap (N_U^- - N_U) \neq \emptyset, \qquad C_\alpha \cap N_U \neq \emptyset.$$

Hence C_α includes several positive parabolic regions, several negative parabolic regions and several elliptic regions simultaneously. Thus we get

Proposition 4. *If the case* (IV) *takes place, C_α are classified into four types* (i), (ii), (iii) *and* (iv) *stated above, and*

1) *if C_α is of the type* (i), *C_α is a positive parabolic region in which every orbit tends to M positively and leaves C_α negatively,*

2) *if C_α is of the type* (ii), *C_α is a negative parabolic region in which every orbit tends to M negatively and leaves C_α positively,*

3) *if C_α is of the type* (iii), *C_α is an elliptic region in which every orbit tends to M both positively and negatively,*

4) *if C_α is of the type* (iv), *C_α is a union of positive parabolic regions, negative parabolic regions and elliptic regions none of which is empty.*

Also at least one C_α must be different from the type (iii).

Summarizing Propositions 1~4, we have the following fundamental Theorem.

Theorem. *Let M be a nonsaddle compact invarint set isolated from minimal sets. Then M has a fundamental system of neighbourhoods whose members have no hyperbolic regions. Let U be any such neighbourhood and $\{C_\alpha\}$ be the connected components of $\bar{U} - M$.*

If $\bar{U}$ is compact and locally connected and $\partial U \neq \emptyset$, then each C_α belongs to one of the following four types and at least one of C_α must be different from the type C):

A) *C_α is a positive parabolic region,*

B) *C_α is a negative parabolic region,*

C) *C_α is an elliptic region,*

D) *C_α is a union of positive parabolic regions, negative parabolic regions and elliptic regions none of which is empty.*

4. Further Study of Elliptic Regions

Here we investigate the structure of $\bar{U}-M$ in more detail where $\bar{U}$ is always supposed to be locally connected.

Proposition 5. *If C_α is of the type D), $C_\alpha-N_\alpha$ is not connected.*

Proof. Evidently we have

$$C_\alpha-N_\alpha=(N_\alpha^+-N_\alpha)\cup(N_\alpha^--N_\alpha)$$

and

$$N_\alpha^+-N_\alpha\neq\emptyset, \qquad N_\alpha^--N_\alpha\neq\emptyset,$$

by Lemma 2. Since

$$(N_\alpha^+-N_\alpha)\cap(N_\alpha^--N_\alpha)=\emptyset,$$

we have only to show that $N_\alpha^+-N_\alpha$ and $N_\alpha^--N_\alpha$ are both open in $C_\alpha-N_\alpha$. But since

$$N_\alpha^+-N_\alpha=N_\alpha^+\cup N_\alpha^--N_\alpha^-=C_\alpha-N_\alpha^-,$$

and N_α^- is closed in C_α (because $N_\alpha^-=N_{\bar{U}}^-\cap C_\alpha$ and $N_{\bar{U}}^-$ is closed in $\bar{U}-M$), $N_\alpha^+-N_\alpha$ is open in C_α. The openness of $N_\alpha^--N_\alpha$ can be proved similarly.

Proposition 6. *If C_α is of the type D), then every elliptic region in C_α intersects ∂U.*

Proof. Since an elliptic region E in C_α is a connected component of N_U which is a closed set in $\bar{U}-M$, E is closed in C_α. As $C_\alpha-E\neq\emptyset$ and C_α is a connected set, E is not open in C_α. Therefore E has a boundary point x which is a cluster point either of $N_\alpha^+-N_\alpha$ or of $N_\alpha^--N_\alpha$.

Suppose that x is a cluster point of $N_\alpha^+-N_\alpha$. Then there exists a sequence $\{x_n\}\subset N_\alpha^+-N_\alpha$ such that $x_n\to x$.

Since $C^-(x_n)\not\subset\bar{U}$, there exists $t_n<0$ such that

$$\pi(x_n, t)\in C_\alpha, \qquad t>t_n,$$

$$\pi(x_n, t_n)\in\partial U\cap C_\alpha.$$

As C_α is closed in $\bar{U}-M$ and ∂U is compact, we may suppose that

$$\pi(x_n, t_n)\to y\in\partial U\cap C_\alpha.$$

If the sequence $\{t_n\}$ is unbounded, we may suppose that $t_n\to-\infty$. Hence $y\in J^-(x)$. As $y\in\partial U$, we have

$$J^{-}(x)\not\subset M.$$

On the other hand, $x\in E$ implies $L^{-}(x)\cap M\neq\emptyset$. So, by Theorem A, M is a saddle set contrary to the assumption.

Therefore $\{t_n\}$ is bounded and we may suppose that $t_n\to T>-\infty$. Then, by the continuity of π, it follows that

$$y=\pi(x, T).$$

As $x\in E$ and E is an invariant set, $y\in E$. Hence $y\in E\cap\partial U$ and E intersects ∂U.

Proposition 7. *Let the number of connected components of $X-M$ be finite. Then every elliptic region of U intersects ∂U if U is chosen sufficiently small.*

Proof. Let $X_1, \cdots, X_n$ be the connected components of $X-M$. We choose an open neighbourhood U of M so that

$$U\not\supset X_1, \cdots, U\not\supset X_n,$$

and let $\{C_\alpha\}$ be the connected components of $\bar{U}-M$. If C_α contains an elliptic region, it must be either of type C) or of type D).

Let C_α be of type C), i.e. C_α itself is an elliptic region. As $\bar{U}$ is locally connected, C_α is open in $\bar{U}-M$. Also C_α is closed in $\bar{U}-M$ as C_α is a connected component of N_U which is closed in $\bar{U}-M$. Therefore C_α is open and closed in $\bar{U}-M$. Consequently if $C_\alpha\subset U$, C_α is open and closed in $X-M$ and hence must coincide with one of $X_1, \cdots, X_n$. But since U has been so chosen that none of $X_1, \cdots, X_n$ is contained in U, this is obviously impossible. Thus we have $C_\alpha\not\subset U$ and C_α intersects ∂U.

If C_α is of type D), then the conclusion follows directly from Proposition 6. Thus we have completed the proof.

From now on, we always assume that $X-M$ has only a finite number of connected components and the neighbourhood U is chosen so small that the conclusion of the Proposition 7 holds.

From Proposition 7, we see that, if $\bar{V}\subset U$, N_V is actually smaller than $N_U\cap\bar{V}$. In fact, let E be an elliptic region with respect to U. Then $E\cap\bar{V}$ does not belong to N_V, because if $y\in E\cap\bar{V}$ is a boundary point of E, then, as $C(y)\cap\partial U\neq\emptyset$, $C(y)$ does not belong to $\bar{V}$ and hence $y\notin N_V$.

Thus we have the following alternative:

1) if U is chosen sufficiently small, then $N_U=\emptyset$, or

2) $N_U\neq\emptyset$ for every neighbourhood U.

In the former case, the situation is quite simple, because, by choosing U sufficiently small, we can avoid the complicated case (IV) and C_α is either of the type A) or of the type B).

So the most difficult part of the study lies in the case when 2) takes place. In this case, we can prove the following

Proposition 8. *If the case* 2) *takes place, then for any neighbourhood U of M, at least one elliptic region has an interior point.*

Proof. Let U be an arbitrary neighbourhood of M and V be a neighbourhood of M with $\bar{V}\subset U$. Then, by assumption, $N_V\neq\emptyset$. If $x\in N_V$, then $C(x)\subset\bar{V}\subset U$ and hence x belongs to some elliptic region with respect to U, E say. Since $C(x)\subset\bar{V}$, $C(x)$ does not intersect ∂U. Therefore, as the proof of Proposition 6 shows, x is not a boundary point of E. Hence E has an interior point. This completes the proof.

It is also clear that, if there exists an elliptic region E (with respect to U) with interior points, then there exists a neighbourhood V of M with $\bar{V}\subset U$ such that $N_V\neq\emptyset$. In fact, let x be an interior point of E, then $C(x)$ deos not meet ∂U since the interior of E is an invariant set. Se we can construct a neighbourhood V of M so that $\bar{V}\subset U$ and $V\supset C(x)$. Then $x\in N_V$ and N_V is not empty.

As a result, we have

Proposition 9. *For the case* 1) *to be realized, it is necessary and sufficient that there exists a neighbourhood U of M such that no elliptic regions have interior points.*

Concerning this, we add one more proposition.

Proposition 10. *Let C_α be of the type* C) *or* D), *i.e. C_α includes an elliptic region. If there exists a neighbourhood V of M such that $\bar{V}\subset U$ and $\bar{V}\cap C_\alpha$ is connected, that at least one elliptic region in C_α has an interior point.*

Proof. First notice that $\bar{V}\cap C_\alpha\neq\emptyset$ since $\bar{C}_\alpha\cap M\neq\emptyset$. Let us put

$$\bar{V}_\alpha=\bar{V}\cap C_\alpha=(\bar{V}-M)\cap C_\alpha.$$

By the assumption on C_α, $N_\alpha^+\neq\emptyset$ and $N_\alpha^-\neq\emptyset$. Let x be a point in N_α^+. Then since $L^+(x)\subset M$, there exists $T>0$ such that $\pi(x,t)\in\bar{V}$ for $t\geqq T$. Then

$$\pi(x, T)\in\bar{V}_\alpha\cap N_V^+$$

which shows that $\bar{V}_\alpha\cap N_V^+\neq\emptyset$. Similarly $\bar{V}_\alpha\cap N_V^-\neq\emptyset$. As we assumed that $G_V=\emptyset$, we have

$$\bar{V}_\alpha=(\bar{V}_\alpha\cap N_V^+)\cup(\bar{V}_\alpha\cap N_V^-).$$

Since $\bar{V}_\alpha$ is connected by assumption and $\bar{V}_\alpha\cap N_V^+$ and $\bar{V}_\alpha\cap N_V^+$ are both nonempty and closed in $\bar{V}_\alpha$, the above relation implies

$$(\bar{V}_\alpha\cap N_V^+)\cap(\bar{V}_\alpha\cap N_V^-)=\bar{V}_\alpha\cap N_V\neq\emptyset.$$

If $y\in\bar{V}_\alpha\cap N_V$, then since $C^+(y)\subset\bar{V}$ and $C^+(y)\subset C_\alpha$,

$$C^+(y)\subset\bar{V}_\alpha.$$

Analogously we have $C^-(y)\subset\bar{V}_\alpha$. Therefore $C(y)\subset\bar{V}_\alpha\subset C_\alpha$ which shows that y belongs to some elliptic region, E say. Since $C(y)\subset\bar{V}_\alpha\subset U, C(y)$ does not meet ∂U. Hence y is not a boundary point of E. Thus E must have an interior point. This completes the proof.

Suppose that M has a fundamental system of neighbourhoods such that $\bar{U}-M$ is connected for every member U of this system. If there exists a neighbourhood with elliptic regions, the case 1) can never take place. In fact, if 1) takes place, there must exist a neighbourhood U of M such that no elliptic regions have interior points (Proposition 9). Let V be a neighbourhood of M with $\bar{V}\subset U$. Since we may suppose that $\bar{U}-M$ and $\bar{V}-M$ are both connected, $C_\alpha=\bar{U}-M$ and $\bar{V}\cap C_\alpha=\bar{V}\cap(\bar{U}-M)=\bar{V}-M$. Hence $\bar{V}\cap C_\alpha$ is connected. So, by Proposition 10, at least one elliptic region in $\bar{U}-M$ must have interior points which is a contradiction.

We conclude this paper by proving two propositions concerning the number of elliptic regions with interior points.

Proposition 11. *The number of elliptic regions with interior points is at most countable.*

Proof. Let $\{E_\alpha\}$ be the totality of elliptic regions with interior points. Let us denote by I and I_α the interior of N_U and E_α respectively. Then we have $I=\cup I_\alpha$, and $I_\alpha\cap I_\beta=\emptyset$ if $\alpha\neq\beta$. As $\bar{U}$ is compact, $\bar{U}$ has a countable base of open sets. Since $\cup I_\alpha$ is a non-overlapping covering of an open set I, the number of I_α must be at most countable.

Proposition 12. *Let U be an arbitrary neighbourhood of M and V be a neighbourhood of M with $\bar{V}\subset U$. Then the member of elliptic regions (with respect to U) which intersects N_V is finite.*

Proof. Let $\{E'_\alpha\}$ be the totality of elliptic regions with respect to U such that $E'_\alpha\cap N_V\neq\emptyset$. Let x be a point of N_V. Then $C(x)\subset\bar{V}\subset U$ and hence $C(x)\cap\partial U=\emptyset$. This implies that x is an interior point of some E'_α. Conversely every E'_α includes a point of N_V by definition which should necessarily be an interior point of E'_α. So if we denote by I'_α the interior of E'_α, every I'_α is nonempty and $\cup I'_\alpha\supset N_V$.

Let W be any open neighbourhood of M with $\bar{W}\subset V$, and consider the set

$$K=N_V\cap(X-W).$$

Then K is nonempty and compact and $\cup I'_\alpha$ is an open convering of K. Therefore we can select a finite covering of K from $\cup I'_\alpha$.

As every I'_α contains a point of N_V, it contains at least one elliptic region E with respect to V. Then since $E\cap\partial V\neq\emptyset$ by Proposition 7 and $\partial V\subset X-W$, E contains a point of K in it. Consequently every I'_α contains a point of K. Moreover we have $I'_\alpha\cap I'_\beta=\emptyset$ if $\alpha\neq\beta$. So, in selecting a finite covering of K from $\cup I'_\alpha$, none of I'_α can be excluded. This means that $\cup I'_\alpha$ is itself a finite covering. Hence the number of E'_α is finite.

Remark. Propositions 1~4 were already published in my book " Iso-Rikigaku (Topological Dynamics) ", Kyôritsu, Tokyo, in Japanese.

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