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A PARAMETRIZATION OF BENDERS DECOMPOSITION

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ABSTRACT

Benders decomposition is a multi-step procedure effective for solving mixed variables programming problems.

In this paper, we shall present a parametrization of Benders decomposition with respect to a parameter on the right hand side of constraints.

1. Preliminaries —Basic Theorems of Benders Decomposition and Parametric Analysis by Primal-Dual Algorithm—

In our analysis, we use two procedures, as subroutines, namely Benders decomposition and a parametric analysis of linear programmings by primal-dual algorithm, which are summarized as follows.

1.1 Basic Theorems of Benders Decomposition

BENDERS (1962) has presented a partitioning procedure for solving mixed variables programming problems of the type

$$Max\,\{c'x+f(y)|Ax+F(y)\leq b,\, 0\leq x\in R^p,\, y\in S\}, \tag{1.1}$$

where $x\in R^p$ (the p-dimensional Euclidean space), $y\in R^q$ and S is an arbitrary subset of R^q. Furthermore, A is an (m,p) matrix, $f(y)$ is a scalar function and $F(y)$ an m-component vector function both defined on S, b and c are fixed vectors in R^m and R^p, respectively, and prime denotes transposition.

His basic idea is a partitioning of the given problem into two subproblems: a programmimg problem (which may be linear, nonlinear, discrete, etc.) defined on S, and a linear programming problem defined on R^p.

In this connection, he defined two sets C and G as follows:
(a) A polyhedral convex cone C in R^{m+1}:

$$C=\{(u_0, u)|A'u-cu_0\geq 0, u\geq 0, u_0\geq 0\}. \tag{1.2}$$

(b) A set G in R^{q+1}:

$$G=\bigcap_{(u_0,u)\in C}\{(x_0, y)|u_0x_0+u'F(y)-u_0f(y)\leq u'b, y\in S\}. \tag{1.3}$$

Then, he states the basic theorem for a partitioning procedure:

Theorem 1.1
(*1*) *Problem* (1.1) *is not feasible if and only if the programming problem*

$$Max\{x_0|(x_0, y)\in G\} \tag{1.4}$$

is not feasible, i.e. if and only if the set G is empty.
(*2*) *Problem* (1.1) *is feasible without having on optimum solution, if and only if Problem* (1.4) *is feasible without having on optimum solution.*
(*3*) *If* $(\bar{x}, \bar{y})$ *is an optimum solution of Problem* (1.1) *and*

$$\bar{x}_0=c'\bar{x}+f(\bar{y}),$$

then $(\bar{x}_0, \bar{y})$ *is an optimum solution of Problem* (1.4) *and* $\bar{x}$ *is an optimum solution of the linear programming problem*

$$Max\{c'x|Ax\leq b-F(\bar{y}), x\geq 0\}. \tag{1.5}$$

(*4*) *If* $(\bar{x}_0, \bar{y})$ *is an optimum solution of Problem* (1.4), *then Problem* (1.5) *is feasible and the optimum value of the objective function in this problem is equal to* $\bar{x}_0-f(\bar{y})$. *If* $\bar{x}$ *is an optimum solution of Problem* (1.5), *then* $(\bar{x}, \bar{y})$ *is an optimum solution of Problem* (1.1), *with optimum value* $\bar{x}_0$ *for the objective function.*

Based on this theorem, he has designed a multi-step procedure for solving Problem (1.1).

The procedure starts from a subset Q of C and solves a programming problem

$$Max\{x_0|(x_0, y)\in G(Q)\}, \tag{1.6}$$

where $G(Q)$ is a set defined by

$$G(Q)=\bigcap_{(u_0, u)\in Q}\{(x_0, y)|u_0x_0+u'F(y)-u_0f(y)\leq u'b, y\in S\}. \tag{1.7}$$

Let an optimum solution of (1.6) be $(\bar{x}_0, \bar{y})$. Then, solve the problem

$$Min\{(b-F(\bar{y}))'u|A'u\geq c, u\geq 0\}. \tag{1.8}$$

Let an optimum solution of (1.8) be $\bar{u}$. Then, we have:

Theorem 1.2
If $(\bar{x}_0, \bar{y})$ *is an optimum solution of Problem* (1.6), *it is also an optimum solution of Problem* (1.4) *if and only if*

$$(b-F(\bar{y}))'\bar{u}=\bar{x}_0-f(\bar{y}).$$

And, if equality holds, we get an optimum solution $(\bar{x}, \bar{y})$ *of Problem* (1.1), *where* $\bar{x}$ *is an optimum solution of the linear programming problem* (1.5).

On the other hand, if

$$(b-F(\bar{y}))'\bar{u}<\bar{x}_0-f(\bar{y}),$$

then extend the set Q by adding a certain vertex of the feasible region of Problem (1.8) and/or a certain extremal ray of the convex cone C. And return to Problem (1.6). Repeat the above procedures until an optimum solution of Problem (1.1) is found or Problem (1.6) and hence (1.1) are decided to have no feasible solution or Problem (1.1) is decided to have no finite optimum solution, since Problem (1.8) has no feasible solution.

1.2 Parametric Programming and Primal-Dual Algorithm

KELLEY (1959) has presented a primal-dual algorithm for solving such a type of parametric programming problems as

(1.9) $$Max\{y'b \mid y'A \leq c'+\lambda d'\},$$

where A is an (m, n) matrix, b and y are m-vectors, c and d are n-vectors and λ is a scalar parameter.

We assume that an optimum solution $\bar{y}$ of Problem (1.9) for $\lambda=\lambda_0$ is known. And we are going to find an optimum solution of Problem (1.9) for λ less than λ_0.

First, define an index set S by

(1.10) $$S=\{j \mid w_j \equiv (c+\lambda_0 d)_j - y'A_j=0, 1\leq j\leq n\},$$

where $(c+\lambda_0 d)_j$ is the j-th element of vector $(c+\lambda_0 d)$ and A_j is the j-th column of matrix A.

And then, define the *restricted problem* as follows:

(1.11) $$Min\{\sigma'b \mid \sigma'A_j \geq d_j (j \in S)\},$$

where σ is an m-vector.

Now, KELLEY's algorithm is as follows:

Step 1. *Solving the restricted problem (1.11)*

Slove Problem (1.11). If it has no optimum solution, then Problem (1.9) has no feasible solution for λ less than λ_0. (*The end.*) Otherwise, let an optimum solution of (1.11) be $\bar{\sigma}$.

Step 2. *Finding the bound of* θ

Let $\beta_j=d_j-\bar{\sigma}'A_j (1\leq j\leq n)$.

Find a positive number θ_0 by

(1.12) $$\theta_0=\begin{cases} min\{w_j/\beta_j \mid \beta_j>0\} \\ \infty (\text{if } \beta_j\leq 0, 1\leq j\leq n). \end{cases}$$

Step 3. *Getting optimum solutions*

Step 3.1. An optimum solution of Problem (1.9) for $\lambda=\lambda_0-\theta$ $(0\leq\theta\leq\theta_0)$, is $\bar{y}'-\theta\bar{\sigma}'$.

Step 3.2. Let $\bar{x}$ be an optimum solution of the dual problem of (1.11). Then, $\bar{x}$ is also an optimum solution of the dual problem of (1.9) for $\lambda=\lambda_0-\theta$ $(0\leq\theta\leq\theta_0)$.

(*This completes one cycle of the algorithm.*)

2. A Parametrization of Benders Decomposition

We consider the following parametric programming problem.

[Problem I(λ)] (with variables x and y)

$$Max\{c'x+f'y\,|\,Ax+Fy\leq b+\lambda d,\ 0\leq x\in R^p,\ y\in S\subset R^q,\ \lambda\in R\},$$

where A is an (m,p) matrix, F an (m,q) matrix, b and d m-vectors, c a p-vector, and f a q-vector.

Notice that the variable y appears in linear forms in our problem.

Now, we define several problems and sets corresponding to those in Section 1.1.

[Problem II($y|\lambda$)] (with variable x)

$$Max\{z_1=c'x\,|\,x\geq 0,\ Ax\leq b+\lambda d-Fy\}.$$

[Problem III ($y|\lambda$)] (with variable u)

$$Min\{z_2=u'(b+\lambda d-Fy)\,|\,A'u\geq c,\ u\geq 0\}.\quad \text{(The dual problem of Problem II)}$$

[Polyhedral Convex Cone C and Set G]

$$C=\{(u_0,u)\,|\,A'u-cu_0\geq 0,\ u\geq 0,\ u_0\geq 0\}.$$

$$G=\bigcap_{(u_0,u)\in C}\{(x_0,y)\,|\,u_0x_0+u'Fy-u_0f'y\leq u'(b+\lambda d),\ y\in S\}.$$

[Sets Q and $G(Q|\lambda)$]

$$Q=\text{A finite subset of } C.$$

$$G(Q|\lambda)=\bigcap_{(u_0,u)\in Q}\{(x_0,y)\,|\,u_0x_0+u'Fy-u_0f'y\leq u'(b+\lambda d),\ y\in S\}.$$

[Problem IV($G(Q|\lambda)$)] (with variables x_0 and y)

$$Max\{x_0\,|\,(x_0,y)\in G(Q|\lambda)\}.$$

Notice that the polyhedral convex cone C has no relation to the parameter λ.

In what follows, we assume that an optimum solution $(\bar{x},\bar{y})$ of Problem $I(\lambda)$ is known for $\lambda=\lambda_0$. (If not, we may use Benders decomposition to find out one.) Also, we can naturally assume, from the above assumption, that we have a set Q and a point $(1,\bar{u})\in Q$ and we have the relation showing the optimality of $\bar{y},\bar{u}$ and $\bar{x}$:

$$min\{z_2=u'(b+\lambda_0d-F\bar{y})\,|\,A'u\geq c,\ u\geq 0\}=\bar{u}'(b+\lambda_0d-F\bar{y})=\bar{x}_0-f'\bar{y}=c'\bar{x},$$

where $\bar{x}_0$ is the optimum value for the objective function of Problem IV $(G(Q|\lambda_0))$. And we are going to find an optimum solution of Problem I(λ) for λ less than λ_0. We will deal with two cases corresponding to the kinds of S, namely, the case $S=R^q$ and the case S=a set of discrete points in R^q.

3. The Case $S=R^q$

3.1 Algorithm

In this section, we will show an algorithm for finding an optimum solution of Problem I(λ) in the case $S=R^q$ for λ less than λ_0, in the knowledge of an optimum solution $(\bar{x}, \bar{y})$ of Problem I(λ_0).

Step 1. *Solving the restricted problem*

Based on the optimum solution $(\bar{x}_0, \bar{y})$ of Problem IV$(G(Q|\lambda_0))$, derive its *restricted problem* corresponding to Problem (1.10) of Section 1 and solve it. If it has no optimal solution, then Problem I(λ) has no feasible solution for λ less than λ_0. (*The end.*) Otherwise, let an optimum solution be $(\bar{\xi}_0, \bar{\eta})$. And let

(3.1) $$x_0(\theta)=\bar{x}_0-\theta\bar{\xi}_0, \quad y(\theta)=\bar{y}-\theta\bar{\eta}$$

Also, determine the range $[\lambda_0-\theta_1, \lambda_0]$ of λ where $(x_0(\theta), y(\theta))$ remains optimal for Problem IV$(G(Q|\lambda))$, using formula (1.12).

Step 2. *Solving Problem I$(\lambda_0-\theta)$*

Step 2.1. Derive the *restricted problem* of Problem II$(y(\theta)|\lambda_0-\theta)$, in the knowledge of its optimum solution $\bar{x}$ for $\theta=0$ and solve it. Let an optimum solution be $\bar{\xi}$ and let

(3.2) $$x(\theta)=\bar{x}-\theta\bar{\xi}.$$

Then, determine the range $[\lambda_0-\theta_2, \lambda_0]$ of λ where $x(\theta)$ remains optimal for Problem II$(y(\theta)|\lambda)$. In this range, $\bar{u}$ remains optimal for Problem III$(y(\theta)|\lambda)$.

Step 2.2. Let

(3.3) $$\theta_0=\min\{\theta_1, \theta_2\}.$$

Step 2.3. An optimum solution of Problem I$(\lambda_0-\theta)$ for $0\leq\theta\leq\theta_0$, is $(x(\theta), y(\theta))$.

Step 3. *Finding next starting solution*

When θ goes out of the range, then $(x(\theta), y(\theta))$ is no longer optimal. Apply Benders decomposition, starting from the present Q and using $y(\theta)$ or $\bar{u}$ if necessary, to find an optimum solution of Problem I$(\lambda_0-\theta_0-\varepsilon)$ where ε is a sufficiently small positive number.

(*This completes one cycle of the algorithm.*)

3.2 Validity of the Algorithm

Proposition 3.1.

If the restricted problem in Step 1 has no optimum solution, then Problem I(λ)

has no feasible solution for λ less than λ_0.

Proof: By Theorem 1.1, Problem $I(\lambda)$ is not feasible if and only if Problem IV$(G|\lambda)$ is not feasible. Since $Q\subset C$, we have $G(Q)\supset G$. Hence, if $G(Q)$ is empty, then also G.
Q.E.D.

This proposition works in Step 1 as the termination criterion.

Proposition 3. 2.
$(x(\theta), y(\theta))$, obtained in Step 1 and Step 2, is an optimum solution of Problem $I(\lambda_0-\theta)$ for $0\leq\theta\leq\theta_0$.

Proof: (i) First, we will demonstrate the equality

$$x(\theta)+\bar{u}'Fy(\theta)-f'y(\theta)=\bar{u}'(b+(\lambda_0-\theta)d). \quad (0\leq\theta\leq\theta_0) \tag{3.4}$$

Let Q_1 be a subset of Q such that $(u_0, u)\in Q_1$ satisfies

$$u_0\bar{x}_0+u'F\bar{y}-u_0f'\bar{y}=u'(b+\lambda_0 d).$$

Of course, $(1, \bar{u})\in Q_1$ by assumption.

Then, the restricted problem in Step 1 is

$$Min\{\xi_0|u_0\xi_0+u'F\eta-u_0f'\eta\geq u'd, (u_0, u)\in Q_1\}.$$

Its optimum solution $(\bar{\xi}_0, \bar{\eta})$ must satisfy, for some $(1, u^*)\in Q_1$, the equality

$$\bar{\xi}_0+u^{*\prime}F\bar{\eta}-f'\bar{\eta}=u^{*\prime}d.$$

We can assume that this $(1, u^*)$ is $(1, \bar{u})$, without losing generality, because u^* is also an optimum solution of Problem III$(\bar{y}|\lambda_0)$.

Thus, we have

$$\bar{\xi}_0+\bar{u}'F\bar{\eta}-f'\bar{\eta}=\bar{u}'d.$$

Therefore,

$$(\bar{x}_0-\theta\bar{\xi}_0)+\bar{u}'F(\bar{y}-\theta\bar{\eta})-f'(\bar{y}-\theta\bar{\eta})=\bar{u}'(b+(\lambda_0-\theta)d).$$

This shows the equality (3.4).

(ii) Since $y(\theta)$ is an optimum solution of Problem IV$(G(Q|\lambda_0-\theta))$ and $\bar{u}$ is an optimum solution of Problem III$(y(\theta)|\lambda_0-\theta)$ in the range $0\leq\theta\leq\theta_0$, and we have the equality (3.4), we can conclude that $(x(\theta), y(\theta))$ is optimal for Problem I$(\lambda_0-\theta)$ $(0\leq\theta\leq\theta_0)$, by Theorem 1.2, $x(\theta)$ being an optimum solution of Problem II$(y(\theta)|\lambda_0-\theta)$.
Q.E.D.

The above propositions demonstrate the validity of the algorithm.

4. The Discrete Variable Case

4.1 Algorithm

We will show an algorithm for solving the parametric Problem I(λ) when S

is a finite set of discrete points in R^q. Similarly to the preceding section, we assume that we have already found an optimum solution $(\bar{x}, \bar{y})$ of Problem I(λ_0), the set Q, an optimum solution $\bar{u}$ of Problem III$(\bar{y}|\lambda_0)$ and an optimum solution $(\bar{x}_0, \bar{y})$ of Problem IV $(G(Q|\lambda_0))$; and we have the equality $\bar{u}'(b+\lambda_0 d-F\bar{y})=\bar{x}_0-f'\bar{y}$.

Step 1. *Finding the bound of decrease of* λ

Step 1.1. Try the parametric analysis of Problem IV$(G(Q|\lambda))$ with respect to λ and determine the range $[\lambda_0-\theta_1, \lambda_0]$ $(\theta_1\geq 0)$ where the solution $\bar{y}$ remains optimal. If there is no feasible solution of Problem IV$(G(Q|\lambda))$ for $\lambda<\lambda_0$, then Problem I(λ) has no feasible solution for λ less than λ_0. (*The end.*)

Step 1.2. Otherwise, by applying the parametric analysis of Problem III$(\bar{y}|\lambda)$ with respect to λ, determine the range $[\lambda_0-\theta_2, \lambda_0]$ $(\theta_2\geq 0)$ where the solution $\bar{u}$ remains optimal.

Step 1.3. Let $\theta_0=\min\{\theta_1, \theta_2\}$.

Step 2. *Determining an optimum solution of Problem I*(λ)

If $\theta_0=0$, go to Step 3. Otherwise, solve Problem II$(\bar{y}|\lambda_0-\theta)$ $(0\leq\theta\leq\theta_0)$. Let an optimum solution be $\bar{x}-\theta\bar{\xi}$, where $\bar{\xi}$ is an optimum solution of the restricted problem II$(\bar{y}|\lambda_0)$. Then, we have an optimum solution $(\bar{x}-\theta\bar{\xi}, \bar{y})$ of Problem I$(\lambda_0-\theta)$ for $0\leq\theta\leq\theta_0$.

Step 3. *Finding next starting solution*

Apply Benders decomposition, starting from the present Q and using $\bar{y}$ or $\bar{u}$ if necessary, to find an optimum solution of Problem I$(\lambda_0-\theta_0-\varepsilon)$ where ε is a sufficiently small positive number.

(*This completes one cycle of the algorithm.*)

4.2 Validity of the Algorithm

Proposition 4.1.

If Problem IV$(G(Q|\lambda))$ has no feasible solution for $\lambda<\lambda_0$ in Step 1.1, then Problem I(λ) has no feasible solution for $\lambda<\lambda_0$.

Proof: The same as Proposition 3.1. *Q.E.D.*

Proposition 4.2.

Let the maximum value for the objective function x_0 of Problem IV$(G(Q|\lambda_0-\theta))$ $(0\leq\theta\leq\theta_0)$ be $x_0(\theta)$. Then, $x_0(\theta)$ satisfies

$$x_0(\theta)=\bar{x}_0-\theta\bar{u}'d.$$

Proof: By the definition of θ_0, $\bar{y}$ is an optimum solution of Problem IV$(G(Q|\lambda_0-\theta))$ $(0\leq\theta\leq\theta_0)$. Therefore, for every $(u_0, u)\in Q$, we have

$$u_0x_0(\theta)+u'F\bar{y}-u_0f'\bar{y}\leq u'(b+(\lambda_0-\theta)d).$$

And, since $(1, \bar{u})\in Q$, we have

$$x_0(\theta)+\bar{u}'F\bar{y}-f'\bar{y}\leq\bar{u}'(b+(\lambda_0-\theta)d).$$

But, for $\theta=0$, from the optimality of $(\bar{x}_0, \bar{y})$ and $(1, \bar{u})$, we have

$$\bar{x}_0+\bar{u}'F\bar{y}-f'\bar{y}=\bar{u}'(b+\lambda_0 d).$$

Hence, the following relation holds:

$$x_0(\theta)\leq\bar{x}_0-\theta\bar{u}'d.$$

Now, we demonstrate the equality. Suppose the contrary. Then, for some $\theta(0\leq\theta\leq\theta_0)$, there exists at least a $(1,\bar{\bar{u}})\in Q$, such that

(4.1) $$x_0(\theta)=-\bar{\bar{u}}'F\bar{y}+f'\bar{y}-\bar{\bar{u}}'(b+(\lambda_0-\theta)d)>-\bar{u}'F\bar{y}+f'\bar{y}-\bar{u}'(b+(\lambda_0-\theta)d).$$

But, since $\bar{u}$ is a feasible solution of Problem $\mathrm{III}(\bar{y}|\lambda_0-\theta)$ and $\bar{\bar{u}}$ is an optimum solution, we have

$$\bar{u}'(b+(\lambda_0-\theta)d-F\bar{y})\leq\bar{\bar{u}}'(b+(\lambda_0-\theta)d-F\bar{y}).$$

This contradicts (4.1) *Q.E.D.*

Proposition 4. 3.
The solution $(\bar{x}-\theta\bar{\xi},\bar{y})$ in Step 2 is an optimum solution of Problem $I(\lambda)$ for $\lambda=\lambda_0-\theta(0\leq\theta\leq\theta_0)$.

Proof: Proposition 4.2 means

$$x_0(\theta)-f'\bar{y}=\bar{u}'(b+(\lambda_0-\theta)d-F\bar{y}).$$

By Theorem 1.2, this shows the optimality of $(\bar{x}-\theta\bar{\xi},\bar{y})$. *Q.E.D.*

The above propositions demonstrate the validity of the algorithm.

Concluding Remarks

Benders decomposition is recognized as an excellent partitioning procedure for solving such a mixed type problem that involves both structured variables and non-structured variables. Our algorithm deals with a parametrization of Benders decomposition in the descending value of the parameter. But we can easily modify the algorithm to be valid for the ascending case. TONE (forthcoming) shows an example of applications of algorithms in this paper.

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