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AN ALGORITHM FOR FINDING A COMMON BASIS OF TWO MATROIDS

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ABSTRACT

A matroid is an axiomization of linear independence of column vectors of a matrix. It was first described by WHITNEY (1935) (TUTTE 1965; HARARY and WELSH 1969; WILSON 1973). This paper proposes an algorithm for finding a common basis of two matroids, $M_1=(E, \mathcal{I}_1)$ and $M_2=(E, \mathcal{I}_2)$, for which several algorithms have been proposed (IRI and TOMIZAWA 1976; GREENE and MAGNANTI 1975; LAWLER 1975). The algorithm generates a sequence of pairs of bases of M_1 and independent sets of the dual matroid M_2^* of M_2, which increases the cardinality of their union. It will be shown that just one of the following cases occurs when this process does not generate such a sequence any more:

(i) A pair whose union is E is obtained so that the corresponding basis of M_1 is a common basis,

(ii) M_1 and M_2 have no common basis.

This algorithm can be considered as a modification of EDMONDS' algorithm (EDMONDS 1968) in IRI and TOMIZAWA's manner (IRI and TOMIZAWA 1976).

Existence Theorem of a Common Basis

Let E be a finite set of undefined objects called *elements*. Let $\mathcal{I}$ be a collection of subsets of E. $M=(E, \mathcal{I})$ is a *matroid* on the domain E if $\mathcal{I}$ satisfies the following axioms;

($I1$) If $I \subseteq J$ and $J \in \mathcal{I}$, then $I \in \mathcal{I}$.

($I2$) If $I, J \in \mathcal{I}$ and $|I|<|J|$, then there is an element $e \in J-I$ such that $I \cup \{e\} \in \mathcal{I}$.

For convenience we shall call a member of $\mathcal{I}$ an *independent set*. A *basis* is a maximal independent set contained in the domain E. A *circuit* is a minimal dependent (not independent) set.

Axiom ($I2$) can be replaced with one of the following axioms ($I2-1$) and ($I2-2$) (EDMONDS 1968).

$(I2-1)$ For any subset X of E, every maximal independent set contained in X has the same number of elements.
We refer to the number of elements asserted in Axiom $(I2-1)$ as the *rank* $r(X)$ of X. It is known that the rank function $r(.)$ satisfies the relation

$$r(X\cup Y)+r(X\cap Y)\leqq r(X)+r(Y) \tag{1}$$

for any subsets X and Y of E (MIRSKY 1971).
$(I2-2)$ The union of any independent set I and any element e contains at most one circuit.
We shall denote the unique circuit asserted in Axiom $(I2-2)$ by $C(e, I, M)$. Let X and Y be subsets of E. The subset of Y consisting of $X\cap Y$ and each element $e\in Y$ such that $r(X\cap Y)=r(X\cap Y\cup\{e\})$ is called the *span* of X in Y, and denoted by $S(X, Y, M)$. We also say that each element in the span of X is *dependent* on X.

By the above definitions we get the following lemmas (IRI and TOMIZAWA 1976).

Lemma 1. Let X and Y be any subsets of E such that $Y\supseteq X$. Let I be a maximal independent set in X. Then

$$S(X, Y, M)=S(I, Y, M).$$

Lemma 2. Let I and J be independent sets and $I\subseteq J$. Then, for each element $e\in S(I, E, M)-I$,

$$C(e, J, M)=C(e, I, M).$$

Lemma 3. Let I be an independent set, $e\in I$, $e'\in S(I, E, M)-I$. Then e is contained in $C(e', I, M)-\{e'\}$ if and only if e' is not in $S(I-\{e\}, E, M)$.

Let $M_1=(E, \mathcal{J}_1)$, $M_2=(E, \mathcal{J}_2)$, $\cdots$, $M_k=(E, \mathcal{J}_k)$ be k matroids defined on the domain E. We shall write

$$\mathcal{J}=\{I: I=I_1\cup I_2\cup\cdots\cup I_k, I_i\in\mathcal{J}_i\}.$$

Then $M=(E, \mathcal{J})$ is a matroid, which is known as the *union matroid* (EDMONDS 1968; MIRSKY 1971). We shall denote it by $M=M_1\cup M_2\cup\cdots\cup M_k$.

We here introduce an important theorem by NASH-WILLIAMS without proof (MIRSKY 1971).

Theorem 1. Let $M_1, M_2, \cdots, M_k$ be matroids defined on the domain E. If $r_1, r_2, \cdots, r_k$ and r denote the rank functions of $M_1, M_2, \cdots, M_k$ and the union matroid $M=M_1\cup M_2\cup\cdots\cup M_k$, respectively, then for each subset Y of E, we have the relation

$$r(Y)=\min_{X\subseteq Y}\{r_1(X)+r_2(X)+\cdots+r_k(X)+|Y-X|\}. \tag{2}$$

Let $M=(E, \mathcal{J})$ be a matroid and let for some subset X of E,

$$\mathcal{J}'=\{I: I\in\mathcal{J}, I\subseteq X\}.$$

Then $(E, \mathcal{I}')$ is also a matroid. This matroid is the *contraction* of M to X and is denoted by $M \times X$. Let $\mathcal{B}$ be the collection of all bases of a matroid M and let

$$\mathcal{I}^* = \{I : I \subseteq E - B \text{ for some } B \in \mathcal{B}\}.$$

Then $(E, \mathcal{I}^*)$ is also a matroid which is called the *dual matroid* of M and is usually denoted by M^* (TUTTE 1965). A basis of M^* is the complement of some basis of M, and vice versa. Let M_1 and M_2 be two matroids defined on the domain E. A *common basis* of M_1 and M_2 is a subset of E which is a basis of both M_1 and M_2. Then it is evident that M_1 and M_2 have a common basis if and only if there are disjoint bases of M_1 and M_2^* whose union is E.

Theorem 2. M_1 and M_2 have a common basis if and only if

$$r_1(E) = r_2(E), \tag{3}$$

and there is no subset X of E such that

$$|X| > r_1(X) + r_2^*(X), \tag{4}$$

where r_2^* is the rank function of M_2^*.

Proof. We first verify the necessity. Let B be a common basis, so that

$$r_1(E) = |B| = r_2(E).$$

Since E-B is a basis of M_2^*, E is the disjoint union of bases of M_1 and M_2^*, that is, E is an independent set of $M_1 \cup M_2^*$. Then, by Theorem 1, for any subset X of E,

$$|E| = r(E) \leqq r_1(X) + r_2^*(X) + |E - X|,$$

then we get the relation (4).

The relation (4) indicates that E is itself an independent set of $M_1 \cup M_2^*$, then there are independent sets of M_1 and M_2^* whose union is E. By (3), each independent set must be a basis of M_1 and M_2^*, respectively, and they are mutually disjoint. Then the basis of M_1 is a common basis. *Q. E. D.*

An Algorithm for Finding a Common Basis of Two Matroids

Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be matroids defined on E. We consider the problem of finding a member of $\mathcal{I}_1 \cap \mathcal{I}_2$ which contains as many elements as possible. If there is a common basis of M_1 and M_2, it is obviously the solution of this problem. Hence we can find a common basis if we solve this problem. Let I be a member of $\mathcal{I}_1 \cap \mathcal{I}_2$ which has maximal cardinality. Then, I is to be represented as the intersection of two bases B_1 and B_2 of M_1 and M_2 as follows;

$$I = B_1 \cap B_2.$$

Since
$$|I|=|B_1\cap B_2|=|B_1-(E-B_2)|=|B_1-B_2^*|$$
$$=|B_1\cup B_2^*|-|B_2^*|,$$

and $|B_2^*|$ is constant, maximizing $|I|$ is equivalent to maximizing $|B_1\cup B_2^*|$, where B_2^* is a basis of the dual matroid M_2^* of M_2. On the other hand, any basis B of $M_1\cup M_2^*$ can be represented as the union of bases of M_1 and M_2^*; $B=B_1\cup B_2^*$. By Axiom $(I2-1)$, each basis has the maximal cardinality of independent sets and conversely any independent sets of maximal cardinality are bases. Thus B is a basis of $M_1\cup M_2^*$ if and only if B_1 and B_2^* are the pair of bases maximizing $|B_1\cup B_2^*|$. Thus the problem of finding a common basis is now reduced to the problem of finding a basis of the union matroid.

We first define a graph $G=(V,L)$ for a basis I_1 of M_1 and a basis I_2^* of $M_2^*\times(E-I_1)$. The vertex-set V is the disjoint union of $V_1=\{e_1', e_2', \cdots, e_n'\}$ and $V_2=\{e_1'', e_2'', \cdots, e_n''\}$ both of which are replicas of $E=\{e_1, e_2, \cdots, e_n\}$, that is, there are one-to-one correspondence F_1 from V_1 onto E and F_2 from V_2 onto E such that $F_1(e_i')=F_2(e_i'')=e_i$. The edge-set L is also the disjoint union of L_{12}, L_{11}, and L_{22}, where L_{12} is the set of all undirected edges (e_i', e_i''), $e_i'\in V_1$, $e_i''\in V_2$, L_{11} the set of all directed edges (e_i', e_j') such that $e_j\in C(e_i, I_1, M_1)-\{e_i\}$, and L_{22} the set of all directed edges (e_i'', e_j'') such that $e_j\in C(e_i, I_2^*, M_2^*)-\{e_i\}$. We write $V_A=F_1^{-1}(S(I_2^*, E, M_2^*)-(I_1\cup I_2^*))\cup F_2^{-1}(S(I_2^*, E, M_2^*)-(I_1\cup I_2^*))$. We also write $V_B=F_2^{-1}(E-S(I_2^*, E, M_2^*))$. A *path* P from a vertex e_{i_1}' or e_{i_1}'' in V_A to a vertex e_{i_m}'' in V_B is a sequence of edges of the form $\{(e_{i_1}', e_{i_1}''), (e_{i_1}'', e_{i_2}''), (e_{i_2}'', e_{i_2}'), \cdots, (e_{i_{m-1}}', e_{i_m}'), (e_{i_m}', e_{i_m}'')\}$ or the form $\{(e_{i_1}'', e_{i_1}'), (e_{i_1}', e_{i_2}'), (e_{i_2}', e_{i_2}''), \cdots. (e_{i_{m-1}}', e_{i_m}'), (e_{i_m}', e_{i_m}'')\}$ in which each edge has one endpoint in common with its predecessor in the sequence and the other endpoint in common with its successor in the sequence, and all edges of $L_{11}\cup L_{22}$ in the sequence are oriented along the sequence. The *length* of the path P is the number of edges in P.

The algorithm introduced here is a natural modification of the algorithm by EDMONDS (1968). The iteration steps are shown as below:
1) Take a basis I_1 of M_1 and a basis I_2^* of $M_2^*\times(E-I_1)$.
2) Construct the graph G for I_1 and I_2^*.
3) Find a path P of the shortest length from an arbitrary vertex of V_A to an arbitrary vertex of V_B.
4) If there is no path from V_A to V_B, terminate the iteration. Otherwise, go to 5).
5) Orient the edges of L_{12} in the path P along P and decompose them into P_1 and P_2, where P_1 is the set of edges from V_2 to V_1 and P_2 from V_1 to V_2. Since P_1 and P_2 are the sets of edges of the form (e_i'', e_i'), and (e_i', e_i'') regard them as the set of vertices e_i' of V_1. Replace I, by

$$I_1\cup F_1(P_1)-F_1(P_2),$$

and I_2^* by

$$I_2^*\cup F_1(P_2)-F_1(P_1),$$

and proceed from 2).

To see the validity of the algorithm, we have to prove the following Theorems 3, 4 and 5.

Theorem 3. Let I_1' and $I_2^{*\prime}$ be the updated sets in 5). Then I_1' is still a basis of M_1, $I_2^{*\prime}$ is a basis of $M_2^* \times (E-I_1')$ and $|I_2^{*\prime}|=|I_2^*|+1$.

Theorem 4. This algorithm terminates by finding a common basis of M_1 and M_2 or indicating that M_1 and M_2 have no common basis when there is no path from V_A to V_B (including the case that $V_A=\phi$ and/or $V_B=\phi$).

Theorem 5. This algorithm terminates by finding a basis of $M_1 \cup M_2^*$ even if M_1 and M_2 have no common basis.

Proofs of the Theorems

In preparation for the proof of Theorem 3, we shall introduce a lemma confirmed by Iri and Tomizawa (1976).

Lemma 4. Let I be an independent set of a matroid M. If there are $2q$ elements $\{e_1, e_2, \cdots, e_q, f_1, f_2, \cdots, f_q\}$ such that $e_i \notin I$, $f_i \in I$ for $1 \leqq i \leqq q$ and

$$f_j \in C(e_j, I, M)-\{e_j\} \qquad 1 \leqq j \leqq q, \tag{5}$$

$$f_j \notin C(e_i, I, M)-\{e_i\} \qquad 1 \leqq i < j \leqq q, \tag{6}$$

then $I'=I \cup \{e_1, e_2, \cdots, e_q\}-\{f_1, f_2, \cdots, f_q\}$ is independent.

Proof. If $q=1$, then the assertion is trivial from Axiom $(I2-2)$. We suppose that the assertion holds when $q \leqq p-1$ as the inductive hypothesis. The set $I''=I \cup \{e_p\}-\{f_p\}$ is independent by (5). To complete the proof, we have to show that the set I'' and the elements $\{e_1, e_2, \cdots, e_{p-1}, f_1, f_2, \cdots, f_{p-1}\}$ satisfies the condition (5) and (6). Since, for any $i<p$,

$$f_p \notin C(e_i, I, M)-\{e_i\},$$

we have, by Lemma 3,

$$e_i \in S(I-\{f_p\}, E, M).$$

Then, by Lemma 2,

$$\begin{aligned} C(e_i, I, M) &= C(e_i, I-\{f_p\}, M) \\ &= C(e_i, I-\{f_p\} \cup \{e_p\}, M) \\ &= C(e_i, I'', M). \end{aligned}$$

Thus the lemma follows. *Q. E. D.*

Proof of Theorem 3. We first observe that Theorem 3 is true when the number of edges of L_{12} in P is even. We call the edges of L_{11} in P along P $g_1, g_2, \cdots, g_r$, the edges of L_{22} in P along P $h_1, h_2, \cdots, h_{r-1}$, as shown in Fig. 1. Let the initial endpoint and terminal endpoint of g_i be e_i' and f_i' and let the initial endpoint and

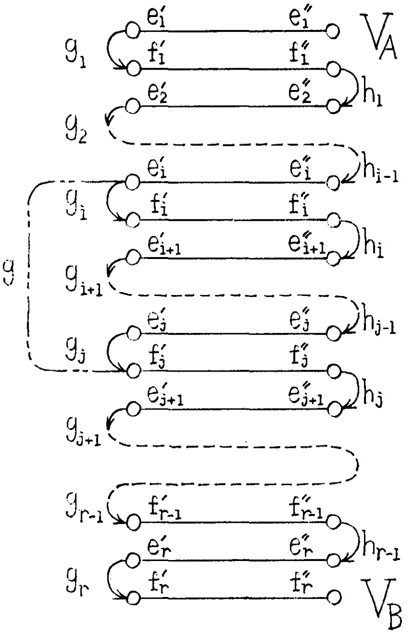

Fig. 1.

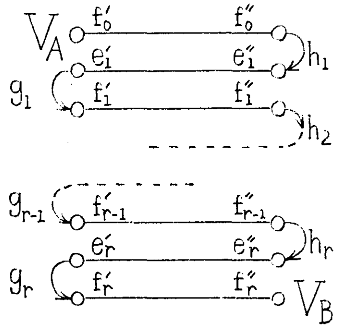

Fig. 2.

terminal endpoint of h_i be f_i'' and e_{i+1}''. Since, by the construction of L_{11},

$$e_i = F_1(e_i') \notin I_1 \qquad 1 \leqq i \leqq r,$$

$$f_i = F_1(f_i') \in I_1 \qquad 1 \leqq i \leqq r,$$

we get

$$|I_1'| = |I_1 \cup F_1(P_1) - F_1(P_2)|$$

$$= |I_1 \cup \{e_1, e_2, \cdots, e_r\} - \{f_1, f_2, \cdots, f_r\}| = |I_1|.$$

Then we have only to verify that I_1' is an independent set of M_1 to prove that I_1' is a basis of M_1. By the construction

$$f_j \in C(e_j, I_1, M_1) - \{e_j\} \qquad 1 \leqq j \leqq r,$$

which implies that $\{e_1, e_2, \cdots, e_r, f_1, f_2, \cdots f_r\}$ satisfies (5). To prove that $\{e_1, e_2, \cdots, e_r, f_1, f_2, \cdots, f_r\}$ satisfies (6) we suppose that

$$f_j \in C(e_i, I_1, M_1) - \{e_i\}$$

for some i and j such that $1 \leqq i < j \leqq r$. Then there should exist an edge $g = (e_i', f_j')$ of L_{11}. Though the path P can be written in the form

$$P = \{\cdots, (e_i'', e_i'), (e_i', f_i'), (f_i', f_i''), \cdots, (e_j'', e_j'), (e_j', f_j'), (f_j', f_j''), \cdots\}$$

we could find a path P'

$$P'=\{\cdots,(e_i'',e_i'),(e_i',f_j'),(f_j',f_j''),\cdots\}$$

of shorter length. This contradicts the choice of P. Hence I_1' is an independent set of M_1.

Next let us consider $I_2^{*\prime}$. By the construction of L_{22}

$$e_{j+1}\in C(f_j,I_2^*,M_2^*)-\{f_j\}\qquad 1\leqq j\leqq r-1,$$

$$e_{j+1}\notin C(f_i,I_2^*,M_2^*)-\{f_i\}\qquad 1\leqq i<j\leqq r-1,$$

then

$$I_2^{*\prime\prime}=I_2^*\cup\{f_1,f_2,\cdots,f_{r-1}\}-\{e_2,e_3,\cdots e_r\}$$

is an independent set of M_2^*, and $|I_2^{*\prime\prime}|=|I_2^*|$. If we recall that $S(I_2^{*\prime\prime},E,M_2^*)=S(I_2^*,E,M_2^*)$, by Lemma 1, then f_r is not in $S(I_2^{*\prime\prime},E,M_2^*)$. Therefore,

$$\begin{aligned}I_2^{*\prime}&=I_2^*\cup F_1(P_2)-F_1(P_1)\\&=I_2^*\cup\{f,f_2,\cdots,f_{r-1}\}-\{e_2,e_3\cdots,e_r\}\cup\{f_r\}\\&=I_2^{*\prime\prime}\cup\{f_r\}\end{aligned}\qquad(7)$$

is an independent set of M_2^* and $|I_2^{*\prime}|=|I_2^*|+1$.

As shown before,

$$I_1'=I_1\cup\{e_1,e_2,\cdots,e_r\}-\{f_1,f_2,\cdots,f_r\},$$

then

$$\begin{aligned}E-I_1'&=(E-I_1)\cup\{f_1,f_2,\cdots,f_r\}-\{e_1,e_2,\cdots,e_r\}\\&\subseteq(E-I_1)\cup\{f_1,f_2,\cdots,f_{r-1}\}\cup\{f_r\}.\end{aligned}$$

By the construction of L_{22}, since $I_2^*\subseteq E-I_1$,

$$\{f_1,f_2,\cdots,f_{r-1}\}\subseteq S(I_2^*,E,M_2^*)\subseteq S(E-I_1,E,M_2^*).$$

Thus

$$\begin{aligned}r_2^*(E-I_1')&\leqq r_2^*((E-I_1)\cup\{f_1,f_2,\cdots,f_{r-1}\}\cup\{f_r\})\\&\leqq r_2^*(E-I_1)+1.\end{aligned}\qquad(8)$$

Hence, by (7) and (8), I_2^* is a basis of $M_2^*\times(E-I_1')$.

The same argument holds for the case where the number of edges of L_{12} is odd, as shown in Fig. 2. *Q.E.D.*

Proof of Theorem 4. We shall confirm Theorem 4 with aid of Theorem 2. There is no path P if one of the following cases occurs:

case (1) Both V_A and V_B are empty.
case (2) V_B is not empty but V_A is empty.
case (3) V_A is not empty but V_B is empty.
case (4) Neither V_A nor V_B is empty but there is no path P from V_A to V_B.

In case (1), since $V_B=F_2^{-1}(E-S(I_2^*, E, M_2^*))$ is empty, we get $E=S(I_2^*, E, M_2^*)$ which implies that I_2^* is a basis of M_2^*. On the other hand, since $V_A=F_1^{-1}(S(I_2^*, E, M_2^*)-(I_1\cup I_2^*))\cup F_2^{-1}(S(I_2^*, E, M_2^*)-(I_1\cup I_2^*))$ is empty, the union of I_1 and I_2^* is $S(I_2^*, E, M_2^*)=E$. Thus I_1 and I_2^* are disjoint bases of M_1 and M_2^*, respectively, and I_1 is a common basis of M_1 and M_2.

In case (2), $V_B\neq\phi$ implies that I_2^* is not a basis of M_2^*. Moreover $V_A=\phi$, that is, $S(I_2^*, E, M_2^*)\subseteq I_1\cup I_2^*$, implies that $I_1\cup I_2^*=E$. Because, otherwise, for any element e in $E-(I_1\cup I_2^*)$, $I_2^*\cup\{e\}$ would be an independent set of M_2^* and $I_2^*\cup\{e\}\subseteq E-I_1$, which contradicts that I_2^* is a basis of $M_2^*\times(E-I_1)$. Therefore,

$$r_1(E)+r_2^*(E)>|I_1|+|I_2^*|=|E|,$$

that is,

$$r_2(E)=|E|-r_2^*(E)<r_1(E).$$

Thus there is no common basis by virtue of (3) of Theorem 2.

In the similar way, we can prove that there is no common basis in case (3). Since $V_B=\phi$ implies that I_2^* is a basis of M_2^*, $E-I_2^*$ is a basis of M_2. For $V_A\neq\phi$,

$$|I_1|+|I_2^*|<|E|,$$

therefore

$$r_2(E)=|E-I_2^*|>|I_1|=r_1(E).$$

Thus there is no common basis.

To complete the proof we have to verify that there is no common basis in case (4). Choose an arbitrary pair of vertices e_0' and e_0'' in V_A. Let the set U be the set of all vertices to which there is a path from one of the pair. Let $U_1=U\cap V_1$ and $U_2=U\cap V_2$. As $F_1(U_1)=F_2(U_2)$, we put $A=F_1(U_1)$. By the construction of L_{11} and L_{22},

$$(U-(\{e_0'\}\cup\{e_0''\}))\cap V_A=\phi,$$

that is,

$$A-\{e_0\}\subseteq I_1\cup I_2^*. \tag{9}$$

By the construction of A and the fact that there is no path from V_A to V_B,

$$A\subseteq S(I_1, A, M_1),$$

$$A\subseteq S(I_2^*, A, M_2^*),$$

which implies that $I_1\cap A$ and $I_2^*\cap A$ are maximal independent sets in A of M_1 and M_2^*, respectively. Hence

$$r_1(A)=|I_1\cap A|,$$

$$r_2^*(A)=|I_2^*\cap A|.$$

Recalling the relation (9), we get

$$|A|=|(I_1\cap A)\cup(I_2^*\cap A)\cup\{e_0\}|$$
$$=|I_1\cap A|+|I_2^*\cap A|+1$$
$$>|I_1\cap A|+|I_2^*\cap A|$$
$$=r_1(A)+r_2^*(A),$$

which violates the condition (4). *Q. E. D.*

Proof of Theorem 5. We shall verify Theorem 5 in each cases as in the proof of Theorem 4. Let $B_1=I_1$ and B_2^* be an arbitrary basis of M_2^* containing I_2^*, where I_1 and I_2^* are the sets obtained when the algorithm terminates. To verify Theorem 5, we have only to show that $|B_1\cup B_2^*|\geqq|B_1'\cup B_2^{*\prime}|$ for any pair of bases B_1' and $B_2^{*\prime}$ of M_1 and M_2^*.

As shown in the proof of Theorem 4, case (1) and (3) imply that I_1 and I_2^* are the disjoint bases of M_1 and M_2^*. Then,

$$|B_1\cup B_2^*|=|I_1\cup I_2^*|=|I_1|+|I_2^*|,$$

which implies that $B_1\cup B_2^*$ is a basis of $M_1\cup M_2^*$.

Let us consider the case (2). Generally,

$$|B_1'\cup B_2^*|\leqq|E|. \tag{10}$$

However, in the case (2), $E=I_1\cup I_2^*$. Thus we get the relation

$$|B_1'\cup B_2^{*\prime}|\leqq|I_1\cup I_2^*|=|B_1\cup B_2^*|.$$

Case (4) is left to prove. Let U_j $(1\leqq j\leqq t)$ be the set of all vertices to which there is a path from e_j' or e_j'' $(1\leqq j\leqq t)$ in V_A. And let $U_{1j}=U_j\cap V_1$ and $U_{2j}=U_j\cap V_2$. As $F_1(U_{1j})=F_2(U_{2j})$, let $A_j'=F_1(U_{1j})$. If some of them intersect, we combine them and denote it by A_i $(1\leqq i\leqq u)$. We assume that A_i is the union of m_i A_j''s, that is, $A_i=A_{j_1}'\cup A_{j_2}'\cup\cdots\cup A_{j_{m_i}}'$. Then

$$A_i\cap A_k=\phi \qquad i\neq k, \tag{11}$$

$$\sum_{i=1}^{u} m_i=t. \tag{12}$$

Now we write $A=\bigcup_{i=1}^{u}A_i$. Since each element of A_i is dependent on both $A_i\cap I_1$ and $A_i\cap I_2^*$,

$$r_1(A_i)=|I_1\cap A_i|,$$
$$r_2^*(A_i)=|I_2^*\cap A_i|.$$

Then, using the property that $I_1\cap I_2^*=\phi$,

$$r_1(A_i)+r_2^*(A_i)=|I_1\cap A_i|+|I_2^*\cap A_i|$$

$$=|(I_1 \cup I_2^*) \cap A_i|$$
$$=|A_i|-|A_i-(I_1 \cup I_2^*)|$$
$$=|A_i|-m_i. \tag{13}$$

If we now recall (1), then we have that

$$r_1(A)=r_1(A_1 \cup A_2 \cup \cdots \cup A_u)$$
$$\leqq r_1(A_1)+r_1(A_2)+\cdots+r_1(A_u)$$
$$=\sum_{i=1}^{u} r_1(A_i),$$

$$r_2^*(A) \leqq \sum_{i=1}^{u} r_2^*(A_i).$$

By (12) and (13),

$$r_1(A)+r_2^*(A) \leqq \sum_{i=1}^{u} (r_1(A_i)+r_2^*(A_i))$$
$$=\sum_{i=1}^{u} (|A_i|-m_i)$$
$$=\sum_{i=1}^{u} |A_i|-t.$$

Hence by (11),

$$r_1(A)+r_2^*(A)+|E-A| \leqq \sum_{i=1}^{u} |A_i|-t+|E|-|A|$$
$$=|A|-t+|E|-|A|=|E|-t.$$

Then using Theorem 1, the rank $r(E)$ of E with respect to $M_1 \cup M_2^*$ satisfies the relation that

$$r(E) \leqq |E|-t. \tag{14}$$

On the other hand $I_1 \cup I_2^*$ is an independent set of $M_1 \cup M_2^*$ and

$$|I_1 \cup I_2^*|=|E|-t.$$

Then,

$$r(E) \geqq |E|-t. \tag{15}$$

(14) and (15) imply that

$$r(E)=|E|-t.$$

For an arbitrary pair of bases B_1' and $B_2^{*\prime}$

$$|B_1' \cup B_2^{*\prime}| \leqq r(E)=|E|-t=|B_1 \cup B_2^*|.$$ *Q. E. D.*

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