Title	An algorithm for finding a common basis of two matroids
Sub Title	
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Publisher	慶応義塾大学工学部
Publication year	1976
Jtitle	Keio engineering reports Vol.29, No.4 (1976. 6) ,p.41- 51
JaLC DOI	
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Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00290004- 0041

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AN ALGORITHM FOR FINDING A COMMON BASIS OF TWO MATROIDS

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(Received, May 10, 1976)

ABSTRACT

A matroid is an axiomization of linear independence of column vectors of a matrix. It was first described by WHITNEY (1935) (TUTTE 1965; HARARY and WELSH 1969; WILSON 1973). This paper proposes an algorithm for finding a common basis of two matroids, $M_1=(E, \mathcal{J}_1)$ and $M_2=(E, \mathcal{J}_2)$, for which several algorithms have been proposed (IRI and TOMIZAWA 1976; GREENE and MAGNANTI 1975; LAWLER 1975). The algorithm generates a sequence of pairs of bases of M_1 and independent sets of the dual matroid M_2^* of M_2 , which increases the cardinality of their union. It will be shown that just one of the following cases occurs when this process does not generate such a sequence any more:

- (i) A pair whose union is E is obtained so that the corresponding basis of M_1 is a common basis,
- (ii) M_1 and M_2 have no common basis.

This algorithm can be considered as a modification of EDMONDS' algorithm (EDMONDS 1968) in IRI and TOMIZAWA'S manner (IRI and TOMIZAWA 1976).

Existence Theorem of a Common Basis

Let E be a finite set of undefined objects called *elements*. Let \mathcal{G} be a collection of subsets of E. $M=(E, \mathcal{G})$ is a *matroid* on the domain E if \mathcal{G} satisfies the following axioms;

(*I*1) If $I \subseteq J$ and $J \in \mathcal{J}$, then $I \in \mathcal{J}$.

(12) If $I, J \in \mathcal{J}$ and |I| < |J|, then there is an element $e \in J-I$ such that $I \cup \{e\} \in \mathcal{J}$. For convenience we shall call a member of \mathcal{J} an *independent set*. A *basis* is a maximal independent set contained in the domain *E*. A *circuit* is a minimal dependent (not independent) set.

Axiom (I2) can be replaced with one of the following axioms (I2-1) and (I2-2) (EDMONDS 1968).

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(I2-1) For any subset X of E, every maximal independent set contained in X has the same number of elements.

We refer to the number of elements asserted in Axiom (I2-1) as the rank r(X) of X. It is known that the rank function r(.) satisfies the relation

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y) \tag{1}$$

for any subsets X and Y of E (MIRSKY 1971).

(I2-2) The union of any independent set I and any element e contains at most one circuit.

We shall denote the unique circuit asserted in Axiom (I2-2) by C(e, I, M). Let X and Y be subsets of E. The subset of Y consisting of $X \cap Y$ and each element $e \in Y$ such that $r(X \cap Y) = r(X \cap Y \cup \{e\})$ is called the *span* of X in Y, and denoted by S(X, Y, M). We also say that each element in the span of X is *dependent* on X.

By the above definitions we get the following lemmas (IRI and TOMIZAWA 1976).

Lemma 1. Let X and Y be any subsets of E such that $Y \supseteq X$. Let I be a maximal independent set in X. Then

$$S(X, Y, M) = S(I, Y, M).$$

Lemma 2. Let I and J be independent sets and $I \subseteq J$. Then, for each element $e \in S(I, E, M) - I$,

$$C(e, J, M) = C(e, I, M).$$

Lemma 3. Let I be an independent set, $e \in I$, $e' \in S(I, E, M) - I$. Then e is contained in $C(e', I, M) - \{e'\}$ if and only if e' is not in $S(I - \{e\}, E, M)$.

Let $M_1=(E, \mathcal{J}_1), M_2=(E, \mathcal{J}_2), \dots, M_k=(E, \mathcal{J}_k)$ be k matroids defined on the domain E. We shall write

$$\mathcal{J} = \{I : I = I_1 \cup I_2 \cup \cdots \cup I_k, I_i \in \mathcal{J}_i\}.$$

Then $M=(E, \mathcal{J})$ is a matroid, which is known as the *union matroid* (EDMONDS 1968; MIRSKY 1971). We shall denote it by $M=M_1\cup M_2\cup\cdots\cup M_k$.

We here introduce an important theorem by NASH-WILLIAMS without proof (M_{IRSKY} 1971).

Theorem 1. Let M_1, M_2, \dots, M_k be matroids defined on the domain *E*. If r_1, r_2, \dots, r_k and *r* denote the rank functions of M_1, M_2, \dots, M_k and the union matroid $M = M_1 \cup M_2 \cup \dots \cup M_k$, respectively, then for each subset *Y* of *E*, we have the relation

$$r(Y) = \min_{X \subseteq Y} \{r_1(X) + r_2(X) + \dots + r_k(X) + |Y - X|\}.$$
(2)

Let $M = (E, \mathcal{J})$ be a matroid and let for some subset X of E,

$$\mathcal{G}' = \{I : I \in \mathcal{G}, I \subseteq X\}.$$

Then (E, \mathcal{G}') is also a matroid. This matroid is the *contraction* of M to X and is denoted by $M \times X$. Let \mathcal{B} be the collection of all bases of a matroid M and let

$$\mathcal{J}^* = \{I : I \subseteq E - B \text{ for some } B \in \mathcal{B}\}.$$

Then (E, \mathcal{J}^*) is also a matroid which is called the *dual matroid* of M and is usually denoted by M^* (TUTTE 1965). A basis of M^* is the complement of some basis of M, and vice versa. Let M_1 and M_2 be two matroids defined on the domain E. A common basis of M_1 and M_2 is a subset of E which is a basis of both M_1 and M_2 . Then it is evident that M_1 and M_2 have a common basis if and only if there are disjoint bases of M_1 and M_2^* whose union is E.

Theorem 2. M_1 and M_2 have a common basis if and only if

$$r_1(E) = r_2(E),$$
 (3)

and there is no subset X of E such that

$$|X| > r_1(X) + r_2^*(X),$$
 (4)

where r_2^* is the rank function of M_2^* .

Proof. We first verify the necessity. Let B be a common basis, so that

$$\boldsymbol{r}_1(E) = |B| = \boldsymbol{r}_2(E).$$

Since E-B is a basis of M_2^* , E is the disjoint union of bases of M_1 and M_2^* , that is, E is an independent set of $M_1 \cup M_2^*$. Then, by Theorem 1, for any subset X of E,

$$|E| = r(E) \leq r_1(X) + r_2^*(X) + |E - X|,$$

then we get the relation (4).

The relation (4) indicates that E is itself an independent set of $M_1 \cup M_2^*$, then there are independent sets of M_1 and M_2^* whose union is E. By (3), each independent set must be a basis of M_1 and M_2^* , respectively, and they are mutually disjoint. Then the basis of M_1 is a common basis. Q. E. D.

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Let $M_1 = (E, \mathcal{J}_1)$ and $M_2 = (E, \mathcal{J}_2)$ be matroids defined on E. We consider the problem of finding a member of $\mathcal{J}_1 \cap \mathcal{J}_2$ which contains as many elements as possible. If there is a common basis of M_1 and M_2 , it is obviously the solution of this problem. Hence we can find a common basis if we solve this problem. Let I be a member of $\mathcal{J}_1 \cap \mathcal{J}_2$ which has maximal cardinality. Then, I is to be represented as the intersection of two bases B_1 and B_2 of M_1 and M_2 as follows;

$$I = B_1 \cap B_2.$$

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Since

$$|I| = |B_1 \cap B_2| = |B_1 - (E - B_2)| = |B_1 - B_2^*|$$
$$= |B_1 \cup B_2^*| - |B_2^*|,$$

and $|B_2^*|$ is constant, maximizing |I| is equivalent to maximizing $|B_1 \cup B_2^*|$, where B_2^* is a basis of the dual matroid M_2^* of M_2 . On the other hand, any basis B of $M_1 \cup M_2^*$ can be represented as the union of bases of M_1 and M_2^* ; $B=B_1 \cup B_2^*$. By Axiom (I2-1), each basis has the maximal cardinality of independent sets and conversely any independent sets of maximal cardinality are bases. Thus B is a basis of $M_1 \cup M_2^*$ if and only if B_1 and B_2^* are the pair of bases maximizing $|B_1 \cup B_2^*|$. Thus the problem of finding a common basis is now reduced to the problem of finding a basis of the union matroid.

We first define a graph G = (V, L) for a basis I_1 of M_1 and a basis I_2^* of $M_2^* \times$ $(E-I_1)$. The vertex-set V is the disjoint union of $V_1 = \{e'_1, e'_2, \dots, e'_n\}$ and $V_2 = \{e''_1, e'_2, \dots, e'_n\}$ e''_2, \dots, e''_n both of which are replicas of $E = \{e_1, e_2, \dots, e_n\}$, that is, there are one-toone correspondence F_1 from V_1 onto E and F_2 from V_2 onto E such that $F_1(e'_i) =$ $F_2(e_i'')=e_i$. The edge-set L is also the disjoint union of L_{12} , L_{11} , and L_{22} , where L_{12} is the set of all undirected edges $(e'_i, e''_i), e'_i \in V_1, e''_i \in V_2, L_{11}$ the set of all directed edges (e'_i, e'_j) such that $e_j \in C(e_i, I_1, M_1) - \{e_i\}$, and L_{22} the set of all directed edges (e''_i, e''_j) such that $e_j \in C(e_i, I_2^*, M_2^*) - \{e_i\}$. We write $V_A = F_1^{-1}(S(I_2^*, E, M_2^*)) - \{e_i\}$ $(I_1 \cup I_2^*)) \cup F_2^{-1}(S(I_2^*, E, M_2^*) - (I_1 \cup I_2^*)). \text{ We also write } V_B = F_2^{-1}(E - S(I_2^*, E, M_2^*)). A = F_2^{-1}(E - S(I_2^*, E, M_2^*)).$ path P from a vertex e'_{i_1} or e''_{i_1} in V_A to a vertex e''_{i_m} in V_B is a sequence of edges of the form $\{(e'_{i_1}, e''_{i_1}), (e''_{i_1}, e''_{i_2}), (e''_{i_2}, e'_{i_2}), \cdots, (e'_{i_{m-1}}, e'_{i_m}), (e'_{i_m}, e''_{i_m})\}$ or the form $\{(e''_{i_1}, e'_{i_1}), (e''_{i_1}, e''_{i_1}), (e'''_{i_1}, e''_{i_1}), (e''_{i_1}, e''_{i_1})$ $(e'_{i_1}, e'_{i_2}), (e'_{i_2}, e''_{i_2}), \dots, (e'_{i_{m-1}}, e'_{i_m}), (e'_{i_m}, e''_{i_m})\}$ in which each edge has one endpoint in common with its predecessor in the sequence and the other endpoint in common with its successor in the sequence, and all edges of $L_{11} \cup L_{22}$ in the sequence are oriented along the sequence. The *length* of the path P is the number of edges in P.

The algorithm introduced here is a natural modification of the algorithm by Edmonds (1968). The iteration steps are shown as below:

1) Take a basis I_1 of M_1 and a basis I_2^* of $M_2^* \times (E-I_1)$.

2) Construct the graph G for I_1 and I_2^* .

3) Find a path P of the shortest length from an arbitrary vertex of V_A to an arbitrary vertex of V_B .

4) If there is no path from V_A to V_B , terminate the iteration. Otherwise, go to 5). 5) Orient the edges of L_{12} in the path P along P and decompose them into P_1 and P_2 , where P_1 is the set of edges from V_2 to V_1 and P_2 from V_1 to V_2 . Since P_1 and P_2 are the sets of edges of the form (e''_i, e'_i) , and (e'_i, e''_i) regard them as the set of vertices e'_i of V_1 . Replace I, by

$$I_1 \cup F_1(P_1) - F_1(P_2),$$

and I_2^* by

$$I_2^* \cup F_1(P_2) - F_1(P_1),$$

and proceed from 2).

To see the validity of the algorithm, we have to prove the following Theorems 3, 4 and 5.

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Theorem 3. Let I'_1 and I'_2 be the updated sets in 5). Then I'_1 is still a basis of M_1 , I'_2 is a basis of $M'_2 \times (E - I'_1)$ and $|I'_2| = |I'_2| + 1$.

Theorem 4. This algorithm terminates by finding a common basis of M_1 and M_2 or indicating that M_1 and M_2 have no common basis when there is no path from V_A to V_B (including the case that $V_A = \phi$ and/or $V_B = \phi$).

Theorem 5. This algorithm terminates by finding a basis of $M_1 \cup M_2^*$ even if M_1 and M_2 have no common basis.

Proofs of the Theorems

In preparation for the proof of Theorem 3, we shall introduce a lemma confirmed by IRI and TOMIZAWA (1976).

Lemma 4. Let I be an independent set of a matroid M. If there are 2q elements $\{e_1, e_2, \dots, e_q, f_1, f_2, \dots, f_q\}$ such that $e_i \notin I$, $f_i \in I$ for $1 \leq i \leq q$ and

$$f_j \in C(e_j, I, M) - \{e_j\} \qquad 1 \le j \le q, \tag{5}$$

$$f_j \notin C(e_i, I, M) - \{e_i\} \qquad 1 \leq i < j \leq q, \tag{6}$$

then $I' = I \cup \{e_1, e_2, \dots, e_q\} - \{f_1, f_2, \dots, f_q\}$ is independent.

Proof. If q=1, then the assertion is trivial from Axiom (I2-2). We suppose that the assertion holds when $q \leq p-1$ as the inductive hypothesis. The set $I'' = I \cup \{e_p\} - \{f_p\}$ is independent by (5). To complete the proof, we have to show that the set I'' and the elements $\{e_1, e_2, \dots, e_{p-1}, f_1, f_2, \dots, f_{p-1}\}$ satisfies the condition (5) and (6). Since, for any i < p,

$$f_p \notin C(e_i, I, M) - \{e_i\},$$

we have, by Lemma 3,

$$e_i \in S(I - \{f_p\}, E, M).$$

Then, by Lemma 2,

$$C(e_i, I, M) = C(e_i, I - \{f_p\}, M)$$

= $C(e_i, I - \{f_p\} \cup \{e_p\}, M)$
= $C(e_i, I'', M).$

Thus the lemma follows.

Proof of Theorem 3. We first observe that Theorem 3 is true when the number of edges of L_{12} in P is even. We call the edges of L_{11} in P along $P g_1, g_2, \dots, g_r$, the edges of L_{22} in P along $P h_1, h_2, \dots, h_{r-1}$, as shown in Fig. 1. Let the initial endpoint and terminal endpoint of g_i be e'_i and f'_i and let the initial endpoint and

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terminal endpoint of h_i be f''_i and e''_{i+1} . Since, by the construction of L_{11} ,

$$e_i = F_1(e'_i) \notin I_1 \qquad 1 \leq i \leq r,$$

$$f_i = F_1(f'_i) \in I_1 \qquad 1 \leq i \leq r,$$

we get

$$|I'_{1}| = |I_{1} \cup F_{1}(P_{1}) - F_{1}(P_{2})|$$

= |I_{1} \cup \{e_{1}, e_{2}, \dots, e_{r}\} - \{f_{1}, f_{2}, \dots, f_{r}\}| = |I_{1}|.

Then we have only to verify that I'_1 is an independent set of M_1 to prove that I'_1 is a basis of M_1 . By the construction

$$f_j \in C(e_j, I_1, M_1) - \{e_j\} \qquad 1 \leq j \leq r,$$

which implies that $\{e_1, e_2, \dots, e_r, f_1, f_2, \dots f_r\}$ satisfies (5). To prove that $\{e_1, e_2, \dots, e_r, f_1, f_2, \dots, f_r\}$ satisfies (6) we suppose that

$$f_j \in C(e_i, I_1, M_1) - \{e_i\}$$

for some *i* and *j* such that $1 \le i < j \le r$. Then there should exist an edge $g = (e'_i, f'_j)$ of L_{11} . Though the path *P* can be written in the form

$$P = \{ \cdots, (e''_i, e'_i), (e'_i, f'_i), (f'_i, f''_i), \cdots, (e''_j, e'_j), (e'_j, f'_j), (f'_j, f''_j), \cdots \}$$

we could find a path P'

$$P' = \{ \cdots, (e''_i, e'_i), (e'_i, f'_j), (f'_j, f''_j), \cdots \}$$

of shorter length. This contradicts the choice of P. Hence I'_1 is an independent set of M_1 .

Next let us consider $I_2^{*\prime}$. By the construction of L_{22}

$$e_{j+1} \in C(f_j, I_2^*, M_2^*) - \{f_j\} \qquad 1 \le j \le r-1,$$

$$e_{j+1} \notin C(f_i, I_2^*, M_2^*) - \{f_i\} \qquad 1 \le i < j \le r-1,$$

then

$$I_2^{*''} = I_2^* \cup \{f_1, f_2, \cdots, f_{r-1}\} - \{e_2, e_3, \cdots e_r\}$$

is an independent set of M_2^* , and $|I_2^{*\prime\prime}| = |I_2^*|$. If we recall that $S(I_2^{*\prime\prime}, E, M_2^*) = S(I_2^*, E, M_2^*)$, by Lemma 1, then f_r is not in $S(I_2^{*\prime\prime}, E, M_2^*)$. Therefore,

$$I_{2}^{*'} = I_{2}^{*} \cup F_{1}(P_{2}) - F_{1}(P_{1})$$

$$= I_{2}^{*} \cup \{f, f_{2}, \dots, f_{r-1}\} - \{e_{2}, e_{3}, \dots, e_{r}\} \cup \{f_{r}\}$$

$$= I_{2}^{*''} \cup \{f_{r}\}$$
(7)

is an independent set of M_2^* and $|I_2^{*'}| = |I_2^*| + 1$.

As shown before,

$$I'_1 = I_1 \cup \{e_1, e_2, \cdots, e_r\} - \{f_1, f_2, \cdots, f_r\},\$$

then

$$E - I_1' = (E - I_1) \cup \{f_1, f_2, \cdots, f_r\} - \{e_1, e_2, \cdots, e_r\}$$
$$\subseteq (E - I_1) \cup \{f_1, f_2, \cdots, f_{r-1}\} \cup \{f_r\}.$$

By the construction of L_{22} , since $I_2^* \subseteq E - I_1$,

$$\{f_1, f_2, \dots, f_{r-1}\} \subseteq S(I_2^*, E, M_2^*) \subseteq S(E - I_1, E, M_2^*).$$

Thus

$$r_{2}^{*}(E-I_{1}') \leq r_{2}^{*}((E-I_{1}) \cup \{f_{1}, f_{2}, \cdots, f_{r-1}\} \cup \{f_{r}\})$$
$$\leq r_{2}^{*}(E-I_{1}) + 1.$$
(8)

Hence, by (7) and (8), I_2^* is a basis of $M_2^* \times (E-I_1')$.

The same argument holds for the case where the number of edges of L_{12} is odd, as shown in Fig. 2. Q. E. D.

Proof of Theorem 4. We shall confirm Theorem 4 with aid of Theorem 2. There is no path P if one of the following cases occurs:

- case (1) Both V_A and V_B are empty.
- case (2) V_B is not empty but V_A is empty.
- case (3) V_A is not empty but V_B is empty.
- case (4) Neither V_A nor V_B is empty but there is no path P from V_A to V_B .

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In case (1), since $V_B = F_2^{-1}(E - S(I_2^*, E, M_2^*))$ is empty, we get $E = S(I_2^*, E, M_2^*)$ which implies that I_2^* is a basis of M_2^* . On the other hand, since $V_A = F_1^{-1}(S(I_2^*, E, M_2^*) - (I_1 \cup I_2^*)) \cup F_2^{-1}(S(I_2^*, E, M_2^*) - (I_1 \cup I_2^*))$ is empty, the union of I_1 and I_2^* is $S(I_2^*, E, M_2^*) = E$. Thus I_1 and I_2^* are disjoint bases of M_1 and M_2^* , respectively, and I_1 is a common basis of M_1 and M_2 .

In case (2), $V_B \neq \phi$ implies that I_2^* is not a basis of M_2^* . Moreover $V_A = \phi$, that is, $S(I_2^*, E, M_2^*) \subseteq I_1 \cup I_2^*$, implies that $I_1 \cup I_2^* = E$. Because, otherwise, for any element e in $E - (I_1 \cup I_2^*)$, $I_2^* \cup \{e\}$ would be an independent set of M_2^* and $I_2^* \cup \{e\} \subseteq E - I_1$, which contradicts that I_2^* is a basis of $M_2^* \times (E - I_1)$. Therefore,

$$r_1(E) + r_2^*(E) > |I_1| + |I_2^*| = |E|,$$

that is,

$$r_2(E) = |E| - r_2^*(E) < r_1(E).$$

Thus there is no common basis by virtue of (3) of Theorem 2.

In the similar way, we can prove that there is no common basis in case (3). Since $V_B = \phi$ implies that I_2^* is a basis of M_2^* , $E - I_2^*$ is a basis of M_2 . For $V_A \neq \phi$,

$$|I_1| + |I_2^*| < |E|,$$

therefore

$$r_2(E) = |E - I_2^*| > |I_1| = r_1(E).$$

Thus there is no common basis.

To complete the proof we have to verify that there is no common basis in case (4). Choose an arbitrary pair of vertices e'_0 and e''_0 in V_A . Let the set U be the set of all vertices to which there is a path from one of the pair. Let $U_1 = U \cap V_1$ and $U_2 = U \cap V_2$. As $F_1(U_1) = F_2(U_2)$, we put $A = F_1(U_1)$. By the construction of L_{11} and L_{22} ,

$$(U - (\{e'_0\} \cup \{e''_0\})) \cap V_A = \phi,$$

that is,

$$A - \{e_0\} \subseteq I_1 \cup I_2^*. \tag{9}$$

By the construction of A and the fact that there is no path from V_A to V_B ,

$$A \subseteq S(I_1, A, M_1),$$

 $A \subseteq S(I_2^*, A, M_2^*),$

which implies that $I_1 \cap A$ and $I_2^* \cap A$ are maximal independent sets in A of M_1 and M_2^* , respectively. Hence

$$r_1(A) = |I_1 \cap A|,$$

 $r_2^*(A) = |I_2^* \cap A|.$

Recalling the relation (9), we get

$$|A| = |(I_1 \cap A) \cup (I_2^* \cap A) \cup \{e_0\}|$$

= |I_1 \cap A| + |I_2^* \cap A| + 1
> |I_1 \cap A| + |I_2^* \cap A|
= r_1(A) + r_2^*(A),

Q.*E*.*D*.

which violates the condition (4).

Proof of Theorem 5. We shall verify Theorem 5 in each cases as in the proof of Theorem 4. Let $B_1 = I_1$ and B_2^* be an arbitrary basis of M_2^* containing I_2^* , where I_1 and I_2^* are the sets obtained when the algorithm terminates. To verify Theorem 5, we have only to show that $|B_1 \cup B_2^*| \ge |B_1' \cup B_2^{*'}|$ for any pair of bases B_1' and $B_2^{*'}$ of M_1 and M_2^* .

As shown in the proof of Theorem 4, case (1) and (3) imply that I_1 and I_2^* are the disjoint bases of M_1 and M_2^* . Then,

$$|B_1 \cup B_2^*| = |I_1 \cup I_2^*| = |I_1| + |I_2^*|,$$

which implies that $B_1 \cup B_2^*$ is a basis of $M_1 \cup M_2^*$.

Let us consider the case (2). Generally,

$$|B_1' \cup B_2^*| \le |E|. \tag{10}$$

However, in the case (2), $E = I_1 \cup I_2^*$. Thus we get the relation

$$|B_1' \cup B_2^{*\prime}| \leq |I_1 \cup I_2^*| = |B_1 \cup B_2^*|.$$

Case (4) is left to prove. Let U_j $(1 \le j \le t)$ be the set of all vertices to which there is a path from e'_j or e''_j $(1 \le j \le t)$ in V_A . And let $U_{1j} = U_j \cap V_1$ and $U_{2j} = U_j \cap V_2$. As $F_1(U_{1j}) = F_2(U_{2j})$, let $A'_j = F_1(U_{1j})$. If some of them intersect, we combine them and denote it by A_i $(1 \le i \le u)$. We assume that A_i is the union of $m_i A'_j s$, that is, $A_i = A'_{j_1} \cup A'_{j_2} \cup \cdots \cup A'_{j_{m_i}}$. Then

$$A_i \cap A_k = \phi \qquad i \neq k, \tag{11}$$

$$\sum_{i=1}^{u} m_i = t. \tag{12}$$

Now we write $A = \bigcup_{i=1}^{u} A_i$. Since each element of A_i is dependent on both $A_i \cap I_1$ and $A_i \cap I_2^*$,

$$r_1(A_i) = |I_1 \cap A_i|,$$

 $r_2^*(A_i) = |I_2^* \cap A_i|.$

Then, using the property that $I_1 \cap I_2^* = \phi$,

$$r_1(A_i) + r_2^*(A_i) = |I_1 \cap A_i| + |I_2^* \cap A_i|$$

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$$= |(I_1 \cup I_2^*) \cap A_i|$$

$$= |A_i| - |A_i - (I_1 \cup I_2^*)|$$

$$= |A_i| - m_i.$$
(13)

If we now recall (1), then we have that

$$r_{1}(A) = r_{1}(A_{1} \cup A_{2} \cup \dots \cup A_{u})$$

$$\leq r_{1}(A_{1}) + r_{1}(A_{2}) + \dots + r_{1}(A_{u})$$

$$= \sum_{i=1}^{u} r_{1}(A_{i}),$$

$$r_{2}^{*}(A) \leq \sum_{i=1}^{u} r_{2}^{*}(A_{i}).$$

By (12) and (13),

$$r_{1}(A) + r_{2}^{*}(A) \leq \sum_{i=1}^{u} (r_{1}(A_{i}) + r_{2}^{*}(A_{i}))$$
$$= \sum_{i=1}^{u} (|A_{i}| - m_{i})$$
$$= \sum_{i=1}^{u} |A_{i}| - t.$$

Hence by (11),

$$r_1(A) + r_2^*(A) + |E - A| \leq \sum_{i=1}^{u} |A_i| - t + |E| - |A|$$
$$= |A| - t + |E| - |A| = |E| - t.$$

Then using Theorem 1, the rank r(E) of E with respect to $M_1 \cup M_2^*$ satisfies the relation that

$$r(E) \leq |E| - t. \tag{14}$$

On the other hand $I_1 \cup I_2^*$ is an independent set of $M_1 \cup M_2^*$ and

$$|I_1 \cup I_2^*| = |E| - t.$$

Then,

$$r(E) \ge |E| - t. \tag{15}$$

(14) and (15) imply that

$$r(E) = |E| - t.$$

For an arbitrary pair of bases B'_1 and B''_2

$$|B_1' \cup B_2^{*'}| \leq r(E) = |E| - t = |B_1 \cup B_2^{*}|. \qquad Q. E. D.$$

An Algorithm for Finding a Common Basis of Two Matroids

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