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DIRECTIONAL CONVEXITY AND THE DIRECTIONAL DISCRETE MAXIMUM PRINCIPLE FOR QUANTIZED CONTROL SYSTEMS

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ABSTRACT

The quantized discrete maximum principle of the Pontryagin type for quantized control systems was given by Dr. Anzai both in linear and nonlinear case. In this paper, the authors relax the convexity requirement, which is important in Anzai's quantized discrete maximum principle, and show that in spite of this relaxed assumption, the similar results can be obtained for the optimal quantized control vector both in linear and nonlinear case.

For this purpose, we introduce the concepts of the directional convexity and describe some properties of it. Then, under the assumption of the directional convexity, we prove the directional quantized discrete maximum principle (which we denote as D.Q.D.M.P.) of the Pontryagin type for quantized control systems.

We consider that this work can be an important basis for the extension to the ε-technique for the quantized discrete maximum principle which gives efficient informations to the abnormal problem.

1. Introduction

Control systems whose controls can take only given discrete-level values are called quantized control systems. Such systems have been considered by many scholars and many works have been done.

Since quantized control systems are optimized using quantized controls, dynamic programming is an ordinary method for this problem. By applying this method, Lewis and Tou (1963), and Arora and Pierre (1973) treated quantized systems in deterministic time-discrete case, Havira and Lewis (1972) in deterministic time-continuous case. Kim and Djaduri (1967) transformed quantized control values to integers and applied integer programming technique. Larson (1967) considered the optimal time-variant quantizer on stochastic systems. As for the subject of the uniform quantization of deviation signals, Betram (1958), Slaughter (1964) and Johnson (1965) treated quantized errors on deterministic systems, and Widrow (1961) on stochastic systems. Limbert and Taft (1967) argued the stability in quantizing the deviations by the non-uniform quantizer. Tsugawa (1973) and Sano (1975) took approaches on these subjects by introducing the concepts of sub-optimality. On the other hand, quantized control systems can be considered as systems with constrained controls. Studies on systems having constraints in controls were done by Desoer and Wing (1961) and Wing and Desoer (1963). Witsenhausen (1966) took this approach in discretizing the state variables.

Anzai (1972) pointed out that basic studies on system structures of quantized control systems have not made so far. Using the theory of algebraic equation and geometry of numbers, he discussed the reachability and the controllability of the systems in comparison with systems having continuous-level controls.

We will mention about the meaning of these systems in the field of systems engineering. With the development of systems engineering, systems which cannot be formulated without controllers having discrete-level controls are increasing. For examples, capital budgeting problems for multi-stage economic systems with economies of scale, control problems with additive automata and linear sequential networks, programmed control problems for traffic control systems, schedule control problems, and control problems for systems with some parallel connected on-off controllers: these are all classified into this category. To handle these systems efficiently is very important.

Method used for these systems had been restricted to programming techniques such as dynamic programming or integer programming. Anzai made a new approach. He paid attention to Halkin's discrete maximum principle which is derived by the geometrical considerations, and using the idea of convex hull, he derived Anzai's Pontryagin type quantized discrete maximum principle which holds for the optimal quantized control variables.

In this paper, the authors derive the similar Pontryagin type directional quantized discrete maximum principle both in linear and nonlinear case by relaxing convexity to directional convexity. The proof is made by introducing the comoving space method which uses the separation theorem of convex sets. In this derivation, there are merits in the relaxation of the convexity requirement and in the free choice of the evaluation axis.

In the following, we make the mathematical preparation about the directional convexity.

2. Directional Convexity

In this section we shall give 4 definitions and 12 theorems for the preparation to sections 3, 4, and 5. These results are derived by Holtzman and Halkin (1966).

Definition 2.1. Let z be a nonzero vector, and A be a set. A is said to be

z-directionally convex, if for each $a, b \in A$, and $\mu \in [0, 1]$, there exists a $\beta \geqq 0$, such that

$$\mu a + (1-\mu)b + \beta z \in A.$$

Definition 2.2. A point a is said to be a z-directional boundary point of a set A if

(1) for every $\varepsilon > 0$, there exists $b \in A$ such that $||a-b|| < \varepsilon$, and
(2) for every $\beta > 0$, $a + \beta z \notin A$.

Definition 2.3. The z-shadow of a set A is the set

$$S = \{a - \lambda z : a \in A, \lambda \geqq 0\}.$$

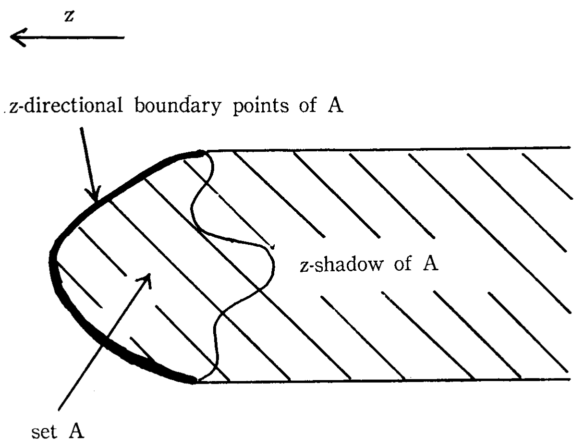

Fig. 1 A z-directional convex set A, z-directional boundary points of A, and the z-shadow of A.

Theorem 2.1. The z-shodow of a z-directionally convex set A is convex.

Proof. Let S be the z-shadow of A. If $a, b \in S$, there exists $\lambda_1, \lambda_2 \geqq 0$, such that $a + \lambda_1 z \in A$ and $b + \lambda_2 z \in A$. Then from the z-directional convexity of A, for every $\mu \in [0, 1]$, there exists a $\beta \geqq 0$ such that

$$\mu(a + \lambda_1 z) + (1-\mu)(b + \lambda_2 z) + \beta z \in A.$$

Let

$$\beta^* = \mu\lambda_1 + (1-\mu)(b + \lambda_2 z) + \beta \geqq 0,$$

then

$$\mu a + (1-\mu)b + \beta^* z \in A.$$

Since S is the z-shadow of A,

$$\mu a + (1-\mu)b \in S.$$

This shows that S is convex.

Definition 2.4. A matrix M is said to be a z-directional matrix, if for each z-directionally convex set A, the set

$$B=\{Mx : x\in A\}$$

is also z-directionally convex.

The following theorems are easy ones. We shall omit the prooves.

Theorem 2.2. If M is a z-directional matrix, then $Mz=\lambda z$ for some real λ.

Theorem 2.3. If $Mz=\lambda z$, $\lambda\geqq 0$, then M is a z-directional matrix.

Theorem 2.4. Suppose that M has a nonzero eigenvalue γ with eigenvector y, i.e., $My=\gamma y$, and that z is linearly independent of y. Then $Mz=\lambda z$, $\lambda\geqq 0$, if and only if M is a z-directional matrix.

Theorem 2.5. If M is nonsingular, then $Mz=\lambda z$, $\lambda\geqq 0$, if and only if M is a z-directional matrix.

Theorem 2.6. If M_1 and M_2 are nonsingular z-directional matrices, and $\alpha_1, \alpha_2\geqq 0$, then $\alpha_1 M_1+\alpha_2 M_2$ is also a z-directional matrix.

Theorem 2.7. If $||M||<1$ and M is a z-directional matrix, then $I+M$ is a z-directional matrix.

Theorem 2.8. If M_1 and M_2 are z-directional matrices, then M_1M_2 is also a z-directional matrix.

Theorem 2.9. If A_1 and A_2 are z-directionally convex sets, then A_1+A_2 is also a z-directionally convex set. Where, A_1+A_2 is defined as the set $\{x : x=a_1+a_2, a_1\in A_1, a_2\in A_2\}$.

Theorem 2.10. If A is a z-directionally convex set, and $\alpha\geqq 0$, then αA is also

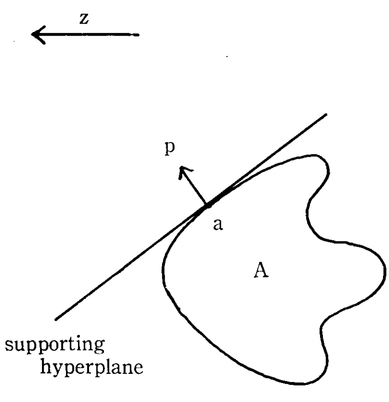

Fig. 2 Existence of a supporting hyperplane at a z-directional boundary point.

a z-directionally convex set. Where, αA is defined as the set $\{x : x=\alpha a, a \in A, \alpha \geqq 0\}$.

Theorem 2.11. Let A be a z-directionally convex set, and S be the z-shadow of A. If a point a is a z-directional boundary point of A, then a is a boundary point of S.

Theorem 2.12. If a is a z-directional boundary point of a z-directionally convex set A, there exists a nonzero vector p such that

$$\langle p, x \rangle \leqq \langle p, a \rangle \qquad \text{for all } x \in A.$$

3. The statement of discrete optimal control problems

In this section we shall formulate the discrete optimal control problems to be studied here both in linear and in nonlinear case and state their foundamental assumptions and ideas.

In linear case.

We are concerned with the system described by the difference equation

$$x(i+1)-x(i)=A(i)x(i)+g(i, u(i)) \quad i=0,1,\dots,k-1, \tag{3.1}$$

where x is an n-state vector, an element of an Euclidean space E^n, u is an r-control vector, an element of an Euclidean space E^r, A is an $n \times n$ matrix defined for every $i=0,1,\dots,k-1$, and g is an n-vector defined for every $i=0,1,\dots,k-1$ and every control $u \in \bar{\Omega}$, a given set of admissible integer control vectors. We are given an initial state vector x_0. We are also given a nonzero vector z, and $\|z\|=1$. Here $\|a\|$ is the norm of the vector a.

We make the following assumptions:

(1) The sets $w(i)=\{g(i, u(i)) | u \in \bar{\Omega}\}$ are bounded for all $i=0,1,\dots,k-1$.

(2) The matrices $I+A(i)$ are nonsingular for all $i=0,1,\dots,k-1$.

(3) The matrices $I+A(i)$ are z-directional matrices for all $i=0,1,\dots,k-1$.

For stating the optimal control problem, some additional notations are given. S is a control strategy:

$$S=\{(i, u(i)) | i=0,1,\dots,k-1\}. \tag{3.2}$$

The strategy S will be called "admissible" if

$$u(i) \in \bar{\Omega} \quad \text{for all } i=0,1,\dots,k-1.$$

F is the set of all admissible strategies.

We shall denote by $x(j, S)$ the value of the state variable at step j corresponding to the solution of (3.1) satisfying

$$x(0, S)=x_0 \tag{3.3}$$

and with the strategy S. The optimal control problem in linear case is to find a strategy $S \in F$ such that, for the given initial condition,

$$x(k, S) \in E^n \tag{3.4}$$

and

$$\langle z, x(k, S)\rangle \text{ is maximized.} \tag{3.5}$$

Here $\langle a, b\rangle$ is the scalar product of vectors a and b. A strategy satisfying the above will be called an optimal strategy and be denoted $\hat{S}$.

For the sets $w(i)$, we introduce z-directional convex hulls of $w(i)$ and let denote $d(i)$ for all $i=0, 1, \dots, k-1$. Then the sets $d(i)$ are bounded, closed and z-directional convex for all $i=0, 1, \dots, k-1$. As for these sets, see Munakata and Kaneko (1975), Memorandum.

Now a concept of "Comoving space along a trajectory" will be introduced here. Let $G(i)$ be an $n\times n$ matrix for $i=0, 1, \dots, k$ and defined by

$$G(k)=I \tag{3.6}$$

$$G(i)-G(i+1)=G(i+1)A(i), \qquad i=0, 1, \dots, k-1. \tag{3.7}$$

From (3.6) and (3.7),

$$G(i)=(I+A(k-1))(I+A(k-2))\cdots(I+A(i)).$$

Since $I+A(j)$ is nonsingular, and a z-directional matrix for $j=0, 1, \dots, k-1$, $G^{-1}(i)$ exists, and $G(i)$ is a z-directional matrix for $i=0, 1, \dots, k$. Let Y be an n-dimensional Euclidean space with elements y. We shall consider the mapping from X into Y defined by the relation

$$y=G(i)[x-x(i, \hat{S})], \tag{3.8}$$

$y(i, S, \hat{S})$ is defined by

$$y(i, S, \hat{S})=G(i)[x(i, S)-x(i, \hat{S})] \tag{3.9}$$

Thus, $y(0, S, \hat{S})=0$, and also exists an inverse mapping.

The set of reachable events for the space X is defined by

$$W(i)=\{x(i, S)|S\in \bar{F}\}, \tag{3.10}$$

where $\bar{F}$ is the set of strategies made from the control vectors to be selected from the set $d(i)$.

The set of reachable events for the space Y is

$$W(i, \hat{S})=\{y(i, S, \hat{S})|S\in \bar{F}\}. \tag{3.11}$$

To the space X, the space Y is called the comoving space along the optimal trajectory. See Fig. 3.

In nonlinear case.

We are concerned with the system described by the difference equation

$$x(i+1)-x(i)=f(i, x(i), u(i)) \qquad i=0, 1, \dots, k-1. \tag{3.12}$$

We are given an initial state vector x_0. We are also given a nonzero vector z,

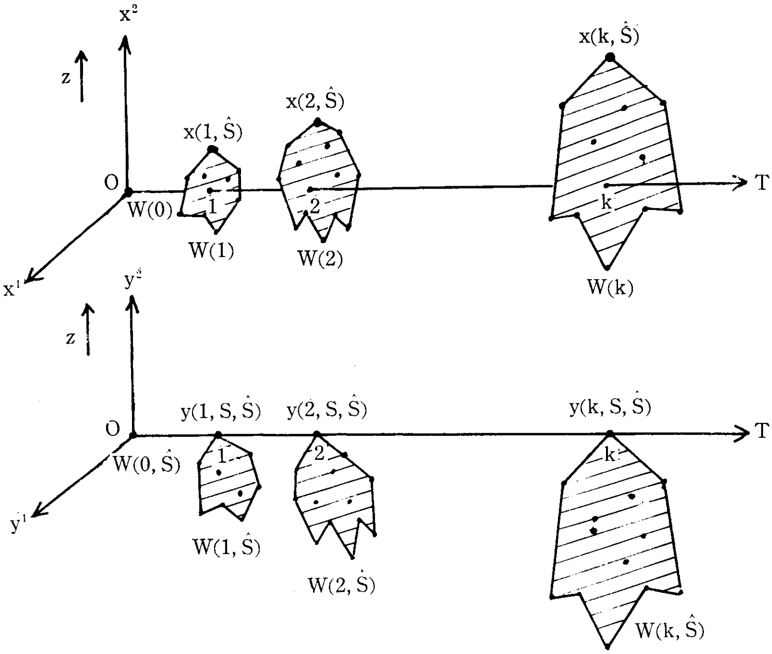

Fig. 3 The sets of reachable events W(i) and W(i, Ŝ) in the case of a two dimensional state space.

and $||z||=1$. Notations are the same in linear case.

For every i, the vector-valued function $f(i, x(i), u(i))$ is given and satisfies the following assumptions:

(a) $f(i, x, u)$ is defined for all real vectors x and $(x, u)\in E^n\times\bar{\Omega}$.

(b) $f(i, x, u)$, for every $u\in\bar{\Omega}$, is twice continuously differentiable with respect to x.

(c) $f(i, x, u)$ and all its first and second partial derivatives with respect to x are uniformly bounded over $A\times\bar{\Omega}$ for any bounded set $A\subset E^n$.

(d) the matrix

$$I+\sum_{j=1}^{n+1}\alpha^j\frac{\partial}{\partial x}f(i, x, y_j)$$

is nonsingular with $y_j\in\bar{\Omega}$, $x\in E^n$, $\sum_{j=1}^{n+1}\alpha^j=1, \alpha^j\geqq 0$.

(e) the set $w(i, x)=\{f(i, x, u)|u\in\bar{\Omega}\}$ is bounded for every $x\in E^n$.

(f) $f(i, x, u)$ is independent of z-directional element of x.

The optimal control problem in nonlinear case is to find optimal control sequences $u(0), u(1), \dots, u(k-1)$ such that, for the given initial condition,

$$\langle z, x(k)\rangle \text{ is maximized.} \tag{3.13}$$

The sequences $\hat{u}(0), \hat{u}(1), \dots, \hat{u}(k-1)$ satisfying the above will be called optimal control sequences and the corresponding $\hat{x}(0), \hat{x}(1), \ldots, \hat{x}(k)$ will be called an optimal trajectory.

For the sets $w(i, x)$, we introduce z-directional convex hulls of $w(i, x)$ and let denote $d(i, x)$ for all $i=0, 1, \ldots, k-1$. Then the sets $d(i, x)$ are bounded, closed and z-directional convex for all $i=0, 1, \ldots, k-1$.

An essential difference between Anzai's and the present paper is that, for the bounded sets $w(i)$ and $w(i, x)$, in the former Anzai introduces the convex hull of them, in the latter we introduce the z-directional convex hull. Then there is the difference that Anzai sets the objective so that the n-th element (last element) of $x(k)$ may be maximized, on the other hand we set the objective to be maximized in an arbitrary direction (vector z). Needless to say, the relaxing of the condition; the convexity requirement to directional convexity, and the n-th direction to an arbitrary direction, greatly extends the practical applicapability.

4. Directional quantized discrete maximum principle

In this section we shall give the D.Q.D.M.P., in both cases, justified for the optimal integer control vectors.

In linear case.

If $\hat{S}$ is an optimal strategy, then there exists a nonzero vector $p(i, \hat{S})$ such that:

1) Maximization of Hamiltonian;

$$H(i, x(i, \hat{S}), \hat{u}(i), p(i+1, \hat{S})) \geqq H(i, x(i, \hat{S}), u, p(i+1, \hat{S})) \quad \text{for } i=0, 1, \ldots, k-1 \text{ and all } u \in \bar{\Omega}. \tag{4.1}$$

2) Adjoint Equation;

$$p(i, \hat{S}) - p(i+1, \hat{S}) = A(i)^T p(i+1, \hat{S}) \quad \text{for } i=0, 1, \ldots, k-1. \tag{4.2}$$

3) Transversality Condition;

$$\langle p(k, \hat{S}), z \rangle \geqq 0. \tag{4.3}$$

Where H is defined by

$$H(i, x, u, p) = \langle A(i)x + g(i, u), p \rangle, \tag{4.4}$$

and $A(i)^T$ is the transpose of the matrix $A(i)$.

In nonlinear case.

If the sequences $\hat{u}(0), \hat{u}(1) \ldots, \hat{u}(k-1)$ and $\hat{x}(0), \hat{x}(1), \ldots, \hat{x}(k)$ are optimal, then there exists a sequence of nonzero vectors $\hat{p}(0), \hat{p}(1) \ldots, \hat{p}(k)$ such that:

1) Maximization of Hamiltonian;

$$\langle f(i, \hat{x}(i), \hat{u}(i)), \hat{p}(i+1) \rangle \geqq \langle f(i, \hat{x}(i), u), \hat{p}(i+1) \rangle \quad \text{for } i=0, 1, \ldots, k-1 \text{ and all } u \in \bar{\Omega}. \tag{4.5}$$

2) Adjoint Equation;

$$\hat{p}(i)-\hat{p}(i+1)=\left[\frac{\partial}{\partial x}f(i,x(i),\hat{u}(i))\Big|x(i)=\hat{x}(i)\right]^T\hat{p}(i+1) \tag{4.6}$$

for $i=0,1,\ldots,k-1$.

3) Transversality Condition;

$$\langle\hat{p}(k),z\rangle>0. \tag{4.7}$$

5. Proof of the directional quantized discrete maximum principle

In this section we shall prove the D.Q.D.M.P. in both cases respectively.
In linear case.

Theorem 5.1. The set $W(k,\hat{S})$ is a z-directional convex hull.

Proof. See Munakata and Kaneko (1975), Discussion Note, and Fig. 4.

Theorem 5.2. If $\hat{S}$ is an optimal strategy, then $x(k,\hat{S})$ is a z-directional boundary point of $W(k)$.

Proof. See the above Discussion Note, and Fig. 4.
This result of Thm. 5.2 is called "Halkin's principle of the optimal evolution."

Theorem 5.3. If $x(k,\hat{S})$ is a z-directional boundary point of $W(k)$, then $y=0$ is a z-directional boundary point of $W(k,\hat{S})$.

Proof. See the above Discussion Note, and Fig. 4.

Theorem 5.4. If $y=0$ is a z-directional boundary point of $W(k,\hat{S})$, then there exists a nonzero vector $\phi(\hat{S})$ such that

$$\langle G(i+1)[g(i,u)-g(i,\hat{u}(i))],\phi(\hat{S})\rangle\leqq 0 \tag{5.1}$$

for all $i=0,1,\ldots,k-1$ and all $u\in\bar{\Omega}$.

Proof. See the above Discussion Note, and Fig. 4.

Theorem 5.5. If there exists a nonzero vector $\phi(\hat{S})$ satisfying (5.1), then there exists a nonzero vector $p(i,\hat{S})$ defined for $i=0,1,\ldots,k$ such that

$$p(i,\hat{S})=G(i)^T\phi(\hat{S}), \tag{5.2}$$

$$H(i,x(i,\hat{S}),\hat{u}(i),p(i+1,\hat{S}))\geqq H(i,x(i,\hat{S}),u,p(i+1,\hat{S})) \tag{5.3}$$

for all $i=0,1,\ldots,k-1$ and all $u\in\bar{\Omega}$,

$$p(i,\hat{S})-p(i+1,\hat{S})=A(i)^Tp(i+1,\hat{S}) \tag{5.4}$$

for all $i=0,1,\ldots,k-1$.

Proof. See the above Discussion Note.

Theorem 5.6. If $\hat{S}$ is an optimal strategy, then there exists a vector $p(i, \hat{S})$ defined for $i=0, 1, \ldots, k$ and satisfying (5.3) and (5.4).

Proof. This Theorem is a direct consequence of Theorem 5.2, 5.3, 5.4 and 5.5.
It remains to prove the transversality condition for the sake of D.Q.D.M.P. in linear case.

Theorem 5.7. If $\hat{S}$ is an optimal strategy, then

$$\langle p(k, \hat{S}), z\rangle \geqq 0. \tag{5.5}$$

Proof. From Theorem 5.5 and Theorem 5.4, we have

$$p(k, \hat{S}) = G(k)^T \phi(\hat{S}) = \phi(\hat{S}),$$

and

$$\langle y, \phi(\hat{S})\rangle \leqq 0 \quad \text{for all } y \in W(k, \hat{S}).$$

Since we have also

$$y = G(k)[x - x(k, \hat{S})] = x - x(k, \hat{S}),$$

then

$$\langle x - x(k, \hat{S}), \phi(\hat{S})\rangle \leqq 0 \quad \text{for all } x \in W(k).$$

So that

$$\langle r - x(k, \hat{S}), \phi(\hat{S})\rangle \leqq 0 \quad \text{for all } r \in z\text{-shadow of } W(k).$$

Since

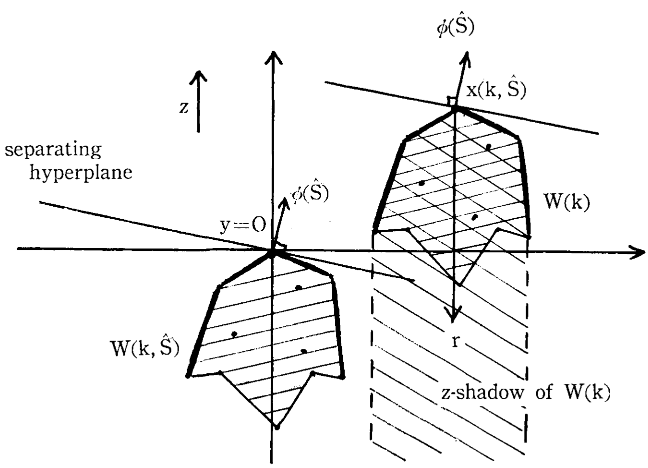

Fig. 4 The relation of W(k), W(k, Ŝ) and a nonzero vector φ(Ŝ) normal to the separating hyperplane and a z-shadow of W(k), at the final step k.

$$r = x(k, \hat{S}) - z \in z\text{-shadow of } W(k),$$

then

$$\langle z, p(k, \hat{S}) \rangle \geqq 0.$$

See Fig. 4.

Lastly the authors give an important theorem relative to an existence of optimality and a quantized control.

Theorem 5.8. In this linear case, there exists an optimal strategy and the optimal strategy $\hat{S} \in F$ is coincident with the optimal strategy in $\bar{F}$.

Proof. See Munakata and Kaneko (1975), Memorandum.

Therefore, the D.Q.D.M.P. in linear case has been derived.

In nonlinear case.

First of all, we shall call a nonlinear directional quantized optimal control problem to be proved here 'this optimal control problem'. In this optimal control problem, we will select the difference vectors $(x(i+1)-x(i))$ from $d(i, x)$: z-directional convex hull of $w(i, x)$. Using the idea of a relaxing problem, Halkin (1966) section 5, Warga (1962), we consider a 'relaxed problem' of this optimal control problem that we allow the difference vectors to be selected from $v(i, x)$: the convex hull of $d(i, x)$, instead of $d(i, x)$ itself. And the relaxed problem is identical to this optimal control problem except the selection of the difference vectors. On the other hand, the quantized discrete maximum principle in nonlinear case is known in Anzai and Munakata (1973). See Appendix B.

Then, our proof of the D.Q.D.M.P. in nonlinear case uses fully the result of Anzai and Munakata (1973), the relaxed problem, and a basic property of optimal control systems. That is to say, Anzai's Q.D.M.P. can be applicable to the relaxed problem because of the convexity of $v(i, x)$, where though there is a difference between Anzai's and the present paper about a performance axis, it can be done very easily to exchange the directional vector e^n for vector z. However Anzai's Q.D.M.P. cannot be applicable to this optimal control problem because $d(i, x)$ is not necessarily convex. But since $d(i, x)$ is z-directional convex, then we can show that the solution to the relaxed problem is also the solution to this optimal control problem.

Thus, we will show the above fact in a simple manner. And it is also easy to show that the solution to this optimal control problem is the solution to the relaxed problem.

Theorem 5.9. If $d(i, x)$: z-directional convex hull of $w(i, x)$, is z-directional convex, then the solution to the relaxed problem of this optimal control problem is also the solution to this optimal control problem.

Proof. See Munakata and Kaneko (1975), Discussion Note.

Theorem 5.10. If $d(i, x)$ is z-directional convex, then the solution to this optimal control problem is also the solution to the relaxed problem.

Proof. Since the condition for $d(i, x)$ is relaxed only, obviously.

As for an existence of the optimal solution and a quantized control, in addition some numerical examples in linear case, see Munakata and Kaneko (1975), Memorandum.

Therefore, the D.Q.D.M.P. in nonlinear case has also derived.

6. Conclusion

We have derived above the directional quantized discrete maximum principle of the Pontryagin type both in linear and nonlinear case. As we have mentioned in Introduction, it is important to find a new technique for quantized control systems. Anzai's quantized discrete maximum principle is a valuable work. But his convexity assumption is a strict condition, and the restriction of the evaluation axis to e^n-direction is inconvenient. The aim of this paper is to improve these points. In short, it is sufficient for us to know the convexity only in the direction of the evaluation axis. To prove this property, we had to prepare the new concepts of directional convexity.

In the end, the numerical algorithm obtained by this principle becomes essentially the two points boundary value problem. For this algorithm, see Munakata and Kaneko (1975) Memorandum.

Appendix A

The condition $\|A(i)\|<1$, $(i=0,1,\dots,k-1)$ is an important assumption in discretizing the essentially continuous system. This condition was given by Halkin for linear systems, and was also given for nonlinear systems. This condition implies that $I+A(i)$ is nonsingular. We prove this fact.

Theorem. If $\|A\|<1$, then $I+A$ is nonsingular.

Proof. Suppose that $(I+A)x=0$ and $x\neq 0$. Then, $x=-Ax$.

$$\|x\|=\|Ax\|\leqq\|A\|\,\|x\|<\|x\|$$

which is a contradiction. Thus, if $(I+A)x=0$, then $x=0$, which implies that $I+A$ is nonsingular.

The condition that $I+A(i)$ is nonsingular is necessary for the existence of the inverse mapping on the comoving space, and is an easy condition to check the system.

Appendix B

In this Appendix, we shall derive Q.D.M.P. (Quantized Discrete Maximum

Principle) in nonlinear case.

The state equations of the system are given by

$$x(i+1)-x(i)=f(i, x(i), u(i)), \quad i=0,1,\dots,k-1. \tag{B.1}$$

Where, $x\in E^n, u\in\bar{\Omega}\subset E^r$, and $\bar{\Omega}$ is the set of admissible integer controls. Assume that vector-valued functions $f(i,x,u)$, $i=0,1,\dots,k-1$, have the following properties.

(1) $f(i,x,u)$ is defined for all $(x,u)\in E^n\times\bar{\Omega}$.

(2) For all $u\in\bar{\Omega}$, $f(i,x,u)$ is twice continuously differentiable with respect to x.

(3) $f(i,x,u)$ and its first and second partial derivatives with respect to x are uniformly bounded over $A\times\bar{\Omega}$, where A is an arbitrary bounded set in E^n.

(4) $I+\dfrac{\partial f(i,x,u)}{\partial x}$ is nonsingular on $E^n\times\bar{\Omega}$.

(5) The set $w(i)=\{f(i,x,u)|u\in\bar{\Omega}\}$ is bounded for each $x\in E^n$.

Now, we state the Quantized Discrete Optimal Control Problem as follows.

For a given initial state x_0, find the control sequence $u(0), u(1),\dots,u(k-1)$, which gives the maximum value of $x^n(k)$, where $x^n(k)$ is the n-th component of the vector $x(k)$.

Let $c(i)$ be the convex hull of $w(i)$, $c(i)$ is bounded, closed, and convex. Then Q.D.M.P. is derived as follows.

If the sequences $\hat{u}(0),\hat{u}(1),\dots,\hat{u}(k-1)$ and $\hat{x}(0),\hat{x}(1),\dots,\hat{x}(k)$ are optimal, then there exists a sequence of nonzero vectors $\hat{p}(0),\hat{p}(1),\dots,\hat{p}(k-1),\hat{p}(k)$ such that

(1)
$$\langle f(i,\hat{x}(i),\hat{u}(i)), \hat{p}(i+1)\rangle \geqq \langle f(i,\hat{x}(i),u),\hat{p}(i+1)\rangle \quad \text{for } i=0,1,\dots,k-1 \text{ and all } u\in\bar{\Omega}. \tag{B.2}$$

(2)
$$\hat{p}(i)-\hat{p}(i+1)=\left[\frac{\partial f(i,x,\hat{u}(i))}{\partial x}\Big|_{x=\hat{x}(i)}\right]^T \hat{p}(i+1) \quad \text{for } i=0,1,\dots,k-1. \tag{B.3}$$

(3)
$$\hat{p}(k)=\pi, \quad \pi^n>0. \tag{B.4}$$

Proof. Define $S(\hat{x}(k))$ as

$$S(\hat{x}(k))=\{x|x^n>\hat{x}^n(k)\}.$$

Let $W(i)$, $i=0,1,\dots,k$, be the reachable set. Then it is clear that $W(k)$ and $S(\hat{x}(k))$ are disjoint. The set $W(k)$ is not necessarily convex. To corner this difficulty, we consider a certain linearized problem which linearize the original problem around the optimal solutions $\hat{u}(0),\hat{u}(1),\dots,\hat{u}(k-1)$, $\hat{x}(0),\hat{x}(1),\dots,\hat{x}(k-1)$. The state equations of this problem are given by

$$x(i+1)-x(i)=f(i,\hat{x}(i),u(i))+\left[\frac{\partial f(i,x,\hat{u}(i))}{\partial x}\Big|_{x=\hat{x}(i)}\right](x(i)-\hat{x}(i)). \tag{B.5}$$

By the linearization lemma, Halkin (1966), $W^+(k)$ and $S^+(\hat{x}(k))$ are separable. Where we define now the sets $W^+(k)$ and $S^+(\hat{x}(k))$ with respect to the linearized problem in the same way as the sets $W(k)$ and $S(\hat{x}(k))$ defined above. That is, there exists a nonzero vector π, such that

$$\langle x-\hat{x}(k), \pi\rangle \geqq 0, \qquad \text{for all } x\in S^{+}(\hat{x}(k)),$$

$$\langle x-\hat{x}(k), \pi\rangle \leqq 0, \qquad \text{for all } x\in W^{+}(k).$$

Let $\hat{p}(\cdot)$ be nonzero vectors defined as

$$\hat{p}(k)=\pi,$$

$$\hat{p}(i+1)-\hat{p}(i)=-\left[\frac{\partial f(i, x, \hat{u}(i))}{\partial x}\bigg|x=\hat{x}(i)\right]^{T}\hat{p}(i+1).$$

As $\hat{x}(k)$ is also the optimal point for the linearized problem, π'' is positive, which proves (B. 4). It is easy to prove that for $i=0, 1, \ldots, k$, and for all $x\in W^{+}(i)$,

$$\langle x-\hat{x}(i), \hat{p}(i)\rangle \leqq 0. \tag{B. 6}$$

Finally, we will prove (B.2). Suppose that there exist some $j\in\{0, 1, \ldots, k-1\}$, $\tilde{u}(j)\in\bar{\Omega}$, and $\varepsilon>0$, such that

$$\langle f(j, \hat{x}(j), \tilde{u}(j)), \hat{p}(j+1)\rangle=\langle f(j, \hat{x}(j), \hat{u}(j)), \hat{p}(j+1)\rangle+\varepsilon.$$

Define $\tilde{x}(j+1)\in W^{+}(j+1)$ as

$$\tilde{x}(j+1)-\hat{x}(j)=f(j, \hat{x}(j), \tilde{u}(j)).$$

Then,

$$\begin{aligned}&\langle f(j, \hat{x}(j), \tilde{u}(j))-f(j, \hat{x}(j), \hat{u}(j)), \hat{p}(j+1)\rangle\\ &=\langle \tilde{x}(j+1)-\hat{x}(j+1), \hat{p}(j+1)\rangle\\ &=\varepsilon>0\end{aligned}$$

This contradicts (B. 6). Hence (B. 2) is proved.

In this case, it is easy to show that integer solutions restricted to

$$\{f(i, x, u)|u\in\bar{\Omega}\}$$

coincides with the relaxed solutions restricted to

$$\{f(i, x, u)|u\in c(i)\}.$$

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