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POSTOPTIMIZATION OF QUADRATIC PROGRAMMING

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ABSTRACT

In this paper, we present a postoptimization analysis of convex quadratic programmings, that is, (i) modification of coefficients or constants of the problem, (ii) addition of a constraint or a variable, and (iii) parameterization of coefficients or constants. The analysis is done mainly by means of the principal pivoting method of DANTZIG and COTTLE.

1. Introduction

We consider the convex quadratic programming:
Minimize $Q=x'Dx/2+c'x$, subject to $Ax\geq b$, $x\geq 0$ where the matrix D is of order (n, n), positive semi-definite and symmetric and the matrix A is of order (m, n) and the symbol $'$ denotes the transposition of vector or matrix. As is well known, the Kuhn-Tucker conditions for the optimal solution of this quadratic programming are the following linear complementarity problem: To find out w and z which satisfy

$$(1.1)\qquad w-Mz=q, \qquad w'z=0, \qquad w, z\geq 0,$$

where $$Z=\begin{bmatrix} x \\ y \end{bmatrix},\quad w=\begin{bmatrix} u \\ v \end{bmatrix},\quad q=\begin{bmatrix} c \\ -b \end{bmatrix},\quad \text{and}\quad M=\begin{bmatrix} D & -A' \\ A & 0 \end{bmatrix}.$$

If we could find w and z, then x-part of z is an optimal solution of the quadratic program. We can solve this problem by the principal pivoting method (abbreviated to PPM, hereafter) of DANTZIG and COTTLE (DANTZIG and COTTLE (1967)).

Suppose we have already had an optimal solution and let the final canonical form of (1.1) be:

$$\bar{w}-\bar{M}\bar{z}=\bar{q}, \tag{1.2}$$

where $\bar{w}$, $\bar{z}$ are the basic and the nonbasic variables, respectively, and $\bar{M}$ is the principal pivot transform of M at the final step. The optimal solution is $(\bar{w},\bar{z})=(\bar{q},0)$. In the following analysis, the optimal basis B and its inverse B^{-1} are supposed to be known.

2. Modification of *b* and/or *c*

This modification causes the change of the vector q in (1.1) which we denote by q^N. At the final step, this makes $\bar{q}$ become $B^{-1}q^N$. Then, we have:

(i) If $B^{-1}q^N\geq 0$, then the present basis is still optimal. And the solution is $(\bar{w},\bar{z})=(B^{-1}q^N,0)$.

(ii) Otherwise, apply PPM to the system $\bar{w}-\bar{M}\bar{z}=B^{-1}q^N$.

[Validity of PPM in case (ii)]

DANTZIG and COTTOLE (1967) have shown that we can apply PPM to (1.1) to find out the nonnegative complementary solution or to see that (1.1) has no solution, so long as the matrix M is positive semi-definite, regardless of q. And the matrix $\bar{M}$ of (1.2) is also positive semi-definite because the principal pivot transform of a positive semi-definite matrix is also positive semi-definite. So, we can apply PPM to solve successfully the system $\bar{w}-\bar{M}\bar{z}=B^{-1}q^N$, $\bar{w}'\bar{z}=0$ and $\bar{w},\bar{z}\geq 0$, or to see the system is infeasible.

3. Addition of a constraint

Let the added constraint be

$$v_{m+1}=a'_{m+1}x-b_{m+1}\geq 0, \tag{3.1}$$

and its dual varible be y_{m+1}.

If present optimal solution also satisfies this constraint, then the problem is over. Otherwise, let the augumented system of (1.1) be

$$w-Mz+\begin{bmatrix}a_{m+1}\\0\end{bmatrix}y_{m+1}=q \tag{3.2}$$

and

$$v_{m+1}-a'_{m+1}x=-b_{m+1}. \tag{3.3}$$

By multiplying (3.2) by B^{-1}, we have

$$\bar{w}-\bar{M}\bar{z}+B^{-1}\begin{bmatrix} a_{m+1} \\ 0 \end{bmatrix} y_{m+1}=\bar{q}. \tag{3.4}$$

And by eliminating the basic variables from (3.3) using (3.4) (this can be done by usual pivoting or substitution), we have

$$v_{m+1}-\bar{a}'_{m+1}\bar{z}+\bar{a}_{m+1,m+1}y_{m+1}=-\bar{b}_{m+1}, \tag{3.5}$$

where $\quad -\bar{b}_{m+1}<0.$

That is, the new system is

$$\begin{bmatrix} w \\ v_{m+1} \end{bmatrix}-\begin{bmatrix} M & -a^N_{m+1} \\ \bar{a}'_{m+1} & \bar{a}_{m+1,m+1} \end{bmatrix}\begin{bmatrix} \bar{z} \\ y_{m+1} \end{bmatrix}=\begin{bmatrix} \bar{q} \\ -\bar{b}_{m+1} \end{bmatrix}, \tag{3.6}$$

where $\quad a^N_{m+1}=B^{-1}\begin{bmatrix} a_{m+1} \\ 0 \end{bmatrix}.$

The augumented matrix of (3.6) is positive semi-definite, because initially the matrix of the system (3.2) and (3.3) is positive semi-definite and we have derived the system (3.6) by some principal pivot transform, keeping y_{m+1} nonbasic and keeping v_{m+1} basic. So, we can apply PPM to the augumented system (3.6).

4. Addition of a variable

Let the new variable be x_{n+1}, the cofficient vector of x_{n+1} in the constraints be a_{n+1} and the added term in the objective fuction be $(d'x)x_{n+1}+d_0x^2_{n+1}/2+c_{n+1}x_{n+1}$.

Then, the augumented system is

$$w^a-M^az^a=q^a, \tag{4.1}$$

where $\quad w^a=\begin{bmatrix} u \\ u_{n+1} \\ v \end{bmatrix}, \quad z^a=\begin{bmatrix} x \\ x_{n+1} \\ y \end{bmatrix}, \quad q^a=\begin{bmatrix} c \\ c_{n+1} \\ -b \end{bmatrix}, \quad M^a=\begin{bmatrix} D & d & -A' \\ d' & d_0 & -a'_{n+1} \\ A & a_{n+1} & 0 \end{bmatrix}$

and M^a is assumed to be still positive semi-definite.

Thus, we have

$$w-Mz-\begin{bmatrix} d \\ a_{n+1} \end{bmatrix} x_{n+1}=q, \tag{4.2}$$

$$u_{n+1}-(d', -a'_{n+1})z-d_0x_{n+1}=c_{n+1}. \tag{4.3}$$

Multiplying (4.2) by B^{-1}, we have

$$\bar{w}-\bar{M}\bar{z}-B^{-1}\begin{bmatrix} d \\ a_{n+1} \end{bmatrix} x_{n+1}=\bar{q}. \tag{4.4}$$

After eliminating the basic variables from (4.4), we have

$$u_{n+1}-(\bar{d}', -\bar{a}'_{n+1})\bar{z}-\bar{d}_0 x_{n+1}=\bar{c}_{n+1}. \qquad (4.5)$$

Thus,
(i) if $\bar{c}_{n+1}\geq 0$, then $(\bar{w}, \bar{z})=(\bar{q}, 0)$, $x_{n+1}=0$ and $u_{n+1}=\bar{c}_{n+1}$ is optimal, (ii) otherwise, apply PPM to (4.4) and (4.5), where the constants on the right hand side except $\bar{c}_{n+1}$ have nonnegative values.

5. Modification of coefficients of a variable

This modification implies changes of coefficients of a variable x_k in A, D and c and may cause the changes of the k-th row and the k-th column of the matrix M and the k-th element of the vector q. We denote the modified M and q by M^N and q^N, respectively. And consequently, the basis B may become B^N which differs from B at most in a certain row and in a certain column. When B^N is regular (as can be decided in the course of the following algorithm), we can get $(B^N)^{-1}$ from B^{-1} as follows.

Suppose B^N differs from B in the p-th row (β_p) and in the q-th column (γ_q). First, let a row vector f_p be

$$f_p=\beta_p B^{-1} \qquad (5.1)$$

and $\quad f_p=(f_{p1}, \cdots, f_{pp}, \cdots, f_{pn+m})$.

If $f_{pp}\neq 0$, then let the matrix J_p be

$$J_p=\begin{bmatrix} e'_1 \\ \vdots \\ e'_{p-1} \\ g_p \\ e'_{p+1} \\ \vdots \\ e'_{n+m} \end{bmatrix}, \qquad (5.2)$$

where e_i is the i-th unit vector and

$$g_p=(-f_{p1}/f_{pp}, \cdots, -f_{pp-1}/f_{pp}, 1/f_{pp}, -f_{pp+1}/f_{pp}, \cdots, -f_{pn+m}/f_{pp}).$$

Then let

$$\tilde{B}^{-1}=B^{-1}J_p. \qquad (5.3)$$

$\tilde{B}^{-1}$ is the inverse of the matrix $\tilde{B}$ which differs from B by the p-th row β_p. Next, let a column vector h_q be

$$h_q=\tilde{B}^{-1}\gamma_q, \qquad (5.4)$$

and $\quad h'_q=(h_{1q}, \cdots, h_{q-1q}, h_{qq}, \cdots, h_{n+mq})$

If $h_{qq}\neq 0$, then let the matrix K_q be

(5.5) $$K_q=(e_1, \cdots, e_{q-1}, t_q, e_{q+1}, \cdots, e_{n+m}),$$

where $$t'_q=(-h_{1q}/h_{qq}, \cdots, -h_{q-1q}/h_{qq}, 1/h_{qq}, -h_{q+1q}/h_{qq}, \cdots, -h_{n+mq}/h_{qq}).$$

Then, we have

(5.6) $$(B^N)^{-1}=K_q\tilde{B}^{-1}.$$

The validity of the above inversion algorithm can be seen easily by calculus or see Simonnard (1966).

Two cases may occur. The first, we have $(B^N)^{-1}$, the second, B^N is singular and is no longer a basis. In the algorithm, the conditions $f_{pp}\neq 0$ and $h_{qq}\neq 0$ guarantee the first case and if either of them equals to zero, we must go to the second case.

(i) First case: Multiplying the modified system by $(B^N)^{-1}$, we have

(5.7) $$\bar{w}-\bar{M}^N\bar{z}=\bar{q}^N.$$

Then, if $\bar{q}^N\geq 0$, then the optimal solution is $(\bar{w}, \bar{z})=(\bar{q}^N, 0)$, otherwise apply PPM to the system (5.7).

(ii) Second case: In this case, add to the system a variable x_{n+1} with the coefficients equal to the modified coefficients of the variable x_k. And make the c-coefficient of x_k sufficiently large. That is, an addition of a variable and a modification of the vector q are done.

Thus, we can solve the case by the preceding algorithms. The variable x_{n+1} takes place of x_k which never remains or becomes a basic variable.

6. Modification of coefficients and constant of a constraint

This modification may cause changes of M in a certain row and in a certain column and a change of an element of q. So, we can solve the modified system by an algorithm quite analogous to the preceding section.

7. Parameterization of vector q

Instead of making q undergo a discrete variation as in section 2, we shall here make q vary continuously as a linear function of a parameter θ.

Set $q=q_0+\theta\delta$, where δ and q_0 are fixed vectors, and let B be the optimal basis for $\theta=0$. The optimal solution is $(\bar{w}, \bar{z})=(B^{-1}q_0, 0)$. Now, we make θ increase from 0. Then, basic solution becomes $(\bar{w}, \bar{z})=(B^{-1}q_0+\theta B^{-1}\delta, 0)$, and

(i) if $B^{-1}\delta\geq 0$, then $(\bar{w}, \bar{z})$ remains an optimal program for every value of $\theta(\geq 0)$,

(ii) otherwise, there exists a critical value θ_1 of θ beyond which $(\bar{w}, \bar{z}) = (B^{-1}q_0 + \theta B^{-1}\delta, 0)$ ceases to be an optimal solution, at least one element of $\bar{w}$ becoming negative;

$$\theta_1 = -\bar{w}_k/(B^{-1}\delta)_k = \min_s \left[-\bar{w}_s/(B^{-1}\delta)_s\right] \qquad (s \varepsilon \{s \mid (B^{-1}\delta)_s < 0\})$$

where $(B^{-1}\delta)_s$ denotes the s-th element of the vector $B^{-1}\delta$. When θ crosses through the value θ_1, one or several of the $\bar{w}$ variables pass through zero from above. Then PPM may be applied. When several of the $\bar{w}$ variables decrease to zero at the same time (degeneracy case), we can make use of the perturbation method or the lexicographical ordering method to avoid cycling phenomena.

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