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A PROOF OF PERRON'S THEOREM ON DIOPHANTINE APPROXIMATION OF COMPLEX NUMBERS

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ABSTRACT

In this paper we give, by defining a new continued fraction algorithm for complex numbers, a constructive proof of PERRON's theorem with some refinements.

A. HURWITZ (1891) proved, using the theory of continued fractions, that for any irrational number θ there exist infinitely many rational integers p, q such that

$$\left|\theta-\frac{p}{q}\right|<\frac{1}{\sqrt{5}\,q^2},$$

where the constant $\sqrt{5}$ cannot be improved if $\theta=\frac{1}{2}(1+\sqrt{5})$. An extension of this theorem to complex numbers was obtained by O. PERRON (1931). He proved the following

THEOREM 1. *For any complex number θ not belonging to the imaginary quadratic field $Q(\sqrt{-3})$ there exist infinitely many integers p, q in $Q(\sqrt{-3})$ such that*

$$\left|\theta-\frac{p}{q}\right|<\frac{1}{\sqrt[4]{13}\,|q|^2}.$$

If $\theta=\frac{1}{2}(\zeta+\sqrt{\zeta^2+4})$, where $\zeta=\frac{1}{2}(1+\sqrt{-3})$, the constant $\sqrt[4]{13}$ cannot be improved.

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PERRON proved it by making use of a lemma on CASSINI's curve. G. POITOU (1953) made some refinements on PERRON's theorem using a certain kind of complex continued fraction algorithm. His results are the following: *The first three constants of approximations over* $Q(\sqrt{-3})$, *corresponding to* MARKOV *numbers, are* $\sqrt[4]{13}$, 2, *and* $\sqrt{\frac{32\sqrt{3}}{13}}$; *every other exceeds* 2.070068; *and* $\sqrt{\frac{28+16\sqrt{3}}{13}}=$ 2.0701693$\cdots$ *is an accumulation point of constants.*

In this paper we give, by defining a new continued fraction algorithm for complex numbers, a constructive proof of PERRON's theorem with some additional refinements. (cf. Theorem 2 and 3.) This algorithm is of simple geometric type and may be considered in some sense as a natural extension of the real one. Moreover through this algorithm we can exhibit some interesting analogous fact between approximations of real numbers and of complex numbers over $Q(\sqrt{-3})$. Indeed, by means of this algorithm *the first badly approximable number* $\frac{1}{2}(\zeta+\sqrt{\zeta^2+4})$ *can be expanded in the from*

$$\frac{\zeta+\sqrt{\zeta^2+4}}{2}=\zeta+\frac{1}{\zeta}+\frac{1}{\zeta}+\cdots,$$

and the second badly approximable number which just attains the second constant 2 *given by* POITOU *is*

$$\zeta+\sqrt{\zeta^2+1}=2\zeta+\frac{1}{2\zeta}+\frac{1}{2\zeta}+\cdots.$$

We note that there are another type of simple geometric continued fraction algorithms for complex numbers defined by HURWITZ (1888). Recently with the help of HURWITZ's algorithm R. B. LAKEIN (1975) gave a constructive proof of FORD's theorem which is an extention of the theorem of HURWITZ to the case of $Q(\sqrt{-1})$.

1. Definition of a complex continued fraction algorithm

Every complex number z can be uniquely written in the form $z=u\zeta+v\bar{\zeta}$, where u and v are real and $\bar{w}$ is the complex conjugate of a complex number w. We put

$$[z]=[u]\zeta+[v]\bar{\zeta},$$

where $[x]$ is the largest rational integer not exceeding a real number x, and define a continued fraction algorithm $(*)$ as follows;

$$(*)\quad\begin{cases} t_n=t_n(z)=\dfrac{1}{t_{n-1}}-\left[\dfrac{1}{t_{n-1}}\right] & (n\geq 1),\quad t_0=z-[z], \\ a_n=a_n(z)=\left[\dfrac{1}{t_{n-1}}\right] & (n\geq 1),\quad a_0=[z]. \end{cases}$$

We note that these procedures terminate, i.e., $t_n=0$ for some $n\geq 0$, if and only if z belongs to $Q(\sqrt{-3})$. Hence every complex number z can be represented in the form

$$z=a_0+\frac{1|}{|a_1}+\cdots+\frac{1|}{|a_n+t_n}\qquad (n\geq 0), \tag{1}$$

provided $t_k\neq 0$ for all $k<n$.

Let N_ζ be the subset of $Q(\sqrt{-3})$ defined by

$$N_\zeta=\{u\zeta+v\bar{\zeta};\ u, v,\ \text{non-negative integers with } u+v\geq 1\}.$$

We put

$$D=\{u\zeta+v\bar{\zeta};\ u,\ v\geq 0,\ u+v\geq 0\},$$

$$X=\{u\zeta+v\bar{\zeta};\ 0\leq u,\ v<1\},$$

and

$$Y=D\setminus\{z^{-1};\ z\in X\}.$$

Thus it is easily seen that

$$t_n\in X\qquad (\mathrm{n}\geq 0), \tag{2}$$

$$a_n\in N_\zeta\subset D\setminus X\qquad (n\geq 1), \tag{3}$$

and

$$|z|\leq\frac{2\sqrt{3}}{3}\qquad (z\in Y), \tag{4}$$

$$|z|\geq\frac{\sqrt{3}}{2}\qquad (z\in D\setminus X). \tag{5}$$

Let $z_1, z_2, \cdots, z_n$ be any n complex numbers in $D\setminus X$. Then we have $z^{-1}\in Y\setminus\{0\}$ and so $z_{n-1}+z_n^{-1}\in D\setminus X$. Repeating this process we get

$$z_1+\frac{1|}{|z_2}+\cdots+\frac{1|}{|z_n}\in D\setminus X \tag{6}$$

and

$$\frac{1|}{|z_1}+\frac{1|}{|z_2}+\cdots+\frac{1|}{|z_n}\in Y\setminus\{0\}. \tag{7}$$

Note that each step of the above argument is well-defined, since the fractions

$z_n^{-1}, z_{n-1}+z_n^{-1}, \cdots$ are different from zero.

Let $a_0, a_1, \cdots$ be any sequence of integral elements of $Q(\sqrt{-3})$ such that a_n belongs to N_ζ whenever $n \geq 1$. Every finite continued fraction

$$a_0+\frac{1}{|a_1}+\cdots+\frac{1}{|a_n} \qquad (n \geq 0)$$

has a canonical representation p_n/q_n, called nth approximant, in the form of an ordinary fraction, i.e.

$$a_0+\frac{1}{|a_1}+\cdots+\frac{1}{|a_n}=\frac{p_n}{q_n},$$

where p_n and q_n are integers in $Q(\sqrt{-3})$. Especially if the sequence $a_0, a_1, \cdots$ is given by the algorithm $(*)$ we call p_n/q_n the nth approximant of z. Thus, from the general theory of finite continued fractions (see O. PERRON (1967)), we have the following formulae;

$$p_n=a_np_{n-1}+p_{n-2}, \qquad q_n=a_nq_{n-1}+q_{n-2} \qquad (n \geq 1) \tag{8}$$

$$a_n+\frac{1}{|a_{n-1}}+\cdots+\frac{1}{|a_1}=\frac{q_n}{q_{n-1}} \quad (n \geq 1), \tag{9}$$

$$p_nq_{n-1}-p_{n-1}q_n=(-1)^{n-1} \qquad (n \geq 0), \tag{10}$$

where $p_{-1}=1$, $q_{-1}=0$, $p_0=a_0$, $q_0=1$. Further if p_n/q_n is the nth approximant of z, then

$$z-\frac{p_n}{q_n}=(-1)^n\left(a_{n+1}+t_{n+1}+\frac{q_{n-1}}{q_n}\right)^{-1}\frac{1}{q_n^2} \tag{11}$$

LEMMA 1. *Let $a_0, a_1, \cdots$ be any infinite sequence of integers in $Q(\sqrt{-3})$ such that a_n belongs to N_ζ whenever $n \geq 1$ and let q_n be the denominator of the nth approximant. Then we have*

$$q_n \longrightarrow \infty \qquad as \qquad n \longrightarrow \infty.$$

Proof. Suppose on the contrary, that $q_n \not\to \infty$ as $n \to \infty$. So we can choose an infinite subsequence $\{q_{n_j}\}$ of $\{q_n\}$ such that $|q_{n_j}|<M$ for all $j \geq 1$, where M is a constant independent of j. But from (4) and (7) we get

$$\left|\frac{p_n}{q_n}\right|<|a_0|+\frac{2\sqrt{3}}{3}$$

and so

$$|p_{n_j}|<\left(|a_0|+\frac{2\sqrt{3}}{3}\right)M,$$

where the right-hand side is also independent of j. It follows from these inequalities that $p_{n_j}/q_{n_j}=p_{n_k}/q_{n_k}$ for some j and k with $j<k$, since the ring of all intergers in $Q(\sqrt{-3})$ is discrete. Hence we have

$$\frac{1\,|}{|\,a_{n_{j+1}}}+\frac{1\,|}{|\,a_{n_{j+2}}}+\cdots+\frac{1\,|}{|\,a_{n_k}}=0,$$

which contradicts (7).

LEMMA 2. *Let z be any complex number not belonging to $Q(\sqrt{-3})$ and let p_n/q_n be its nth approximant. Then we have*

$$z=\lim_{n\to\infty}\frac{p_n}{q_n}.$$

Proof. From (11) as well as (2), (5), (6), and (9) we have

$$\left|z-\frac{p_n}{q_n}\right|<\frac{2\sqrt{3}}{3}\frac{1}{|q_n|^2}.$$

The lemma follows at once from Lemma 1.

LEMMA 3. *Let $b_0, b_1, \cdots$ be any infinite sequence of integers in $Q(\sqrt{-3})$ such that b_n belongs to N_ζ whenever $n\geq 1$, and let p_n/q_n be its nth approximant. Then p_n/q_n converges to some complex number which belongth to b_0+Y.*

Proof. Let $m>n\geq 1$. Replacing z and t_{n+1} in (11) by p_m/q_m and $\frac{1\,|}{|\,b_{n+1}}+\frac{1\,|}{|\,b_{n+2}}+\cdots+\frac{1\,|}{|\,b_m}$ respectively, we have

$$\frac{p_m}{q_m}-\frac{p_n}{q_n}=(-1)^n\left(b_{n+1}+\frac{1}{b_{n+2}}+\cdots+\frac{1}{b_n}+\frac{q_n-1}{q_n}\right)^{-1}\frac{1}{q_n^2}$$

which tends to zero as $n\to\infty$. And from (7) the limit belongs to b_0+Y.

By means of Lemma 2 and 3 the algorithm $(*)$ well-defines a complex continued fraction expansion of a complex number. This complex continued fraction algorithm is natural extension of the ordinary real one, since both algorithms coincide with each other when z is real. As a corollary of Lemma 2 we remark that any two different complex numbers have different continued fraction expansions defined by the algorithm $(*)$. But at the same time Lemma 3 suggests the existance of some complex number z for which there is a sequence $b_0, b_1, \cdots$ different from $a_0(z), a_1(z), \cdots$, but satisfying the conditions in Lemma 3, whose nth approximant converges to z. Indeed we can show that there are uncountably many such complex numbers. We omit the proof of it, since about these phenomena we shall need in the sequel no more information than the next lemma. A sequence $b_0, b_1, \cdots$ is said to be admissible if there is a complex number z such that $a_n(z)=b_n$ for all $n\geq 0$.

LEMMA 4. *Let $a_0, a_1, \cdots$ be any admissible sequence.*

(a) *If $a_n=\zeta$ [or $a_n=\bar{\zeta}$] for some $n\geq 1$, then $a_{n+1}\neq 2\zeta$ [or $a_{n+1}\neq 2\bar{\zeta}$].*

(b) *If $a_n=a_{n+1}=\zeta$ [or $a_n=a_{n+1}=\bar{\zeta}$], then $a_{n+2}\neq\bar{\zeta}$ [or $a_{n+2}=\zeta$].*

Proof of (a). Let $a_n=\zeta$ for some $n\geq 1$. From (2) we have

$$t_{n-1}=(\zeta+t_n)^{-1}\in X.$$

This implies in particular

$$\operatorname{Re}\left(\left(\zeta+t_n\right)^{-1}\sqrt{\zeta}\right)<\frac{\sqrt{3}}{2}$$

or equivalently

$$\left|(\zeta+t_n)-\left(\frac{2}{3}\zeta+\frac{1}{3}\bar{\zeta}\right)\right|>\frac{\sqrt{3}}{3}, \tag{12}$$

where $\sqrt{\zeta}=\frac{1}{2}(\sqrt{3}+\sqrt{-1})$ $\left(\text{and } \sqrt{\bar{\zeta}}=\frac{1}{2}(\sqrt{3}-\sqrt{-1})\right)$.

Suppose that $a_{n+1}=2\zeta$. We have again from (2)

$$\operatorname{Im}(t_n^{-1})=\operatorname{Im}(2\zeta+t_{n+1})>\frac{\sqrt{3}}{2}$$

and so

$$\left|t_n-\left(-\frac{1}{3}\zeta+\frac{1}{3}\bar{\zeta}\right)\right|<\frac{\sqrt{3}}{3};$$

a contradiction. Considering the complex conjugates of the above inequalities we have the second part of (a) stated in the brackets.

Proof of (b). Let $a_n=a_{n+1}=\zeta$, for some $n\geq 1$. Suppose that $a_{n+2}=\bar{\zeta}$. Then we have from (2)

$$\operatorname{Im}(a_{n+2}+t_{n+2})=\operatorname{Im}(\bar{\zeta}+t_{n+2})<0$$

or equivalently

$$\operatorname{Im}(t_{n+1})=\operatorname{Im}\left((\bar{\zeta}+t_{n+2})^{-1}\right)>0.$$

And so

$$\operatorname{Im}(t_n^{-1})=\operatorname{Im}(\zeta+t_{n+1})>\frac{\sqrt{3}}{2}.$$

Hence we have also

$$\left|t_n-\left(-\frac{1}{3}\zeta+\frac{1}{3}\bar{\zeta}\right)\right|<\frac{\sqrt{3}}{3},$$

which contradicts (12). Similarly we have the second part of (b).

2. Inequalities

In this section we give a series of inequalities needful later. Let $t\in X$ and

$s \in Y$. Then the following inequalities (a)—(g) can readily be verified.

(a) $|(a+s)-(\zeta+2)| \leq \sqrt{3} \quad (a=1, \zeta, 2\zeta)$.
(b) $|(a+s)-2| \leq 2 \quad (a \in N_\zeta, \ |a| \leq \sqrt{3})$.
(c) $|(a+s)-3| \leq 3 \quad (a \in N_\zeta, \ |a| \leq 2)$.
(d) $|t| \leq 1$.
(e) $\sqrt{\bar{\zeta}}\, s + \sqrt{\zeta}\, \bar{s} \geq 0$.
(f) $|s| \leq \frac{2\sqrt{3}}{3}$.
(g) $|s-1| \leq 1$.

Each inequality regarded as a circle with variable t or s transforms into another circle through a linear transformation

$$w = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \cdots + \frac{1|}{|a_n + z}$$

or

$$= \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} z$$

in matrix notation. For completeness we give here elementary formulae of circles transformed by a linear transformation. Let

$$w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d},$$

where a, b, c, d, be complex numbers with $ad-bc=\pm 1$. Let $|z-\gamma|=r$ be a circle. Then it transforms into

$$|w-h^{-1}\sigma| = |h|^{-1} r \qquad \text{if} \quad h \neq 0,$$

$$\bar{\sigma} w + \sigma \bar{w} = l \qquad \text{if} \quad h = 0,$$

where $\quad h=|d+\gamma c|^2 - r^2|c|^2, \quad \sigma=(\gamma a+b)(\bar{\gamma}\bar{c}+\bar{d}) - r^2 a\bar{c}, \quad$ and $\quad l=|b+\gamma a|^2 - r^2|a|^2$.

Let $\bar{\gamma} z + \gamma \bar{z} = 0$ be a straight line. Then it transforms into

$$|w-k^{-1}\tau| = k^{-1}|\gamma| \qquad \text{if} \quad k \neq 0,$$

$$\bar{\tau} w + \tau \bar{w} = m \qquad \text{if} \quad k = 0,$$

where $\quad k=\gamma c\bar{d} + \bar{\gamma}\bar{c} d, \quad \tau=\bar{\gamma} b\bar{c} + \gamma a\bar{d}, \quad$ and $\quad m=\gamma a\bar{b} + \bar{\gamma}\bar{a} b$.

Using these formulae we obtain the following inequalities. The inequality (a) transforms into

$$\operatorname{Re}\left(\sqrt{\bar{\zeta}} \frac{1}{a+s}\right) \geq \frac{\sqrt{3}}{6} \qquad (a=1, \zeta, 2\zeta). \tag{13}$$

The inequality (b) transforms into

$$\mathrm{Re}\left(\frac{1}{a+s}\right)\geq\frac{1}{4}\qquad(a\in N_\zeta,\ |a|\leq\sqrt{3}),\tag{14}$$

$$\left|\frac{1|}{|1}+\frac{1|}{|\zeta}+\frac{1|}{|a+s}-\frac{5}{7}\right|\leq\frac{2}{7}\qquad(a\in N_\zeta,\ |a|<\sqrt{3}).\tag{15}$$

The inequality (c) transforms into

$$\mathrm{Re}\left(\frac{1}{a+s}\right)\geq\frac{1}{6}\qquad(a\in N_\zeta,\ |a|\leq 2).\tag{16}$$

The inequality (d) transforms into

$$\mathrm{Re}\left(\zeta\frac{1}{\zeta+1}\right)\geq\frac{1}{2},\tag{17}$$

$$\mathrm{Re}\left(\frac{1}{1+t}\right)\geq\frac{1}{2},\tag{18}$$

$$\left|\frac{1|}{|2\zeta+\bar{\zeta}}+\frac{1|}{|2\zeta+t}-\left(\frac{1}{2}\zeta+\frac{3}{5}\bar{\zeta}\right)\right|\leq\frac{1}{10},\tag{19}$$

$$\left|\frac{1|}{|\bar{\zeta}}+\frac{1|}{|\bar{\zeta}}+\frac{1|}{|\bar{\zeta}}+\frac{1|}{|1+t}-\left(\zeta+\frac{3}{4}\bar{\zeta}\right)\right|\leq\frac{1}{4}.\tag{20}$$

The inequality (e) transforms into

$$\left|\frac{1|}{|1}+\frac{1|}{|\zeta^j+s}-\left(\frac{5}{6}\zeta+\frac{2}{3}\bar{\zeta}\right)\right|\leq\frac{\sqrt{3}}{6}\qquad(j=0,1),\tag{21}$$

$$\left|\frac{1|}{|\zeta}+\frac{1|}{|\zeta+s}-\left(\frac{5}{6}\zeta+\frac{1}{6}\bar{\zeta}\right)\right|\leq\frac{\sqrt{5}}{6},\tag{22}$$

$$\left|\frac{1|}{|\zeta}+\frac{1|}{|\zeta^j+s}-\left(\frac{2}{3}\zeta+\frac{4}{3}\bar{\zeta}\right)\right|\leq\frac{\sqrt{3}}{3}\qquad(j=0,1),\tag{23}$$

$$\left|\frac{1|}{|\bar{\zeta}}+\frac{1|}{|\bar{\zeta}}+\frac{1|}{|1+s}-\left(\frac{7}{6}\zeta+\frac{5}{6}\bar{\zeta}\right)\right|\leq\frac{\sqrt{3}}{6}.\tag{24}$$

The inequality (f) transforms into

$$\left|\frac{1|}{|2\zeta}+\frac{1|}{|2\zeta}+\frac{1|}{|2\zeta+s}-\left(\frac{4}{23}\zeta+\frac{13}{23}\bar{\zeta}\right)\right|\leq\frac{\sqrt{3}}{46},\tag{25}$$

$$\left|\frac{1|}{|\zeta}+\frac{1|}{|\bar{\zeta}}+\frac{1|}{|1+s}-\left(\frac{3}{5}\zeta+\frac{7}{5}\bar{\zeta}\right)\right|\leq\frac{2\sqrt{3}}{5},\tag{26}$$

$$\mathrm{Re}\left(\sqrt{\bar{\zeta}}\left(\frac{1|}{|\bar{\zeta}}+\frac{1|}{|1}+\frac{1|}{|\zeta+s}\right)\right)\geq\frac{5\sqrt{3}}{12}.\tag{27}$$

The inequality (g) transforms into

$$\left|\frac{1|}{|2\zeta}+\frac{1|}{|\zeta+s}-\left(\frac{2}{9}\zeta+\frac{5}{9}\bar{\zeta}\right)\right|\le\frac{1}{9}, \tag{28}$$

$$\left|\frac{1|}{|\zeta}+\frac{1|}{|1}+\frac{1|}{|\bar{\zeta}+s}-\left(\frac{2}{5}\zeta+\frac{9}{10}\bar{\zeta}\right)\right|\le\frac{1}{10}, \tag{29}$$

$$\left|\frac{1|}{|\bar{\zeta}}+\frac{1|}{|1}+\frac{1|}{|1+s}-\left(\frac{13}{16}\zeta+\frac{5}{16}\bar{\zeta}\right)\right|\le\frac{1}{16}, \tag{30}$$

$$\left|\frac{1|}{|\zeta}+\frac{1|}{|\bar{\zeta}}+\frac{1|}{|\bar{\zeta}+s}-\left(\frac{2}{9}\zeta+\frac{2}{3}\bar{\zeta}\right)\right|\le\frac{1}{9}, \tag{31}$$

$$\left|\frac{1|}{|\bar{\zeta}}+\frac{1|}{|\bar{\zeta}}+\frac{1|}{|\bar{\zeta}}+\frac{1|}{|\bar{\zeta}+s}-\left(\frac{9}{10}\zeta+\frac{1}{2}\bar{\zeta}\right)\right|\le\frac{1}{10}, \tag{32}$$

We may use, if necessary, the complex conjugate of these inequalities, since $X=\bar{X}$ and $Y=\bar{Y}$ where $\bar{X}$ and $\bar{Y}$ are the complex conjugates of X and Y respectively. Later we also require the next lemma.

LEMMA 5. *Let w be a complex number satisfying the inquality*

$$|w-(u\zeta+v\bar{\zeta})|\le r,$$

where u and v are real. Then we have

$$\mathrm{Re}\,(w)\ge(u+v)/2-r,$$

$$\mathrm{Re}\,(\zeta w)\ge v-u/2-r, \qquad \mathrm{Re}\,(\bar{\zeta}w)\ge u-v/2-r,$$

$$\mathrm{Re}\,(\sqrt{\bar{\zeta}}\,w)\ge\sqrt{3}\,v/2-r, \qquad \mathrm{Re}\,(\sqrt{\bar{\zeta}}\,w)\ge\sqrt{3}\,u/2-r.$$

Proof. Clear.

3. Proof of the theorem

PERRON's proof of the second part of the theorem, that is, the statement that the constant cannot be improved, is very brief, besides, the elabolate lemma on CASSINI's curve (O. PERRON (1931)) is not required. So we prove only the first part of the theorem. Let $a_0, a_1, \cdots$ be the partial denominators of a complex number θ not belonging to $Q(\sqrt{-3})$. Put, in (11)

$$d_n=d_n(\theta)=a_{n+1}+t_{n+1}+s_n, \qquad s_n=q_{n-1}/q_n$$

and define

$$d(\theta)=\limsup_{n\to\infty}|d_n(\theta)|.$$

To prove the first part of the theorem it is enough to show that

$$d(\theta)\geq \sqrt[4]{13}=1.8988\cdots \tag{33}$$

for all complex number θ not belonging to $Q(\sqrt{-3})$. But this follows from the next inequality: Let $a_0, a_1, \cdots$ be any admissible sequence. Then we have

$$\max(|d_k|\,;\ n-12\leq k\leq n+6)>1.9 \tag{34}$$

for all $n>12$, provided $(a_n, a_{n+1}, a_{n+2})\neq(\zeta,\zeta,\zeta)$ or $(\bar\zeta,\bar\zeta,\bar\zeta)$. We need now what prepared in § 2, to get the inequality (33). By the definition we have easily

$$t_{n+1}=\frac{1\ |}{|\ a_{n+2}}+\frac{1\ |}{|\ a_{n+3}}+\cdots+\frac{1\ |}{|\ a_j+t_j} \qquad (j\geq n+2),$$

$$s_n=\frac{1\ |}{|\ a_n}+\frac{1\ |}{|\ a_{n-1}}+\cdots+\frac{1\ |}{|\ a_{k+1}+s_k} \qquad (k\leq n-1),$$

and $t_m\in X$, $s_m\in Y$ $(m\geq 1)$. At the same time we have

$$|d_n|\geq \mathrm{Re}\,(\rho a_{n+1})+\mathrm{Re}\,(\rho t_{n+1})+\mathrm{Re}\,(\rho s_n),$$

where $\rho=\zeta^j$ $\left(i=0, \pm\frac{1}{2}, \pm 1\right)$. Hence we may use the inequalities (13)—(32) together with Lemma 5 in order to estimate the absolute value of d_n from below.

Proof of the inequality (33). The proof shall be done in 14 steps A—N. In each step we give an estimate of $|d_n|$ only for the first case, since the conjugate case written in the brackets can be obtained by the conjugate argument of the first. We may assume without loss of generality that $|a_n|\leq 2$ and $a_n\neq 2$, since we have clearly $|d_n|>2$ if $|a_{n+1}|>2$ or $a_{n+1}=2$.

A. $a_{n+1}=2\zeta+\bar\zeta$ [or $\zeta+2\bar\zeta$].

A₁. Let $a_n=2\bar\zeta$ or $a_{n+2}=2\bar\zeta$. Noticing that

$$\mathrm{Re}\,(\zeta^j z)\geq 0 \qquad (j=\pm 1/2), \tag{35}$$

for all complex numbers z in D, we have from (13),

$$\mathrm{Re}\,(\sqrt{\bar\zeta}\,d_n)\geq\sqrt{3}+\sqrt{3}/6>1.9.$$

A₂. Let $|a_n|\leq\sqrt{3}$ or $|a_{n+2}|\leq\sqrt{3}$. From (14) and (16) we have

$$\mathrm{Re}\,(d_n)\geq 3/2+1/6+1/4>1.9.$$

There remains only the next case.

A₃. Let $a_n=a_{n+2}=2\zeta$. From (19) and (35) we have

$$\mathrm{Re}\,(\sqrt{\zeta}\,d_n)\geq\sqrt{3}+\sqrt{3}/4-1/10>1.9.$$

In any case we have

$$|d_n|>1.9.$$

In what follows we may assume that $a_n=2\zeta, 2\bar{\zeta}$ or $|a_n|=1$ for all $n\geq 1$.

B. $a_{n+1}=2\zeta$ [or $2\bar{\zeta}$].

B_1. Let $a_n=\zeta$. This case is excluded, by Lemma 4.

B_2. Let $a_n=\bar{\zeta}, 1, 2\bar{\zeta}$, or $a_{n+2}=\bar{\zeta}, 1, 2\bar{\zeta}$. From (13) and (35) we have

$$\mathrm{Re}\,(\sqrt{\bar{\zeta}}\,d_n)\geq\sqrt{3}+\sqrt{3}/6>1.9.$$

B_3. Let $a_n=2\zeta$ and $a_{n+2}=\zeta$. Moreover if $a_k\neq 2\zeta$ for some k $(n-3\leq k\leq n-1)$, then we have by A_2

$$\max\,(|d_k|\,;\ n-3\leq k\leq n-1)>1.9.$$

Otherwise, i.e., $a_{n-3}=a_{n-2}=a_{n-1}=2\zeta$, we have from (25) and (28)

$$\mathrm{Re}\,(d_{n-1})\geq\sqrt{3}+(\sqrt{3}-1)/9+3\sqrt{3}/46>1.9.$$

B_4. Let $a_n=a_{n+2}=2\zeta$. If $a_k\neq 2\zeta$ for some k with $n-2\leq k\leq n+4$, then we have from B_1-B_3

$$\max\,(|d_k|\,;\ n-2\leq k\leq n+1)>1.9.$$

Otherwise, i.e., $a_{n-2}=a_{n-1}=a_{n+3}=a_{n+4}=2\zeta$, then we have from (25)

$$\mathrm{Re}\,(\sqrt{\bar{\zeta}}\,d_n)\geq\sqrt{3}+2\cdot 3\sqrt{3}/46>1.9.$$

In any case we have

$$\max\,(|d_k|\,;\ n-3\leq k\leq n+1)>1.9.$$

From A and B we may assume that $|a_n|=1$ for all $n\geq 1$. The rest of the proof employs all possible cases of 3 consecutive partial denominators (a_n, a_{n+1}, a_{n+2}) with norm 1.

C. $(a_n, a_{n+1}, a_{n2})=(1, 1, 1)$. From (18) and (21) we have

$$\mathrm{Re}\,(d_n)\geq 1+1/2+(3/4-\sqrt{3}/6)>1.9.$$

D. $(a_n, a_{n+1}, a_{n+2})=(\zeta, \bar{\zeta}, \zeta)$. [or $(\bar{\zeta}, \zeta, \bar{\zeta})$].

D_1. Let $a_{n-1}=\zeta$ or 1. From (17) and (23) we have

$$\mathrm{Re}\,(\zeta d_n)\geq 1+1/2+(1-\sqrt{3}/3)>1.9.$$

D_2. Let $a_{n-1}=\bar{\zeta}$. From (17) and the conjugate of (22) we have

$$\mathrm{Re}\,(\zeta d_n)\geq 1+1/2+(3/4-\sqrt{3}/6)>1.9.$$

In either case we have

$$|d_n|>1.9.$$

E. $(a_n, a_{n+1}, a_{n+2})=(\zeta, 1, \zeta)$. [or $(\bar{\zeta}, 1, \bar{\zeta})$].

E$_1$. Let $a_{n-1}=\zeta$ or 1. This case is divided further into the following cases;

a_{n-1}	a_n	a_{n+1}	a_{n+2}	a_{n+3}	a_{n+4}	
ζ or 1	ζ	1	ζ	ζ or 1		; 1°, say,
				$\bar\zeta$	ζ	; $\lvert d_{n+2}\rvert>1.9$ (by D),
					$\bar\zeta$	; 2°, say,
					1	; 3°, say.

For each case we have the following estimates;

1°. $\mathrm{Re}(\sqrt{\zeta}d_n)\geq\sqrt{3}/2+\sqrt{3}/3+\sqrt{3}/3>1.9$ (by (23)).

2°. $\mathrm{Re}(\sqrt{\zeta}d_n)\geq\sqrt{3}/2+(\sqrt{3}/3-1/9)+\sqrt{3}/3>1.9$ (by (23), (31)).

3°. $\mathrm{Re}(\sqrt{\zeta}d_n)\geq\sqrt{3}/2+3\sqrt{3}/10+\sqrt{3}/3>1.9$ (by (23), (26)).

In any case we have

$$\max(|d_k|;\ n\leq k\leq n+2)>1.9.$$

E$_2$. Let $a_{n+3}=\zeta$ or 1. This case can be reduced to E$_1$, since the above estimates are also valid if we replace $a_{n-1}, a_n, \cdots, a_{n+4}$ by $a_{n+3}, a_{n+2}, \cdots, a_{n-2}$ respectively. (Note that the condition $(a_n, a_{n+1}, a_{n+2})=(\zeta, 1, \zeta)$ is invertible.) Hence we have

$$\max(|d_k|;\ n-2\leq k\leq n)>1.9.$$

E$_3$. Let $a_{n-1}=a_{n+3}=\bar\zeta$. The next table gives all possible cases.

a_{n-2}	a_{n-1}	a_n	a_{n+1}	a_{n+2}	a_{n+3}	a_{n+4}	
ζ	$\bar\zeta$	ζ	1	ζ	$\bar\zeta$		; $\lvert d_{n-2}\rvert>1.9$ (by (D)),
$\bar\zeta$							; excluded by Lemma 4,
1	$\bar\zeta$	ζ	1	ζ	$\bar\zeta$	ζ	; $\lvert d_{n+2}\rvert>1.9$ (by D),
						$\bar\zeta$	; 1°, say,
						1	; 2°, say.

1° $\mathrm{Re}(\sqrt{\zeta}d_n)\geq\sqrt{3}/2+(3\sqrt{3}-1)/9+3\sqrt{3}/10>1.9$ (by (26), (31)).

2°. $\mathrm{Re}(\sqrt{\zeta}d_n)\geq\sqrt{3}/2+2\cdot3\sqrt{3}/10>1.9$ (by (26)).

In any case we have

$$\max(|d_k|;\ n-2\leq k\leq n+2)>1.9.$$

Thus we have from E$_1$, E$_2$, and E$_3$

$$\max(|d_k|;\ n-2\leq k\leq n+2)>1.9.$$

F. $(a_n, a_{n+1}, a_{n+2})=(\zeta, 1, 1)$ [or $(\bar{\zeta}, 1, 1)$].

a_{n-3}	a_{n-2}	a_{n-1}	a_n	a_{n+1}	a_{n+2}	
		ζ				; F_1, say,
		1				
	ζ					; $\lvert d_{n-2}\rvert>1.9$ (by D),
	$\bar{\zeta}$		ζ	1	1	; excluded,
ζ		$\bar{\zeta}$				; F_2, say,
1	1					; F_3, say,
$\bar{\zeta}$						; $\max(\lvert d_k\rvert;\ n-5\leq k\leq n-1)>1.9$ (by E)

F_1. $\mathrm{Re}\,(d_n)\geq 1+1/2+(1-\sqrt{3}/3)>1.9$ (by (18), (23)).

F_2. $\mathrm{Re}\,(\sqrt{\bar{\zeta}}\,d_n)\geq\sqrt{3}/2+\sqrt{3}/4+5\sqrt{3}/12>1.9$ (by (21), (27)).

F_3. $\mathrm{Re}\,(\sqrt{\bar{\zeta}}\,d_n)\geq\sqrt{3}/2+\sqrt{3}/4+(13\sqrt{3}/32-1/16)>1.9$ (by (21), (30)).

In any case we have

$$\max(\lvert d_k\rvert;\ n-5\leq k\leq n-1)>1.9.$$

G. $(a_n, a_{n+1}, a_{n+2})=(1, 1, \zeta)$ [or $(1, 1, \bar{\zeta})$].

$$\max(\lvert d_k\rvert;\ n-6\leq k\leq n-1)>1.9 \quad \text{(by C, F)}$$

H. $(a_n, a_{n+1}, a_{n+2})=(\bar{\zeta}, \zeta, 1)$ [or $(\zeta, \bar{\zeta}, 1)$].

a_{n-2}	a_{n-1}	a_n	a_{n+1}	a_{n+2}	a_{n+3}	
	ζ					; $\lvert d_{n-1}\rvert>1.9$ (by D),
	$\bar{\zeta}$	$\bar{\zeta}$	ζ	1		; excluded,
$\bar{\zeta}$	1					; $\max(\lvert d_k\rvert;\ n-4\leq k\leq n)>1.9$ (by E),
1	1					; $\max(\lvert d_k\rvert;\ n-8\leq k\leq n-3)>1.9$ (by G),
		$\bar{\zeta}$	ζ	1	ζ	; $\max(\lvert d_k\rvert;\ n-1\leq k\leq n+3)>1.9$ (by E),
					1	; $\max(\lvert d_k\rvert;\ n-4\leq k\leq n+3)>1.9$ (by F),
ζ	1	$\bar{\zeta}$	ζ	1	$\bar{\zeta}$	; H_1, say.

H_1. $\mathrm{Re}\,(\sqrt{\bar{\zeta}}\,d_{n-1})\geq\sqrt{3}/2+5\sqrt{3}/12+(5\sqrt{3}-4)/14>1.9$ (by (15), (27)).

In any case we have

$$\max(\lvert d_k\rvert;\ n-8\leq k\leq n+3)>1.9.$$

I. $(a_n, a_{n+1}, a_{n+2})=(1, \zeta, \bar{\zeta})$ [or $(1, \bar{\zeta}, \zeta)$].

a_{n-1}	a_n	a_{n+1}	a_{n+2}	a_{n+3}	a_{n+4}	a_{n+5}	a_{n+6}	
ζ	1	ζ	$\bar{\zeta}$					; $\max(\lvert dk\rvert;\ n-3\leq k\leq n+1)>1.9$ (by E),
1	1	ζ	$\bar{\zeta}$					; $\max(\lvert d_k\rvert;\ n-7\leq k\leq n-2)>1.9$ (by G),
	1	ζ	$\bar{\zeta}$	ζ				; $\lvert d_{n+1}\rvert>1.9$ (by D),
	1	ζ	$\bar{\zeta}$	1				; $\max(\lvert d_k\rvert;\ n-7\leq k\leq n+4)>1.9$ (by H),
$\bar{\zeta}$	1	ζ	$\bar{\zeta}$	$\bar{\zeta}$	ζ			; excluded,
$\bar{\zeta}$	1	ζ	$\bar{\zeta}$	$\bar{\zeta}$	$\bar{\zeta}$	ζ		; excluded,
$\bar{\zeta}$	1	ζ	$\bar{\zeta}$	$\bar{\zeta}$	$\bar{\zeta}$	$\bar{\zeta}$	ζ	; excluded,
$\bar{\zeta}$	1	ζ	$\bar{\zeta}$	$\bar{\zeta}$	$\bar{\zeta}$	$\bar{\zeta}$	$\bar{\zeta}$	; I_1, say,
$\bar{\zeta}$	1	ζ	$\bar{\zeta}$	$\bar{\zeta}$	$\bar{\zeta}$	$\bar{\zeta}$	1	; I_2, say,
$\bar{\zeta}$	1	ζ	$\bar{\zeta}$	$\bar{\zeta}$	$\bar{\zeta}$	1		; I_3, say,
$\bar{\zeta}$	1	ζ	$\bar{\zeta}$	$\bar{\zeta}$	1			; I_4, say.

I_1. $\mathrm{Re}(\sqrt{\zeta}\,d_{n+1})\geq\sqrt{3}/2+(\sqrt{3}/4-1/10)+5\sqrt{3}/12>1.9$ (by (27), (32)).

I_2. $\mathrm{Re}(\sqrt{\zeta}\,d_{n+1})\geq\sqrt{3}/2+(3\sqrt{3}/8-1/4)+5\sqrt{3}/12>1.9$ (by (20), (27)).

I_3. $\mathrm{Re}(\sqrt{\zeta}\,d_{n+1})\geq\sqrt{3}/2+\sqrt{3}/4+5\sqrt{3}/12>1.9$ (by (24), (27)).

I_4. $\mathrm{Re}(\sqrt{\bar{\zeta}}\,d_n)\geq\sqrt{3}/2+5\sqrt{3}/12+(5\sqrt{3}-4)/14>1.9$ (by (15), (24)).

In any case we have

$$\max(|d_k|;\ n-7\leq k\leq n+4)>1.9.$$

J. $(a_n, a_{n+1}, a_{n+2})=(\bar{\zeta}, \zeta, \zeta)$ [or $(\zeta, \bar{\zeta}, \bar{\zeta})$].

From D, I, and Lemma 4 we have

$$\max(|d_k|;\ n-8\leq k\leq n+3)>1.9.$$

K. $(a_n, a_{n+1}, a_{n+2})=(1, \zeta, 1)$ [or $(1, \bar{\zeta}, 1)$].

a_{n-2}	a_{n-1}	a_n	a_{n+1}	a_{n+2}	a_{n+3}	
	ζ	1	ζ	1		; $\max(\lvert d_k\rvert;\ n-3\leq k\leq n+1)>1.9$ (by E),
	1	1	ζ	1		; $\max(\lvert d_k\rvert:\ n-7\leq k\leq n-2)>1.9$ (by G),
		1	ζ	1	ζ	; $\max(\lvert d_k\rvert;\ n-1\leq k\leq n+3)>1.9$ (by F),
		1	ζ	1	1	; $\max(\lvert d_k\rvert;\ n-4\leq k\leq n)>1.9$ (by F),
ζ	$\bar{\zeta}$	1	ζ	1		; $\max(\lvert d_k\rvert;\ n-10\leq k\leq n+1)>1.9$ (by H),
$\bar{\zeta}$	$\bar{\zeta}$	1	ζ	1	$\bar{\zeta}$	; K_1, say.
1	$\bar{\zeta}$	1	ζ	1	$\bar{\zeta}$	; K_1, say.

K_1. $\mathrm{Re}\,(d_{n-1}) \geq 1 + 11/20 + (1-\sqrt{3}/3) > 1.9$ (by (23), (29)).

In any case we have

$$\max\,(|d_k|\,;\; n-10 \leq k \leq n+3) > 1.9.$$

L. $(a_n, a_{n+1}, a_{n+2}) = (\zeta, 1, \bar\zeta)$ [or $(\bar\zeta, 1, \zeta)$].

a_{n-1}	a_n	a_{n+1}	a_{n+2}	a_{n+3}	a_{n+4}	a_{n+5}	
$\bar\zeta$	ζ	1	$\bar\zeta$				; $\max\,(\lvert d_k\rvert\,;\; n-9 \leq k \leq n+2) > 1.9$ (by H),
1							; $\max\,(\lvert d_k\rvert\,;\; n-11 \leq k \leq n+2) > 1.9$ (by K),
ζ	ζ	1	$\bar\zeta$	ζ			; $\max\,(\lvert d_k\rvert\,;\; n-6 \leq k \leq n+5) > 1.9$ (by I),
				$\bar\zeta$	ζ		; excluded,
					$\bar\zeta$	ζ	; excluded,
						$\bar\zeta$	; L_1, say,
						1	; L_2, say,
					1		; L_3, say,
				1			; $\max\,(\lvert d_k\rvert\,;\; n-9 \leq k \leq n+2) > 1.9$ (by K).

L_1. $\mathrm{Re}\,(d_n) \geq 1 + 3/5 + (1-\sqrt{3}/3) > 1.9$ (by (23), (32)).

L_2. $\mathrm{Re}\,(d_n) \geq 1 + 5/8 + (1-\sqrt{3}/3) > 1.9$ (by (20), (23)).

L_3. $\mathrm{Re}\,(d_n) \geq 1 + (1-\sqrt{3}/6) + (1-\sqrt{3}/3) > 1.9$ (by (23), (24)).

In any case we have

$$\max\,(|d_k|\,;\; n-10 \leq k \leq n+6) > 1.9.$$

M. $(a_n, a_{n+1}, a_{n+2}) = (\zeta, \zeta, 1)$ [or $(\bar\zeta, \bar\zeta, 1)$].

From E, F, and L we have

$$\max\,(|d_k|\,;\; n-10 \leq k \leq n+6) > 1.9.$$

N. $(a_n, a_{n+1}, a_{n+2}) = (1, \zeta, \zeta)$ [or $(1, \bar\zeta, \bar\zeta)$].

From E, G, and L we have

$$\max\,(|d_k|\,;\; n-12 \leq k \leq n+4) > 1.9.$$

Since the cases $(a_n, a_{n+1}, a_{n+2}) = (\zeta, \zeta, \bar\zeta)$ [and $(\bar\zeta, \bar\zeta, \zeta)$] are excluded by Lemma 4, we have employed all possible cases except (ζ, ζ, ζ) and $[(\bar\zeta, \bar\zeta, \bar\zeta)]$. In any case we find the inequality (34), as required.

Now we proove the inequality (33). Let $a_0, a_1, \cdots$ be the partial denominators of a complex number given arbitrary. If $a_n = \zeta$, $a_{n+1} = \zeta$, $a_{n+2} = \zeta, \cdots$ or $a_n = \bar\zeta$, $a_{n+1} = \bar\zeta$, $a_{n+2} = \bar\zeta, \cdots$ for all sufficiently large n. Then we have, in either case

$$\lim_{n\to\infty} |d_n| = \sqrt[4]{13}\,.$$

Otherwise, we can choose infinitely many 3 consecutive partial denominators (a_n, a_{n+1}, a_{n+2}) different from (ζ, ζ, ζ) or $(\bar{\zeta}, \bar{\zeta}, \bar{\zeta})$. So by the inequality (34) we have

$$\limsup_{n\to\infty} |d_n| > 1.9 > \sqrt[4]{13}\,.$$

As a result we obtain the inequality (33).

We can deduce some refinements of the theorem 1 from the above proof.

THEOREM 2. *Let θ be any complex number not belonging to $Q(\sqrt{-3})$ and let $a_0, a_1, \cdots$ be its partial denominators defined by the algorithm* $(*)$. *A necessary and sufficient condition that*

$$d(\theta) = \sqrt[4]{13}$$

is that $(a_n, a_{n+1}, \cdots) = (\zeta, \zeta, \cdots)$ or $(\bar{\zeta}, \bar{\zeta}, \cdots)$ for some $n \geq 0$. If θ is not such number, then

$$d(\theta) > 1.9,$$

i.e. the first constant $\sqrt[4]{13}$ is isolated.

E. BOREL (1903) improved the theorem of HURWITZ as follows:

Let θ be any irrational number whose nth approximant is denoted by p_n/q_n. Then for any $n > 1$ at least one of 3 consecutive approximants p_n/q_n, p_{n+1}/q_{n+1}, p_{n+2}/q_{n+2} satisfies the inequality

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\,q^2}\,.$$

In this sense we give a partial improvement of PERRON's theorem.

THEOREM 3. *Let F_n be the number of partial denominators p_n/q_n $(1 \leq k \leq n)$ such that*

$$\left|\theta - \frac{p_n}{q_n}\right| < \frac{1}{\sqrt[4]{13}\,|q_n|^2}\,.$$

If $(a_n, a_{n+1}, a_{n+2}) \neq (\zeta, \zeta, \zeta)$ or $(\bar{\zeta}, \bar{\zeta}, \bar{\zeta})$ for all sufficiently large n. Then

$$F_n > \frac{n}{19} - B,$$

where B is a constant depending possibly on θ.

REFERENCES

BOREL, E. (1903): Contribution à l'analyse arithmétique du continu, J. Math. Pures Appl. (5) **9**, 329–375.

HURWITZ, A. (1888): Über die Entwicklung Komplexer Grössen in Kettenbrüche, Acta Math. **11**, 187–200.

HURWITZ, A. (1891): Über die angenaherte Darstellung der Irrationalzahlen durch rational Brüche, Math. Ann. **39**, 279–284.

LAKEIN, R.B. (1975): A continued fraction proof of FORD's theorem on complex rational approximations, J. Reine Angew. Math. **27**, 1–13.

PERRON, O. (1931): Über einen Approximationssatz von HURWITZ und über die Approximation einer Komplexen Zahl durch Zahlen des Körpers der dritten Einheitswurzeln, Sitzber. Bayer Akad. Wiss., 129–154.

PERRON, O. (1967): Die Lehre von den Kettenbrüchen, 2 Aufl. Chelsea.

POITOU, G. (1953): Sur l'approximation des nombres complexes par les nombres des corps imaginaires quadratiques dénués d'idéaux non principaux, particulièrement lorsque vaut l'algorithme d'Euclide, Ann. Sci. Ecole Norm. Sup. (3) **70**, 199–265.