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ANALYTIC WEAKLY STATIONARY PROCESSES

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ABSTRACT

Some basic results on weakly stationary processes which are mean analytic at the origin, on a half-plane or the entire complex plane, are given.

1. Introduction

Let $X(t, \omega)$, $-\infty < t < \infty$ be a stochastic process of the second order, that is, a process with $E|X(t, \omega)|^2 < \infty$, $-\infty < t < \infty$. If there is a stochastic process $Y(t, \omega)$ of the second order such that $E|[X(t+h, \omega) - X(t, \omega)]/h - Y(t, \omega)|^2 \rightarrow 0$ as $h \rightarrow 0$, for a t, then $X(t, \omega)$ is called mean differentiable at t and $Y(t, \omega)$ the mean derivative. We denote it simply by $X'(t, \omega)$. The mean derivative $X^{(n)}(t, \omega)$ of the *n*-th order of $X(t, \omega)$ is defined in an obvious way.

Suppose that $X(t, \omega)$ has the mean derivatives of all orders at $t=t_0$. If the Taylor series

$$\sum_{n=0}^{\infty} X^{(n)}(t_0, \omega) \frac{(t-t_0)^n}{n!}$$
(1.1)

converges in quadratic mean to $X(t, \omega)$ in $|t-t_0| < \delta$ for some $\delta > 0$, then $X(t, \omega)$ is called mean analytic at $t=t_0$. (BALYAEV [4]). This definition is equivalent to the following one. $X(t, \omega)$ is mean analytic at $t=t_0$, if there is a random function $X_1(z, \omega)$ defined in some neighbourhood D of t_0 in the complex plane, which is mean analytic in D, that is $[X_1(z+h, \omega)-X_1(z, \omega)]/h$ converges in quadratic mean as $h\to 0$ to a random function $X_1(z, \omega)$ for each $z \in D$ and is such that $X_1(t, \omega)=X(t, \omega)$ with probability one for each real t contained in D.

In this definition if the mean analyticity in D of $X_1(z, \omega)$ is replaced by the

almost sure analyticity in D, namely if $X_1(z, \omega)$ is analytic in D as a function of a complex variable z with probability one, then $X(t, \omega)$ is called almost surely analytic at $t=t_0$.

We are, in this paper, concerned with weakly stationary processes and give some rather basic properties of them when they are analytic in the above sense.

2. Some known results

LOÈVE [6] studied the mean analyticity of a stochastic process of the second order and gave basic results one of which takes the following form in the case of a weakly stationary process.

I. In order that a weakly stationary process is everywhere mean analytic if and only if its covariance function is analytic at the origin.

LOÈVE [6] also gave a condition for almost sure analyticity for a second order process. Later BALYAEV [4] improved the result. BALYAEV's theorem turns out to be the following theorem for a weakly stationary process.

II. If a weakly stationary process has the covariance function analytic at the origin, then the process is almost surely analytic at the origin.

Hence in view of the above result, this yields to: If a weakly stationary process is mean analytic at the origin, then it is almost surely analytic at the origin.

As a matter of fact, this is rather a special case of a more general theorem which states:

III. If a random function $f(z, \omega)$ with finite second moment for each $z \in D$, D being a domain in the complex plane, is mean analytic in D, then it is almost surely analytic in D, that is there is a random function $f_1(z, \omega)$ which is analytic in D as a function of z with probability one and is such that $f_1(z, \omega)=f(z, \omega)$ with probability one for each $z \in D$.

This is included in BALYAEV [4] in the local form and was shown by ARNOLD [2].

3. Strip of mean analyticity

We assume throughout in this paper that $X(t, \omega)$, $-\infty < t < \infty$, is a weakly stationary process with $EX(t, \omega)=0$, $-\infty < t < \infty$ and the convariance function

$$\rho(u) = EX(t+u, \omega)X(t, \omega)$$
$$= \int_{-\infty}^{\infty} e^{iu\lambda} dF(\lambda).$$
(3.1)

 $X(t, \omega)$ admits the representation

$$X(t,\omega) = \int_{-\infty}^{\infty} e^{it\lambda} \hat{\xi}(d\lambda,\omega), \qquad (3.2)$$

where $\xi(S, \omega)$, S being a Borel set, is a random measure with

$$E\xi(S,\omega) = 0. \tag{3.2}$$

$$E|\xi(S,\omega)|^2 = F(S),$$
 (3.4)

F(S) being a bounded measure generated by $F(\lambda)$.

We first of all mention the following theorem which is well known in the theory of analytic characteristic functions (LUKACS [7], KAWATA [5]).

Theorem 3.1. If the covariance function (3,1) of a weakly stationaly process is analytic at u=0, then there is a strip

$$-\alpha < \operatorname{Im} z < \beta, \qquad \alpha, \beta > 0$$
 (3.5)

such that there exists a function $\rho(z)$ which is analytic in the strip (3.5), is identical with $\rho(u)$ on the real axis and has the representation

$$\rho(z) = \int_{-\infty}^{\infty} e^{iz\lambda} dF(\lambda) \tag{3.6}$$

there, where

$$\int_{-\infty}^{\infty} e^{-y\lambda} dF(\lambda) < \infty, \quad \text{for} \quad -\alpha < y < \beta.$$
(3.7)

 $-i\alpha$ and $i\beta$ are singularities of $\rho(z)$.

Either α or β , or the both may be infinite. The strip (3.5) is called the strip of analyticity of $\rho(u)$. Corresponding to this theorem, we have

Theorem 3.2. If a weakly stationary process $X(t, \omega)$ with representation (3.2) is mean analytic at t=0, then there are a strip

$$-\alpha_1 < \operatorname{Im} z < \beta_1, \qquad \alpha_1, \beta_1 > 0 \tag{3.8}$$

and a random function $X_1(z, \omega)$ which is mean analytic in (3.8), has the representation

$$X_{1}(z,\omega) = \int_{-\infty}^{\infty} e^{iz\lambda} \xi(d\lambda,\omega)$$
(3.9)

and is identical with $X(t, \omega)$ with probability one for each real z=t, and $X_1(z, \omega)$ is not mean analytic at $z=-i\alpha_1$, $z=i\beta_1$.

The proof is carried out in a way similar to that of Theorem 3.1 in the following manner.

Consider

$$Y_{1}(z,\omega) = \int_{0}^{\infty} e^{iz\lambda} \xi(d\lambda,\omega)$$
(3.10)

and suppose that the integral on the right hand side exists at $z=z_1=t_1+iy_1$, or $e^{-y_1\lambda} \in L^2(dF)$. Then obviously $e^{-y_\lambda} \in L^2(dF)$ for $y > y_1$ and hence (3.10) exists for all z with $\operatorname{Im} z > y_1$. Therefore there should exist $-\alpha_1$ such that the integral in (3.10) exists for all z with $\operatorname{Im} z > -\alpha_1$, and does not exist for any z with $\operatorname{Im} z < -\alpha_1$. α_1 should be nonnegative since $F(\lambda)$ is of bounded variation over $(-\infty, \infty)$.

It is easy to show that $Y_1(z, \omega)$ is differentiable in mean arbitrary number of times and

$$Y_{1}^{(n)}(z,\omega) = i^{n} \int_{0}^{\infty} e^{iz\lambda} \lambda^{n} \hat{\xi}(d\lambda,\omega)$$
(3.11)

for $\text{Im } z > -\alpha_1$. Moreover $Y_1(z, \omega)$ is mean analytic in $\text{Im } z > -\alpha_1$, because as we easily verify,

$$E\left|\sum_{n=0}^{N} \frac{(z-z_{1})^{n}}{n!} Y_{1}^{(n)}(z,\omega) - Y_{1}(z,\omega)\right|^{2}$$
$$= \int_{0}^{\infty} \left|\sum_{n=0}^{N} \frac{(z-z_{1})^{n}}{n!} (i\lambda)^{n} - e^{i\lambda(z-z_{1})}\right|^{2} e^{-2y_{1}\lambda} dF(\lambda)$$

for any z, z_1 such that $y_1 = \text{Im } z_1 > -\alpha_1$, $|z - z_1| < y_1 + \alpha_1$, and the right hand side converges to zero as $N \rightarrow \infty$.

It is obvious that $Y_1(z, \omega)$ is not mean analytic at $z = -i\alpha_1$.

In a similar way, we see that, for

$$Y_2(z,\omega) = \int_{-\infty}^0 e^{iz\lambda} \hat{\varsigma}(d\lambda,\omega), \qquad (3.12)$$

there exists a $\beta_1 \ge 0$ such that $Y_2(z, \omega)$ is mean analytic in $\text{Im } z < \beta_1$ and is not at $z = i\beta_1$.

Now $X(t, \omega)$ is supposed to be mean analytic at t=0 and then it is easy to see that the both $Y_1(z, \omega)$ and $Y_2(z, \omega)$ are mean analytic at z=0. Hence $\alpha_1 > 0$, $\beta_1 > 0$ and $X_1(z, \omega) \equiv Y_1(z, \omega) + Y_2(z, \omega)$ is mean analytic in $-\alpha_1 > \text{Im } z > \beta_1$. Obviously $X_1(t, \omega) = X(t, \omega)$ holds with probability one for each real t. This concludes the proof.

The strip (3.8) is called the *strip of mean analitycity* of $X(t, \omega)$.

In view of the equivalence of existence of the integrals $\int_{-\infty}^{\infty} e^{-2y\lambda} dF(\lambda)$ and $\int_{-\infty}^{\infty} e^{i\lambda z} \zeta(d\lambda, \omega)$ where $\operatorname{Im} z = y$, we may conclude

$$\alpha_1 = \frac{1}{2} \alpha, \qquad \beta_1 = \frac{1}{2} \beta, \qquad (3.13)$$

where α , β are those in Theorem 3.1.

Because of III in 1 we have

Corollary 3.1. If a weakly stationary process $X(t, \omega)$ is mean analytic at the origin, then there is a random function $X_2(z, \omega)$ which is analytic (as a function of z) with probability one in the strip of mean analyticity and is identical with $X(t, \omega)$ on $-\infty < t < \infty$ with probability one for each t.

The domain of almost sure analyticity of a weakly stationary process is not smaller and actually may be larger than the strip of mean analyticity, as the following example shows.

Let the probability space be [0,1] in which Lebesgue measurable sets are considered and the probability is taken to be the Lebesgue measure. Define for $\omega \in [0,1]$, n=1,2...

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$$\begin{aligned} \xi_n(\omega) &= -2^{n/2} e^{-ny_0 2}, & \text{for} \quad 2^{-n} - 2^{-(n+1)} < \omega < 2^{-n}, \\ &= 2^{n/2} e^{-ny_0 /}, & \text{for} \quad 2^{-n} < \omega < 2^{-n} + 2^{-(n+1)}, \\ &= 0, & \text{elsewhere,} \end{aligned}$$

 y_0 being any fixed positive number. Let

$$X(z,\omega) = \sum_{n=1}^{\infty} \xi_n(\omega) e^{inz}$$
(3.14)

which is weakly startionary for z=t. $\{\xi_n(\omega)\}\$ is a sequence of orthogonal random variables. Since

$$\rho(u) = \sum_{n=1}^{\infty} e^{-ny_0} e^{inu}$$

 $\rho(z)$ has the strip of analyticity, $\operatorname{Im} z > -y_0$ and $X(t, \omega)$ has the strip of mean analyticity $\operatorname{Im} z > -y_0/2$. On the other hand for each $0 < \omega < 1$, $\xi_n(\omega) = 0$ for sufficiently large *n* and hence $X(z, \omega)$ is a trigonometric polynominal for each ω . Hence it is analytic for each $\omega \in (0, 1)$, in the whole complex plane.

4. Boundary of an analytic random function

Let $X(t, \omega)$ be a weakly stationary process as in 3. If there is a random function $\varphi(z, \omega)$ which is mean analytic in the half-plane Im z>0 such that

$$E|\varphi(z,\omega) - X(t,\omega)|^2 \to 0$$
 as $y \to 0+$

for every $-\infty < t < \infty$, where z=t+iy, then $X(t, \omega)$ is called the *boundary* of $\varphi(z, \omega)$, or the *boundary process* of a mean analytic random function.

Suppose that the spectral distribution function $F(\lambda)$ of $X(t, \omega)$ is constant for $\lambda < 0$. Then as we saw in the proof of Theorem 3.1, $X(t, \omega)$ is the boundary of $X_1(z, \omega) = \int_0^\infty e^{iz\lambda} \xi(d\lambda, \omega)$. In fact $E |X_1(z, \omega) - X(t, \omega)|^2 = \int_0^\infty \left| e^{-y\lambda} - 1 \right| dF(\lambda)$ converges to zero as $y \to 0+$. We write $X_1(z, \omega)$ for $\varphi(z, \omega)$ without confusion.

Since for every x and every y > 0

$$\int_{-\infty}^{\infty} \left[\int_{0}^{\infty} \left| \frac{y}{(t-x)^2 + y^2} e^{it\lambda} \right|^2 dF(\lambda) \right]^{1/2} dt < \infty,$$

by a theorem analogous to the FUBINI-TONELLI Theorem (See Rozanov [8] p. 12), we have

$$\frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty \frac{y}{(t-u)^2 + y^2} e^{iu\lambda} du \right] \xi(d\lambda, \omega)$$
$$= \frac{1}{\pi} \int_{-\infty}^\infty du \frac{y}{(t-u)^2 + y^2} \int_0^\infty e^{iu\lambda} \xi(d\lambda, \omega)$$

for y > 0, $-\infty < t < \infty$, namely

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(t-u)^2 + y^2} X(u,\omega) du = \int_{0}^{\infty} e^{it\lambda - y\lambda} \xi(d\lambda,\omega)$$
$$= \int_{0}^{\infty} e^{iz\lambda} \xi(d\lambda,\omega), \qquad z = t + iy.$$
(4.2)

Hence we have shown

Theorem 4.1. If the spectral distribution function $F(\lambda)$ of a weakly stationary process $X(t, \omega)$ is constant for $\lambda < 0$, then $X(t, \omega)$ is the boundary of the random function

$$X(z,\omega) = \int_0^\infty e^{iz\lambda} \xi(d\lambda,\omega)$$
(4.3)

which is mean analytic in Im z > 0 and is represented by the Poisson integral

$$X(z,\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(t-u)^2 + y^2} X(u,\omega) du.$$
(4.4)

The converse of this theorem is true in the following sense.

Theorem 4.2. If a weakly stationary process $X(t, \omega)$ is the boundary of a random function $X(z, \omega)$ which is mean analytic in the upper half-plane and is represented by the Poisson integral (4.4) of $X(t, \omega)$, then $F(\lambda)$ is constant for $\lambda < 0$. We also have the following seemingly more general theorem.

Theorem 4.3. If a weakly stationary process $X(t, \omega)$ is the boundary of a random function $X(z, \omega)$ which is mean analytic in the upper half-plane with $E |X(z, \omega)|^2$ bounded in Im z > 0, then $F(\lambda)$ is constant for $\lambda < 0$.

We note that if (4.4) is true, then $E|X(z,\omega)|^2$ is bounded in Im z>0, because the variance of the Poisson integral in (4.4) is easily seen to be $\int_{0}^{\infty} e^{-2y_{2}} dF(\lambda)$.

We shall prove Theorem 4.3. Consider

$$EX(z, \omega)\overline{X(0, \omega)} = \rho(z)$$

which is seen to be analytic in $\operatorname{Im} z > 0$ from the analyticity of $X(z, \omega)$. Since $X(z, \omega)$ converges to $X(t, \omega)$ in quadratic mean as $y \to 0 + (z=t+iy)$, $\rho(z)$ converges to $\rho(t)$ as $y \to 0+$. Hence $g(z)=\rho(z)-\int_0^\infty e^{iz\lambda}dF(\lambda)$ is analytic in $\operatorname{Im} z>0$ and converges to $\int_{-\infty}^0 e^{it\lambda}dF(\lambda)$ as $y\to 0$. On the other hand, $g_1(z)=\int_{-\infty}^0 e^{iz\lambda}dF(\lambda)$ is analytic in $\operatorname{Im} z<0$ and converges to $\int_{-\infty}^0 e^{it\lambda}dF(\lambda)$ as $y\to 0-$. Therefore $g_1(z)$ is analytic cally continued to the upper half-plane and consequently analytic in the whole plane.

Now from the assumption that $E |X(z, \omega)|^2$ is bounded in Im z > 0, $\rho(z)$ is bounded in Im z > 0 and then g(z) is bounded in Im z > 0. Hence $g_1(z)$ is, in the upper half-plane, bounded. Note that $g_1(z)$ is bounded in $\text{Im } z \leq y$ from the form

of the integral. Therefore $g_1(z)$ should be a constant for all z, which implies that $\int_{-\infty}^{0} e^{it\lambda} dF(\lambda)$ is constant and this in turn implies that $F(\lambda)$ is constant for $\lambda < 0$.

5. Order of an entire weakly stationary process

Suppose that a weakly stationary process $X(t, \omega)$ is mean analytic in the whole plane, that is, there is a random function $X_1(z, \omega)$ which is mean analytic in the whole plane such that $X(t, \omega)$, $X_1(t, \omega)$ with probability one for each real t. We write $X_1(z, \omega)$ as before $X(z, \omega)$. Then we should have

$$\int_{-\infty}^{\infty} e^{-y\lambda} dF(\lambda) < \infty \quad \text{for} \quad -\infty < y < \infty, \tag{5.1}$$

 $F(\lambda)$ being the spectral distribution function of $X(t, \omega)$. Let us call $\rho(z) = \int_{-\infty}^{\infty} e^{iz\lambda} dF(\lambda), \ z = t + iy$, the *entire covariance function* of $X(t, \omega)$.

Let us put, for r > 0

$$M_m(\mathbf{r}) = \max_{|z| \le r} E |X(z, \omega)|^2$$
(5.2)

which is equal to

$$\max_{\substack{|z| \le r}} \int_{-\infty}^{\infty} e^{-2y\lambda} dF(\lambda)$$
$$= \max \left[\rho(2ir), \rho(-2ir)\right]$$
$$= \max_{\substack{|z| = 2r}} |\rho(z)|.$$

We define the *mean order* ρ_m of $X(z, \omega)$ by

$$\rho_m = \limsup_{r \to \infty} \frac{\log \log M_m(r)}{\log r}.$$
(5.3)

 ρ_m is no more than the order of $\rho(z)$.

From the known theorem on analytic characteristic functions we have that if the spectral distribution $F(\lambda)$ is nondegenerate,

$$\rho_m \ge 1. \tag{5.4}$$

From III in 1, $X(z, \omega)$ is almost surely entire. We may then define

$$M_a(\mathbf{r}, \boldsymbol{\omega}) = \max_{|\boldsymbol{z}|=r} |X(\boldsymbol{z}, \boldsymbol{\omega})| \tag{5.5}$$

and

$$\rho_a(\omega) = \limsup_{r \to \infty} \frac{\log \log M_a(r, \omega)}{\log r}$$
(5.6)

with probability one. We call $\rho_a(\omega)$ the almost sure order of $X(z, \omega)$.

ARNOLD [3] has extensively studied on the order $\rho(\omega)$ of an entire random power series $f(z, \omega) = \sum_{n=0}^{\infty} a_n(\omega) z^n$ (that is, an entire power series with probability one). The order $\rho(\omega)$ is given by (5.6) with $M_a(r, \omega)$ in which $X(z, \omega)$ is replaced by $f(z, \omega)$ and is also given in the form

$$\rho(\omega) = \limsup_{n \to \infty} \frac{n \log n}{\log(1/|a_n(\omega)|)}.$$
(5.7)

One of ARNOLD's results [3] (Satz 1) is that, for every $x \ge 0$

$$P(\rho(\omega) \le x) = 1 \tag{5.8}$$

is equivalent to

$$P(\operatorname{limsup}|a_n(\omega)| \ge n^{-n/(x_+,\varepsilon)}) = 0$$
(5.9)

for all $\varepsilon > 0$.

From this, Borel-Cantelli lemma gives us that if

$$\sum_{n=1}^{\infty} P([a_n(\omega)] \ge n^{-n/(x+\epsilon)}) < \infty$$
(5.10)

for all $\varepsilon > 0$, then (5.8) is true.

Applying this result to a weakly stationary process, we have the following theorem.

Theorem 5.1. If the spectral distribution function $F(\lambda)$ of a mean entire weakly stationary process satisfies that, for $x \ge 0$

$$\sum_{n=1}^{\infty} n^{2n/(x+\epsilon)} \frac{1}{(n!)^2} \int_{-\infty}^{\infty} \lambda^{2n} dF(\lambda) < \infty$$
(5.11)

for all $\varepsilon > 0$, then

$$P(\rho_a(\omega) \le x) = 1. \tag{5.12}$$

In fact, if $\sum a_n(\omega)z^n$ is the power series expansion of $X(z, \omega)$, then $a_n(\omega) = X^{(n)}(0, \omega)/n! = (i^n/n!) \int_{-\infty}^{\infty} \lambda^n \xi(d\lambda, \omega)$ and by Chebyshev inequality,

$$P(|a_n(\omega)| \ge n^{-n/(x+\epsilon)}) \le n^{2n/(x+\epsilon)} E |a_n(\omega)|^2 = n^{2n/(x+\epsilon)} \frac{1}{(n!)^2} \int_{-\infty}^{\infty} \lambda^{2n} dF(\lambda):$$

Hence from (5.11), (5.10) follows.

From Theorem 5.1 we have, taking 1 for x, the following theorem.

Theorem 5.2. If the spectral distribution $F(\lambda)$ of a mean entire weakly stationary process satisfies that

$$\int_{-\infty}^{\infty} \lambda^{2n} dF(\lambda) \leq C n^{4n} \tag{5.13}$$

for some sequence $\{A_n\}$ with $A_n = O(n)$ and some constant C, then

$$P(\rho_a(\omega) \leq 1) = 1. \tag{5.14}$$

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Actually if (5.13) holds, then the series in (5.11) is, because of STIRLING formula, not larger than

$$C_1\sum_{n=1}^{\infty}n^{2n/(1+\varepsilon)-(2n+1)}e^{2n}n^{\varepsilon_nn}$$

for some $\varepsilon_n \rightarrow 0$, C_1 being a constant and this is

$$\leq C \sum_{n=1}^{\infty} n^{-1} e^{2n} n^{\frac{-2\varepsilon}{\varepsilon+1}n+n\varepsilon} n^{\varepsilon}$$

which is convergent for every $\varepsilon > 0$ and this proves the theorem.

We remark in connection with (5.13), that for every entire weakly stationary process,

$$\int_{-\infty}^{\infty} \lambda^{2n} dF(\lambda) = o(n^{2n}) \tag{5.15}$$

as $n \to \infty$. In fact, we have, choosing any $\varepsilon_n \to 0$, the left hand side of (5.15) is

$$\int_{|\lambda| \ge \epsilon_n n} \lambda^{2n} dF(\lambda) + \int_{|\lambda| < n\epsilon_n} \lambda^{2n} dF(\lambda)$$

$$\le (2n)! \int_{|\lambda| \ge \epsilon_n n} \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} dF(\lambda) + (n\epsilon_n)^{2n} [F(\infty) - F(-\infty)]$$

$$= (2n)! \int_{|\lambda| \ge n\epsilon_n} e^{\lambda} dF(\lambda) + n^{2n} o(1)$$

$$= (2n)! O(1) + o(n^{2n}) = o(n^{2n})$$

because of (5.1) and the STIRLING formula.

As a particular case of Theorem 5.2 a band limited weakly stationary process has an almost sure order not greater than one.

6. Relationship between ρ_n and $\rho_a(\omega)$

For a mean entire weakly stationary process, we shall give the following result.

Theorem 6.1.

$$P(\rho_a(\omega) \le \rho_m) = 1, \qquad \qquad)6.1)$$

where ρ_m and $\rho_a(\omega)$ are the mean order and the almost sure order respectively of a given mean entire weakly stationary process.

Proof. Suppose without loss of generality that $\rho_m < \infty$. From Theorem 5.1. it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{2n/(\rho_{m+1})} \frac{1}{(n!)^2} \int_{-\infty}^{\infty} \lambda^{2n} dF(\lambda) < \infty$$
(6.2)

for any $\varepsilon > 0$, where $F(\lambda)$ is the spectral distribution function as before.

Take up any $\varepsilon_1 > 0$ such that $0 < \varepsilon_1 < \varepsilon$. Since the order of an entire (nonrandom)

power series $\sum_{0}^{\infty} a_n z^n$ is given by $\limsup_{n \to \infty} n \log n / (\log 1 / |a_n|)$, ρ_m is given by

$$\rho_m = \limsup_{n \to \infty} \frac{n \log n}{\log |n! / \rho^{(n)}(0)|}.$$
(6.3)

Hence there is an n_0 such that for $n \ge n_0$

$$\frac{n!}{\rho^{(n)}(0)} > n^{n/(\rho_{m+\epsilon_1})},$$

that is

$$n! n^{-n/(\rho+\epsilon_1)} > \left| \int_{-\infty}^{\infty} \lambda^n dF(\lambda) \right|.$$

Therefore

$$\sum_{n=n_0}^{\infty} n^{2n/(\rho_m+\epsilon)} \frac{1}{(n!)^2} \int_{-\infty}^{\infty} \lambda^{2n} dF(\lambda)$$

$$\leq \sum_{n=n_0}^{\infty} n^{2n/(\rho_m+\epsilon)} \frac{1}{(n!^2)} (2n)! (2n)^{-2n/(\rho_m+\epsilon_1)}$$

$$\leq C \sum_{n=n_0}^{\infty} n^{2n[(\rho_m+\epsilon)^{-1}-(\rho_m+\epsilon_1)^{-1}]} 2^{2n[1-(\rho_m+\epsilon_1)^{-1}]} n^{-1/2}$$

for some constant C and the last series obviously convergent. This proves the theorem.

Finally we give

Theorem 6.2. For a nondegenerate mean entire Gaussian stationary process, we have

$$P(\rho_a(\omega) = \rho_m) = 1. \tag{6.4}$$

Proof. Suppose first $\rho_m < \infty$. From (5.2) and the following lines, and (5.3), we we have, for every $\varepsilon > 0$,

$$\limsup_{r \to \infty} \max \left[\rho(2ir), \rho(-2ir) \right] \exp\left(-r^{\rho_{m^{-\epsilon}}}\right)$$
$$= \limsup_{r \to \infty} \max\left(\int_{-\infty}^{\infty} e^{2r\lambda} dF(\lambda), \int_{-\infty}^{\infty} e^{-2r\lambda} dF(\lambda) \right) \exp\left(-r^{\rho_{m^{-\epsilon}}}\right)$$
(6.5)
$$= \infty.$$
 (6.6)

Write the maximum in (6.5) by $\int_{-\infty}^{\infty} \exp(\theta(r)2r\lambda)dF(\lambda)$, where $\theta(r)$ assumes the values 1 or -1 depending on r.

It is sufficient to show that for any $\varepsilon > 0$ and a certain sequence $\{r_k\}$ tending to infinity,

$$\lim_{k \to \infty} |X((\theta(\mathbf{r}_k)\mathbf{r}_k, \omega)| \exp(-\mathbf{r}_k^{\theta_{m-\epsilon}}) = \infty,$$
(6.7)

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with probability one, where $X(t, \omega)$ is a given stationary Gaussian process.

Let $\{r_k\}$ be a sequence such that

$$\exp\left[(-2r_k)^{\rho_m-\epsilon}\right]\int_{-\infty}^{\infty}\exp\left(\theta(r_k)2r_k\lambda\right)dF(\lambda)\to\infty,\tag{6.8}$$

 $(r_k \rightarrow \infty)$. The existence of such an $\{r_k\}$ follows from (6.6).

In order to show (6.7) it is sufficient to prove

$$P(\bigcap_{\substack{G>0}}\{|\int_{-\infty}^{\infty}\exp\left(\theta(r_k)2r_k\lambda\right)\xi(d\lambda,\omega)| > G\exp\left(r_k^{\theta_m-\epsilon}\right), \text{ i.o.}\}) = 1.$$
(6.9)

The left hand side is equal to

$$\lim_{G\to\infty n\to\infty} P(\bigcup_{k\geq n} \{ | \int_{-\infty}^{\infty} \exp\left(\theta(r_k)r_k\lambda\right)\xi(d\lambda,\omega) > G\exp\left(r_k^{\theta_{m-\epsilon}}\right) \})$$

$$\geq \lim_{G\to\infty n\to\infty} P(| \int_{-\infty}^{\infty} \exp\left(\theta(r_n)r_n\lambda\right)\xi(d\lambda,\omega) | > G\exp\left(r_n^{\theta_{m-\epsilon}}\right)).$$

Since the integral in this expression depends on Gaussian distribution, the last one is

$$\lim_{G\to\infty n\to\infty}\lim_{-\sqrt{2\pi\sigma_n}}\int_{|y|>G\exp{(r_n^{\rho_m-\varepsilon})}}e^{-y^2/(2\sigma_n^2)}dy,$$

where

$$\sigma_n^2 = E \left| \int_{-\infty}^{\infty} e^{\theta(r_n)r_n\lambda} \xi(d\lambda, \omega) \right|^2 = \int_{-\infty}^{\infty} e^{2\theta(r_n)r_n\lambda} dF(\lambda)$$

which is from (6.8)

 $\geq G_1 \exp{(2r_n)^{\rho_{m^{-\varepsilon}}}},$

 G_1 being an arbitrarily large number when *n* is large. Then (6.10) is

$$\lim_{G \to n} \lim_{n \to \infty} \int_{|u| > G \exp(r_n^{\rho} m^{-\epsilon})\sigma_n^{-1}} e^{-u^2/2} du$$
$$\geq \lim_{G \to \infty} \lim_{n \to \infty} \frac{1}{\sqrt{2\pi}} \int_{|u| > J_n} e^{-u^2/2} du,$$

where $J_n = G/G_1^{1/2} \exp(r_n^{\rho_m - \epsilon}) \exp(-(2r_n)^{\rho_m - \epsilon})$. Since $J_n \to 0$ as $n \to \infty$, the last limit is 1. This proves (6.9). Hence the theorem is proved. The case $\rho_m = \infty$ is also shown in a similar way if $\rho_m - \epsilon$ is replaced by an arbitrarily large number.

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