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ANALYTIC WEAKLY STATIONARY PROCESSES

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ABSTRACT

Some basic results on weakly stationary processes which are mean analytic at the origin, on a half-plane or the entire complex plane, are given.

1. Introduction

Let $X(t, \omega)$, $-\infty<t<\infty$ be a stochastic process of the second order, that is, a process with $E|X(t, \omega)|^2<\infty$, $-\infty<t<\infty$. If there is a stochastic process $Y(t, \omega)$ of the second order such that $E|[X(t+h, \omega)-X(t, \omega)]/h-Y(t, \omega)|^2\to 0$ as $h\to 0$, for a t, then $X(t, \omega)$ is called mean differentiable at t and $Y(t, \omega)$ the mean derivative. We denote it simply by $X'(t, \omega)$. The mean derivative $X^{(n)}(t, \omega)$ of the n-th order of $X(t, \omega)$ is defined in an obvious way.

Suppose that $X(t, \omega)$ has the mean derivatives of all orders at $t=t_0$. If the Taylor series

$$\sum_{n=0}^{\infty} X^{(n)}(t_0, \omega)\frac{(t-t_0)^n}{n!} \tag{1.1}$$

converges in quadratic mean to $X(t, \omega)$ in $|t-t_0|<\delta$ for some $\delta>0$, then $X(t, \omega)$ is called mean analytic at $t=t_0$. (BALYAEV [4]). This definition is equivalent to the following one. $X(t, \omega)$ is mean analytic at $t=t_0$, if there is a random function $X_1(z, \omega)$ defined in some neighbourhood D of t_0 in the complex plane, which is mean analytic in D, that is $[X_1(z+h, \omega)-X_1(z, \omega)]/h$ converges in quadratic mean as $h\to 0$ to a random function $X_1'(z, \omega)$ for each $z\in D$ and is such that $X_1(t, \omega)=X(t, \omega)$ with probability one for each real t contained in D.

In this definition if the mean analyticity in D of $X_1(z, \omega)$ is replaced by the almost sure analyticity in D, namely if $X_1(z, \omega)$ is analytic in D as a function of a complex variable z with probability one, then $X(t, \omega)$ is called almost surely analytic at $t=t_0$.

We are, in this paper, concerned with weakly stationary processes and give some rather basic properties of them when they are analytic in the above sense.

2. Some known results

LOÈVE [6] studied the mean analyticity of a stochastic process of the second order and gave basic results one of which takes the following form in the case of a weakly stationary process.

I. *In order that a weakly stationary process is everywhere mean analytic if and only if its covariance function is analytic at the origin.*

LOÈVE [6] also gave a condition for almost sure analyticity for a second order process. Later BALYAEV [4] improved the result. BALYAEV's theorem turns out to be the following theorem for a weakly stationary process.

II. *If a weakly stationary process has the covariance function analytic at the origin, then the process is almost surely analytic at the origin.*

Hence in view of the above result, this yields to: If a weakly stationary process is mean analytic at the origin, then it is almost surely analytic at the origin.

As a matter of fact, this is rather a special case of a more general theorem which states:

III. *If a random function $f(z, \omega)$ with finite second moment for each $z \in D$, D being a domain in the complex plane, is mean analytic in D, then it is almost surely analytic in D, that is there is a random function $f_1(z, \omega)$ which is analytic in D as a function of z with probability one and is such that $f_1(z, \omega)=f(z, \omega)$ with probability one for each $z \in D$.*

This is included in BALYAEV [4] in the local form and was shown by ARNOLD [2].

3. Strip of mean analyticity

We assume throughout in this paper that $X(t, \omega)$, $-\infty<t<\infty$, is a weakly stationary process with $EX(t, \omega)=0$, $-\infty<t<\infty$ and the convariance function

$$\rho(u)=EX(t+u, \omega)\overline{X(t, \omega)}$$

$$=\int_{-\infty}^{\infty} e^{iu\lambda} dF(\lambda). \tag{3.1}$$

$X(t, \omega)$ admits the representation

$$X(t, \omega)=\int_{-\infty}^{\infty} e^{it\lambda} \xi(d\lambda, \omega), \tag{3.2}$$

where $\xi(S, \omega)$, S being a Borel set, is a random measure with

$$E\xi(S, \omega)=0. \tag{3.2}$$

$$E|\xi(S, \omega)|^2=F(S), \tag{3.4}$$

$F(S)$ being a bounded measure generated by $F(\lambda)$.

We first of all mention the following theorem which is well known in the theory of analytic characteristic functions (LUKACS [7], KAWATA [5]).

Theorem 3.1. *If the covariance function* (3.1) *of a weakly stationaly process is analytic at* $u=0$, *then there is a strip*

$$-\alpha<\operatorname{Im} z<\beta, \qquad \alpha, \beta>0 \tag{3.5}$$

such that there exists a function $\rho(z)$ *which is analytic in the strip* (3.5), *is identical with* $\rho(u)$ *on the real axis and has the representation*

$$\rho(z)=\int_{-\infty}^{\infty} e^{iz\lambda}dF(\lambda) \tag{3.6}$$

there, where

$$\int_{-\infty}^{\infty} e^{-y\lambda}dF(\lambda)<\infty, \qquad \text{for} \qquad -\alpha<y<\beta. \tag{3.7}$$

$-i\alpha$ *and* $i\beta$ *are singularities of* $\rho(z)$.

Either α or β, or the both may be infinite. The strip (3.5) is called the strip of analyticity of $\rho(u)$. Corresponding to this theorem, we have

Theorem 3.2. *If a weakly stationary process* $X(t, \omega)$ *with representation* (3.2) *is mean analytic at* $t=0$, *then there are a strip*

$$-\alpha_1<\operatorname{Im} z<\beta_1, \qquad \alpha_1, \beta_1>0 \tag{3.8}$$

and a random function $X_1(z, \omega)$ *which is mean analytic in* (3.8), *has the representation*

$$X_1(z, \omega)=\int_{-\infty}^{\infty} e^{iz\lambda}\xi(d\lambda, \omega) \tag{3.9}$$

and is identical with $X(t, \omega)$ *with probability one for each real* $z=t$, *and* $X_1(z, \omega)$ *is not mean analytic at* $z=-i\alpha_1$, $z=i\beta_1$.

The proof is carried out in a way similar to that of Theorem 3.1 in the following manner.

Consider

$$Y_1(z, \omega)=\int_{0}^{\infty} e^{iz\lambda}\xi(d\lambda, \omega) \tag{3.10}$$

and suppose that the integral on the right hand side exists at $z=z_1=t_1+iy_1$, or $e^{-y_1\lambda}\in L^2(dF)$. Then obviously $e^{-y\lambda}\in L^2(dF)$ for $y>y_1$ and hence (3.10) exists for all z with $\operatorname{Im} z>y_1$. Therefore there should exist $-\alpha_1$ such that the integral in (3.10) exists for all z with $\operatorname{Im} z>-\alpha_1$, and does not exist for any z with $\operatorname{Im} z<-\alpha_1$. α_1 should be nonnegative since $F(\lambda)$ is of bounded variation over $(-\infty, \infty)$.

It is easy to show that $Y_1(z, \omega)$ is differentiable in mean arbitrary number of times and

$$Y_1^{(n)}(z, \omega)=i^n\int_0^\infty e^{iz\lambda}\lambda^n\xi(d\lambda, \omega) \tag{3.11}$$

for $\mathrm{Im}\, z>-\alpha_1$. Moreover $Y_1(z, \omega)$ is mean analytic in $\mathrm{Im}\, z>-\alpha_1$, because as we easily verify,

$$E\left|\sum_{n=0}^N \frac{(z-z_1)^n}{n!} Y_1^{(n)}(z, \omega)-Y_1(z, \omega)\right|^2$$

$$=\int_0^\infty\left|\sum_{n=0}^N \frac{(z-z_1)^n}{n!}(i\lambda)^n-e^{i\lambda(z-z_1)}\right|^2 e^{-2y_1\lambda}dF(\lambda)$$

for any z, z_1 such that $y_1=\mathrm{Im}\, z_1>-\alpha_1$, $|z-z_1|<y_1+\alpha_1$, and the right hand side converges to zero as $N\to\infty$.

It is obvious that $Y_1(z, \omega)$ is not mean analytic at $z=-i\alpha_1$.

In a similar way, we see that, for

$$Y_2(z, \omega)=\int_{-\infty}^0 e^{iz\lambda}\xi(d\lambda, \omega), \tag{3.12}$$

there exists a $\beta_1\geqq 0$ such that $Y_2(z, \omega)$ is mean analytic in $\mathrm{Im}\, z<\beta_1$ and is not at $z=i\beta_1$.

Now $X(t, \omega)$ is supposed to be mean analytic at $t=0$ and then it is easy to see that the both $Y_1(z, \omega)$ and $Y_2(z, \omega)$ are mean analytic at $z=0$. Hence $\alpha_1>0$, $\beta_1>0$ and $X_1(z, \omega)\equiv Y_1(z, \omega)+Y_2(z, \omega)$ is mean analytic in $-\alpha_1>\mathrm{Im}\, z>\beta_1$. Obviously $X_1(t, \omega)=X(t, \omega)$ holds with probability one for each real t. This concludes the proof.

The strip (3.8) is called the *strip of mean analitycity* of $X(t, \omega)$.

In view of the equivalence of existence of the integrals $\int_{-\infty}^\infty e^{-2y\lambda}dF(\lambda)$ and $\int_{-\infty}^\infty e^{i\lambda z}\xi(d\lambda, \omega)$ where $\mathrm{Im}\, z=y$, we may conclude

$$\alpha_1=\frac{1}{2}\alpha, \qquad \beta_1=\frac{1}{2}\beta, \tag{3.13}$$

where α, β are those in Theorem 3.1.

Because of **III** in **1** we have

Corollary 3.1. *If a weakly stationary process $X(t, \omega)$ is mean analytic at the origin, then there is a random function $X_2(z, \omega)$ which is analytic (as a function of z) with probability one in the strip of mean analyticity and is identical with $X(t, \omega)$ on $-\infty<t<\infty$ with probability one for each t.*

The domain of almost sure analyticity of a weakly stationary process is not smaller and actually may be larger than the strip of mean analyticity, as the following example shows.

Let the probability space be [0, 1] in which Lebesgue measurable sets are considered and the probability is taken to be the Lebesgue measure. Define for $\omega\in[0, 1]$, $n=1, 2. \ldots$

$$\begin{aligned} \xi_n(\omega) &= -2^{n/2}e^{-ny_0 2}, && \text{for } 2^{-n}-2^{-(n+1)}<\omega<2^{-n}, \\ &= 2^{n/2}e^{-ny_0/}, && \text{for } 2^{-n}<\omega<2^{-n}+2^{-(n+1)}, \\ &= 0, && \text{elsewhere,} \end{aligned}$$

y_0 being any fixed positive number. Let

$$X(z, \omega)=\sum_{n=1}^{\infty} \xi_n(\omega)e^{inz} \tag{3.14}$$

which is weakly startionary for $z=t$. $\{\xi_n(\omega)\}$ is a sequence of orthogonal random variables. Since

$$\rho(u)=\sum_{n=1}^{\infty} e^{-ny_0}e^{inu}$$

$\rho(z)$ has the strip of analyticity, $\operatorname{Im} z>-y_0$ and $X(t, \omega)$ has the strip of mean analyticity $\operatorname{Im} z>-y_0/2$. On the other hand for each $0<\omega<1$, $\xi_n(\omega)=0$ for sufficiently large n and hence $X(z, \omega)$ is a trigonometric polynominal for each ω. Hence it is analytic for each $\omega\in(0,1)$, in the whole complex plane.

4. Boundary of an analytic random function

Let $X(t, \omega)$ be a weakly stationary process as in **3.** If there is a random function $\varphi(z, \omega)$ which is mean analytic in the half-plane $\operatorname{Im} z>0$ such that

$$E|\varphi(z, \omega)-X(t, \omega)|^2 \to 0 \qquad \text{as } y \to 0+$$

for every $-\infty<t<\infty$, where $z=t+iy$, then $X(t, \omega)$ is called the *boundary* of $\varphi(z, \omega)$, or the *boundary process* of a mean analytic random function.

Suppose that the spectral distribution function $F(\lambda)$ of $X(t, \omega)$ is constant for $\lambda<0$. Then as we saw in the proof of Theorem 3.1, $X(t, \omega)$ is the boundary of $X_1(z, \omega)=\int_0^{\infty} e^{iz\lambda}\xi(d\lambda, \omega)$. In fact $E|X_1(z, \omega)-X(t, \omega)|^2=\int_0^{\infty}\left|e^{-y\lambda}-1\right|dF(\lambda)$ converges to zero as $y\to 0+$. We write $X_1(z, \omega)$ for $\varphi(z, \omega)$ without confusion.

Since for every x and every $y>0$

$$\int_{-\infty}^{\infty}\left[\int_0^{\infty}\left|\frac{y}{(t-x)^2+y^2}e^{it\lambda}\right|^2 dF(\lambda)\right]^{1/2}dt<\infty,$$

by a theorem analogous to the FUBINI-TONELLI Theorem (See ROZANOV [8] p. 12), we have

$$\frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty}\frac{y}{(t-u)^2+y^2}\,e^{iu\lambda}du\right]\xi(d\lambda, \omega)$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}du\frac{y}{(t-u)^2+y^2}\int_0^{\infty}e^{iu\lambda}\xi(d\lambda, \omega)$$

for $y>0$, $-\infty<t<\infty$, namely

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(t-u)^2+y^2}X(u,\omega)du=\int_0^{\infty}e^{it\lambda-y\lambda}\xi(d\lambda,\omega)$$

$$=\int_0^{\infty}e^{iz\lambda}\xi(d\lambda,\omega), \qquad z=t+iy. \tag{4.2}$$

Hence we have shown

Theorem 4.1. *If the spectral distribution function $F(\lambda)$ of a weakly stationary process $X(t,\omega)$ is constant for $\lambda<0$, then $X(t,\omega)$ is the boundary of the random function*

$$X(z,\omega)=\int_0^{\infty}e^{iz\lambda}\xi(d\lambda,\omega) \tag{4.3}$$

which is mean analytic in $\operatorname{Im} z>0$ *and is represented by the Poisson integral*

$$X(z,\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(t-u)^2+y^2}X(u,\omega)du. \tag{4.4}$$

The converse of this theorem is true in the following sense.

Theorem 4.2. *If a weakly stationary process $X(t,\omega)$ is the boundary of a random function $X(z,\omega)$ which is mean analytic in the upper half-plane and is represented by the Poisson integral* (4.4) *of $X(t,\omega)$, then $F(\lambda)$ is constant for $\lambda<0$.*

We also have the following seemingly more general theorem.

Theorem 4.3. *If a weakly stationary process $X(t,\omega)$ is the boundary of a random function $X(z,\omega)$ which is mean analytic in the upper half-plane with $E|X(z,\omega)|^2$ bounded in* $\operatorname{Im} z>0$, *then $F(\lambda)$ is constant for $\lambda<0$.*

We note that if (4.4) is true, then $E|X(z,\omega)|^2$ is bounded in $\operatorname{Im} z>0$, because the variance of the Poisson integral in (4.4) is easily seen to be $\int_0^{\infty}e^{-2y\lambda}dF(\lambda)$.

We shall prove Theorem 4.3. Consider

$$EX(z,\omega)\overline{X(0,\omega)}=\rho(z)$$

which is seen to be analytic in $\operatorname{Im} z>0$ from the analyticity of $X(z,\omega)$. Since $X(z,\omega)$ converges to $X(t,\omega)$ in quadratic mean as $y\to 0+$ $(z=t+iy)$, $\rho(z)$ converges to $\rho(t)$ as $y\to 0+$. Hence $g(z)=\rho(z)-\int_0^{\infty}e^{iz\lambda}dF(\lambda)$ is analytic in $\operatorname{Im} z>0$ and converges to $\int_{-\infty}^{0}e^{it\lambda}dF(\lambda)$ as $y\to 0$. On the other hand, $g_1(z)=\int_{-\infty}^{0}e^{iz\lambda}dF(\lambda)$ is analytic in $\operatorname{Im} z<0$ and converges to $\int_{-\infty}^{0}e^{it\lambda}dF(\lambda)$ as $y\to 0-$. Therefore $g_1(z)$ is analytically continued to the upper half-plane and consequently analytic in the whole plane.

Now from the assumption that $E|X(z,\omega)|^2$ is bounded in $\operatorname{Im} z>0$, $\rho(z)$ is bounded in $\operatorname{Im} z>0$ and then $g(z)$ is bounded in $\operatorname{Im} z>0$. Hence $g_1(z)$ is, in the upper half-plane, bounded. Note that $g_1(z)$ is bounded in $\operatorname{Im} z\leqq y$ from the form of the integral. Therefore $g_1(z)$ should be a constant for all z, which implies that $\int_{-\infty}^{0} e^{it\lambda}dF(\lambda)$ is constant and this in turn implies that $F(\lambda)$ is constant for $\lambda<0$.

5. Order of an entire weakly stationary process

Suppose that a weakly stationary process $X(t, \omega)$ is mean analytic in the whole plane, that is, there is a random function $X_1(z, \omega)$ which is mean analytic in the whole plane such that $X(t, \omega)$, $X_1(t, w)$ with probability one for each real t. We write $X_1(z, \omega)$ as before $X(z, \omega)$. Then we should have

$$\int_{-\infty}^{\infty} e^{-y\lambda}dF(\lambda)<\infty \qquad \text{for} \qquad -\infty<y<\infty, \tag{5.1}$$

$F(\lambda)$ being the spectral distribution function of $X(t, \omega)$. Let us call $\rho(z)=\int_{-\infty}^{\infty} e^{iz\lambda}dF(\lambda)$, $z=t+iy$, the *entire covariance function* of $X(t, \omega)$.

Let us put, for $r>0$

$$M_m(r)=\max_{|z|\leqq r} E\,|X(z, \omega)|^2 \tag{5.2}$$

which is equal to

$$\begin{aligned}&\max_{|z|\leqq r}\int_{-\infty}^{\infty} e^{-2y\lambda}dF(\lambda)\\ &=\max\,[\rho(2ir), \rho(-2ir)]\\ &=\max_{|z|=2r}|\rho(z)|.\end{aligned}$$

We define the *mean order* ρ_m of $X(z, \omega)$ by

$$\rho_m=\limsup_{r\to\infty}\frac{\log\log M_m(r)}{\log r}. \tag{5.3}$$

ρ_m is no more than the order of $\rho(z)$.

From the known theorem on analytic characteristic functions we have that if *the spectral distribution* $F(\lambda)$ is nondegenerate,

$$\rho_m\geqq 1. \tag{5.4}$$

From **III** in **1**, $X(z, \omega)$ is almost surely entire. We may then define

$$M_a(r, \omega)=\max_{|z|=r}|X(z, \omega)| \tag{5.5}$$

and

$$\rho_a(\omega)=\limsup_{r\to\infty}\frac{\log\log M_a(r, \omega)}{\log r} \tag{5.6}$$

with probability one. We call $\rho_a(\omega)$ the *almost sure order* of $X(z, \omega)$.

ARNOLD [3] has extensively studied on the order $\rho(\omega)$ of an entire random power series $f(z,\omega)=\sum_{n=0}^{\infty}a_n(\omega)z^n$ (that is, an entire power series with probability one). The order $\rho(\omega)$ is given by (5.6) with $M_a(r,\omega)$ in which $X(z,\omega)$ is replaced by $f(z,\omega)$ and is also given in the form

$$\rho(\omega)=\limsup_{n\to\infty}\frac{n\log n}{\log(1/|a_n(\omega)|)}. \tag{5.7}$$

One of ARNOLD's results [3] (Satz 1) is that, *for every* $x\geqq 0$

$$P(\rho(\omega)\leqq x)=1 \tag{5.8}$$

is equivalent to

$$P(\limsup|a_n(\omega)|\geqq n^{-n/(x+\varepsilon)})=0 \tag{5.9}$$

for all $\varepsilon>0$.

From this , Borel-Cantelli lemma gives us that *if*

$$\sum_{n=1}^{\infty}P([a_n(\omega)|\geqq n^{-n/(x+\varepsilon)})<\infty \tag{5.10}$$

for all $\varepsilon>0$, *then* (5.8) *is true.*

Applying this result to a weakly stationary process, we have the following theorem.

Theorem 5.1. *If the spectral distribution function $F(\lambda)$ of a mean entire weakly stationary process satisfies that, for* $x\geqq 0$

$$\sum_{n=1}^{\infty}n^{2n/(x+\varepsilon)}\frac{1}{(n!)^2}\int_{-\infty}^{\infty}\lambda^{2n}dF(\lambda)<\infty \tag{5.11}$$

for all $\varepsilon>0$, *then*

$$P(\rho_a(\omega)\leqq x)=1. \tag{5.12}$$

In fact, if $\sum a_n(\omega)z^n$ is the power series expansion of $X(z,\omega)$, then $a_n(\omega)=X^{(n)}(0,\omega)/n!=(i^n/n!)\int_{-\infty}^{\infty}\lambda^n\xi(d\lambda,\omega)$ and by Chebyshev ineqality,

$$P(|a_n(\omega)|\geqq n^{-n/(x+\varepsilon)})\leqq n^{2n/(x+\varepsilon)}E\,|a_n(\omega)|^2=n^{2n/(x+\varepsilon)}\frac{1}{(n!)^2}\int_{-\infty}^{\infty}\lambda^{2n}dF(\lambda):$$

Hence from (5.11), (5.10) follows.

From Theorem 5.1 we have, taking 1 for x, the following theorem.

Theorem 5.2. *If the spectral distribution $F(\lambda)$ of a mean entire weakly stationary process satisfies that*

$$\int_{-\infty}^{\infty}\lambda^{2n}dF(\lambda)\leqq Cn^{An} \tag{5.13}$$

for some sequence $\{A_n\}$ with $A_n=O(n)$ *and some constant* C, *then*

$$P(\rho_a(\omega)\leqq 1)=1. \tag{5.14}$$

Actually if (5.13) holds, then the series in (5.11) is, because of STIRLING formula, not larger than

$$C_1 \sum_{n=1}^{\infty} n^{2n/(1+\varepsilon)-(2n+1)} e^{2n} n^{\varepsilon_n n}$$

for some $\varepsilon_n \to 0$, C_1 being a constant and this is

$$\leqq C \sum_{n=1}^{\infty} n^{-1} e^{2n} n^{\frac{-2\varepsilon}{\varepsilon+1} n + n\varepsilon_n}$$

which is convergent for every $\varepsilon > 0$ and this proves the theorem.

We remark in connection with (5.13), that for every entire weakly stationary process,

$$\int_{-\infty}^{\infty} \lambda^{2n} dF(\lambda) = o(n^{2n}) \tag{5.15}$$

as $n \to \infty$. In fact, we have, choosing any $\varepsilon_n \to 0$, the left hand side of (5.15) is

$$\int_{|\lambda| \geqq \varepsilon_n n} \lambda^{2n} dF(\lambda) + \int_{|\lambda| < n\varepsilon_n} \lambda^{2n} dF(\lambda)$$

$$\leqq (2n)! \int_{|\lambda| \geqq \varepsilon_n n} \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} dF(\lambda) + (n\varepsilon_n)^{2n} [F(\infty) - F(-\infty)]$$

$$= (2n)! \int_{|\lambda| \geqq n\varepsilon_n} e^{\lambda} dF(\lambda) + n^{2n} o(1)$$

$$= (2n)!\, O(1) + o(n^{2n}) = o(n^{2n})$$

because of (5.1) and the STIRLING formula.

As a particular case of Theorem 5.2 a band limited weakly stationary process has an almost sure order not greater than one.

6. Relationship between ρ_n and $\rho_a(\omega)$

For a mean entire weakly stationary process, we shall give the following result.

Theorem 6.1.

$$P(\rho_a(\omega) \leqq \rho_m) = 1, \qquad)6.1)$$

where ρ_m and $\rho_a(\omega)$ are the mean order and the almost sure order respectively of a given mean entire weakly stationary process.

Proof. Suppose without loss of generality that $\rho_m < \infty$. From Theorem 5.1. it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{2n/(\rho_m+\varepsilon)} \frac{1}{(n!)^2} \int_{-\infty}^{\infty} \lambda^{2n} dF(\lambda) < \infty \tag{6.2}$$

for any $\varepsilon>0$, where $F(\lambda)$ is the spectral distribution function as before.

Take up any $\varepsilon_1>0$ such that $0<\varepsilon_1<\varepsilon$. Since the order of an entire (nonrandom) power series $\sum_0^\infty a_n z^n$ is given by $\limsup_{n\to\infty} n\log n/(\log 1/|a_n|)$, ρ_m is given by

$$\rho_m=\limsup_{n\to\infty}\frac{n\log n}{\log|n!/\rho^{(n)}(0)|}. \tag{6.3}$$

Hence there is an n_0 such that for $n\geqq n_0$

$$\frac{n!}{\rho^{(n)}(0)}>n^{n/(\rho_m+\varepsilon_1)},$$

that is

$$n!\,n^{-n/(\rho+\varepsilon_1)}>\left|\int_{-\infty}^{\infty}\lambda^n dF(\lambda)\right|.$$

Therefore

$$\sum_{n=n_0}^{\infty} n^{2n/(\rho_m+\varepsilon)}\frac{1}{(n!)^2}\int_{-\infty}^{\infty}\lambda^{2n}dF(\lambda)$$

$$\leqq\sum_{n=n_0}^{\infty} n^{2n/(\rho_m+\varepsilon)}\frac{1}{(n!^2)}(2n)!\,(2n)^{-2n/(\rho_m+\varepsilon_1)}$$

$$\leqq C\sum_{n=n_0}^{\infty} n^{2n[(\rho_m+\varepsilon)^{-1}-(\rho_m+\varepsilon_1)^{-1}]}2^{2n[1-(\rho_m+\varepsilon_1)^{-1}]}n^{-1/2},$$

for some constant C and the last series obviously convergent. This proves the theorem.

Finally we give

Theorem 6.2. *For a nondegenerate mean entire Gaussian stationary process, we have*

$$P(\rho_a(\omega)=\rho_m)=1. \tag{6.4}$$

Proof. Suppose first $\rho_m<\infty$. From (5.2) and the following lines, and (5.3), we we have, for every $\varepsilon>0$,

$$\limsup_{r\to\infty}\max[\rho(2ir),\rho(-2ir)]\exp(-r^{\rho_m-\varepsilon})$$

$$=\limsup_{r\to\infty}\max\left(\int_{-\infty}^{\infty}e^{2r\lambda}dF(\lambda),\int_{-\infty}^{\infty}e^{-2r\lambda}dF(\lambda)\right)\exp(-r^{\rho_m-\varepsilon}) \tag{6.5}$$

$$=\infty. \tag{6.6}$$

Write the maximum in (6.5) by $\int_{-\infty}^{\infty}\exp(\theta(r)2r\lambda)dF(\lambda)$, where $\theta(r)$ assumes the values 1 or -1 depending on r.

It is sufficient to show that for any $\varepsilon>0$ and a certain sequence $\{r_k\}$ tending to infinity,

$$\lim_{k\to\infty}|X((\theta(r_k)r_k,\omega)|\exp(-r_k^{\rho_m-\varepsilon})=\infty, \tag{6.7}$$

with probability one, where $X(t, \omega)$ is a given stationary Gaussian process.

Let $\{r_k\}$ be a sequence such that

$$\exp[(-2r_k)^{\rho_m-\varepsilon}]\int_{-\infty}^{\infty}\exp(\theta(r_k)2r_k\lambda)dF(\lambda)\to\infty, \tag{6.8}$$

$(r_k\to\infty)$. The existence of such an $\{r_k\}$ follows from (6.6).

In order to show (6.7) it is sufficient to prove

$$P(\bigcap_{G>0}\{|\int_{-\infty}^{\infty}\exp(\theta(r_k)2r_k\lambda)\xi(d\lambda, \omega)|>G\exp(r_k^{\rho_m-\varepsilon}), \text{ i.o.}\})=1. \tag{6.9}$$

The left hand side is equal to

$$\lim_{G\to\infty}\lim_{n\to\infty}P(\bigcup_{k\geqq n}\{|\int_{-\infty}^{\infty}\exp(\theta(r_k)r_k\lambda)\xi(d\lambda, \omega)>G\exp(r_k^{\rho_m-\varepsilon})\})$$

$$\geqq\lim_{G\to\infty}\lim_{n\to\infty}P(|\int_{-\infty}^{\infty}\exp(\theta(r_n)r_n\lambda)\xi(d\lambda, \omega)|>G\exp(r_n^{\rho_m-\varepsilon})).$$

Since the integral in this expression depends on Gaussian distribution, the last one is

$$\lim_{G\to\infty}\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}\sigma_n}\int_{|y|>G\exp(r_n^{\rho_m-\varepsilon})}e^{-y^2/(2\sigma_n^2)}dy,$$

where

$$\sigma_n^2=E\left|\int_{-\infty}^{\infty}e^{\theta(r_n)r_n\lambda}\xi(d\lambda, \omega)\right|^2=\int_{-\infty}^{\infty}e^{2\theta(r_n)r_n\lambda}dF(\lambda)$$

which is from (6.8)

$$\geqq G_1\exp(2r_n)^{\rho_m-\varepsilon},$$

G_1 being an arbitrarily large number when n is large. Then (6.10) is

$$\lim_{G\to n}\lim_{n\to\infty}\int_{|u|>G\exp(r_n^{\rho_m-\varepsilon})\sigma_n^{-1}}e^{-u^2/2}du$$

$$\geqq\lim_{G\to\infty}\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{|u|>J_n}e^{-u^2/2}du,$$

where $J_n=G/G_1^{1/2}\exp(r_n^{\rho_m-\varepsilon})\exp(-(2r_n)^{\rho_m-\varepsilon})$. Since $J_n\to 0$ as $n\to\infty$, the last limit is 1. This proves (6.9). Hence the theorem is proved. The case $\rho_m=\infty$ is also shown in a similar way if $\rho_m-\varepsilon$ is replaced by an arbitrarily large number.

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