Title	A note on local limit theorems for densities
Sub Title	
Author	前島, 信(Maejima, Makoto)
Publisher	慶応義塾大学工学部
Publication year	1975
Jtitle	Keio engineering reports Vol.28, No.6 (1975.) ,p.57- 63
JaLC DOI	
Abstract	The author shows some conditions on local limit theorems for densities of sums of independent random variables and gives a generalization of the renewal density theorem.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00280006-0057

KEIO ENGINEERING REPORTS
VOL. 28, NO. 6, pp. 57-63, 1975

A NOTE ON LOCAL LIMIT THEOREMS FOR DENSITIES

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(Received Mar. 3, 1975)

ABSTRACT

The author shows some conditions on local limit theorems for densities of sums of independent random variables and gives a generalization of the renewal density theorem.

1. Introduction

Let $\{X_k, k=1, 2, \cdots\}$ be a sequence of independent random variables with $EX_k=0$ and finite variances $EX_k^2=\sigma_k^2$, which satisfies the Lindeberg condition. Set $S_n=\sum_{k=1}^n X_k$ and $s_n^2=\sum_{k=1}^n \sigma_k^2$. When the density of S_n/s_n exists, we write it $p_n(x)$. We shall consider the conditions under which the local limit theorem for densities holds:

$$\lim_{n\to\infty} \sup_x x^\beta |p_n(x)-\phi(x)|=0 \tag{1}$$

for $0\leqq\beta\leqq2$, where $\phi(x)=(2\pi)^{-1/2}\exp(-x^2/2)$. The local limit theorem of such a type was studied by SMITH (1953) for independent identically distributed random variables. We consider in this paper the case of independent, but not necessarily identically distributed random variables. Such a case has been studied by STATULYAVICHUS (1965) for $\beta=0$, and by SURVILA (1964) for $0\leqq\beta\leqq2$ and others. In their conditions, however, the existence of the density $q_k(x)$ of X_k is proposed for all k, and even some boundness conditions on $q_k(x)$ are supposed. For example, the result due to SURVILA is as follows: For the sequence $\{X_k\}$ of independent random variables with $EX_k=0$ and finite variances $EX_k^2=\sigma_k^2$ which have the densities $q_k(x)$, suppose that (i) the Lindeberg condition is satisfied, (ii) there exists a C_1 such that $s_n^2<C_1 n$, and (iii) there exists a C_2 such that for all $k, q_k(x)<C_2$. Then (1) holds for $0\leqq\beta\leqq2$.

In this paper we shall derive the conditions different from theirs, that is, the conditions not on the densities but on the characteristic functions. Further we shall remark on a generalization of the renewal density theorem in the renewal theory.

We mention here the recent work of Basu (1974), who gave the local limit theorem (1) for $\beta\geqq 2$, under the Lindeberg condition of order β and the smoothing subsequence condition due to Smith (1953), as will be dicussed in the last section.

2. A Theorem

Denote the characteristic function of X_k by $f_k(t)$. Our result is

Theorem 1. *Let* $\{X_k, k=1,2,\cdots\}$ *be a sequence of independent random variables with* $EX_k=0$ *and finite variances* $EX_k^2=\sigma_k^2$, *and suppose that*

(a) $\{X_k\}$ *satisfies the Lindeberg condition,*

(b) $f_\nu(t)\in L_1(-\infty,\infty)$ *for some* ν,

(c) *for some* $\varepsilon>0$, *there exists a sequence of positive constants* $\{c_k\}$ *such that for* $|t|\geqq\varepsilon$, $|f_k(t)|\leqq c_k(\varepsilon)\leqq 1$, *and*

(c1) $s_n^2<C\sum_{k=1}^n(1-c_k^2)$ *for large* n, C *being a positive constant,*

(c2) $\prod_{k=1}^n c_k=o(n^{-1}s_n^{-1})$ *as* $n\to\infty$.

Then (1) *holds for* $\beta=0$. *If we add the condition*

(b′) $f_\nu^{(\gamma)}(t)\in L_1(-\infty,\infty), \gamma=1,2,$

we have (1) *for* $0\leqq\beta\leqq 2$.

The condition (c) may be satisfied under the condition of rather simple form; for some $\varepsilon>0$, there exists a $c=c(\varepsilon)<1$ independent of k such that $|f_k(t)|<c$ for $|t|\geqq\varepsilon$, and $s_n^2<Cn$ for all n.

Proof. Denoting the characteristic function of S_n/s_n by $\theta_n(t)$, we have $\theta_n(t)=\prod_{k=1}^n f_k(t/s_n)$, and for large n with $n\geqq\nu$, $\theta_n(t)\in L_1(-\infty,\infty)$, because of the condition (b). Hence the density $p_n(x)$ of S_n/s_n exists and $p_n(x)$ is given $(2\pi)^{-1}\int_{-\infty}^{\infty}e^{-itx}\theta_n(t)dt$. Put the characteristic function of the standard normal density $\phi(x)$ by $\theta(t)$. $\theta(t)$ is then integrable over $(-\infty,\infty)$.

We first show the case of $\beta=0$. We have, for a suitably fixed $A>0$,

$$\begin{aligned}\sup_x|p_n(x)-\phi(x)|&<\int_{-\infty}^{\infty}|\theta_n(t)-\theta(t)|dt\\&=\int_{|t|<A}|\theta_n(t)-\theta(t)|dt+\int_{|t|\geqq A}|\theta(t)|dt\\&\quad+\int_{A\leqq|t|<\varepsilon s_n}|\theta_n(t)|dt+\int_{|t|\geqq\varepsilon s_n}|\theta_n(t)|dt\\&\equiv I_1+I_2+I_3+I_4,\end{aligned}$$

say, where ε is the one stated in the condition (c). By the Lindeberg condition, $I_1 \to 0$ as $n \to \infty$. Since $\theta(t)$ is integrable, for any $\eta > 0$, there exists an $A(\eta)$ such that $I_2 < \eta$ for $A \geqq A(\eta)$.

In order to estimate I_3, we use the well-known inequality relation due to CRAMÉR (1937, p. 26) which is stated as

Lemma 1. *If $f(t)$ is a characteristic function such that $|f(t)| \leqq \tau < 1$ for all $|t| \geqq R$, then for $|t| < R$, $|f(t)| \leqq 1 - (1-\tau^2)t^2/8R^2$.*

Note that this relation trivially holds for $\tau = 1$. Thus the condition (c) implies that for $|t| < \varepsilon$,

$$|f_k(t)| \leqq \exp\{\alpha_k t^2\}, \tag{2}$$

where $\alpha_k = (1-c_k)/8\varepsilon^2$. Hence

$$I_3 = \int_{A \leqq |t| < \varepsilon s_n} \left| \prod_{k=1}^{n} f_k(t/s_n) \right| dt$$

$$\leqq \int_{A \leqq |t| < \varepsilon s_n} \exp\left\{ (-1/8\varepsilon^2 s_n^2) \sum_{k=1}^{n} (1-c_k^2) t^2 \right\} dt$$

$$< \int_{A \leqq |t|} \exp\{-t^2/8\varepsilon^2 C\} dt,$$

by the condition (c1). Thus we can make I_3 as small as we desire by choosing A sufficiently large. For I_4,

$$I_4 = \int_{|t| \geqq \varepsilon s_n} |f_\nu(t/s_n)| \cdot \left| \prod_{\substack{k=1 \\ k \neq \nu}}^{n} f_k(t/s_n) \right| dt$$

$$\leqq \prod_{\substack{k=1 \\ k \neq \nu}}^{n} c_k \cdot s_n \int_{|t| \geqq \varepsilon} |f_\nu(t)| dt.$$

Since $f_\nu(t) \in L_1(-\infty, \infty)$ and $\Pi_{k=1}^{n} c_k \cdot s_n \to 0$ as $n \to \infty$, $I_4 \to 0$ as $n \to \infty$. Therefore we conclude the statement for $\beta = 0$.

Next we shall prove $\lim_{n\to\infty} \sup_x x^\beta |p_n(x) - \phi(x)| = 0$ for $0 < \beta \leqq 2$. Clearly, it is sufficient to show the case of $\beta = 2$.

Noticing that $f_k'(\cdot)$ and $f_k''(\cdot)$ exist, we have

$$\theta_n''(t) = s_n^{-2} \left[\sum_{j=1}^{n} f_j''(t/s_n) \prod_{\substack{k=1 \\ k \neq j}}^{n} f_k(t/s_n) \right.$$

$$\left. + \sum_{i=1}^{n} f_i'(t/s_n) \sum_{\substack{j=1 \\ j \neq i}}^{n} f_j'(t/s_n) \prod_{\substack{k=1 \\ k \neq j \neq i}}^{n} f_k(t/s_n) \right].$$

Under the condition (b) and (b′), $\theta_n''(t) \in L_1(-\infty, \infty)$ for large n. Since $\theta_n''(t) = -\int_{-\infty}^{\infty} e^{itx} x^2 p_n(x) dx$ and $\theta''(t) = -\int_{-\infty}^{\infty} e^{itx} x^2 \phi(x) dx$, we have

$$\sup_x x^2 |p_n(x) - \phi(x)| < \int_{-\infty}^{\infty} |\theta_n''(t) - \theta''(t)| dt \equiv I,$$

say. Hence, it suffices to prove that $I\to 0$ as $n\to\infty$. We have

$$I\leqq I_1+I_2+I_3+I_4,$$

where

$$I_1=\int_{|t|<A}|\theta_n''(t)-\theta''(t)|dt, \qquad I_2=\int_{|t|\geqq A}|\theta''(t)|dt,$$

$$I_3=\int_{A\leqq|t|<\varepsilon s_n}|\theta_n''(t)|dt, \qquad I_4=\int_{|t|\geqq\varepsilon s_n}|\theta_n''(t)|dt.$$

For I_1, we use the following lemma due to SMITH (1953).

Lemma 2. *Let $\{g_n(x)\}$ be a sequence of densities with zero mean and unit variance and suppose that the sequence tends uniformly to a density function $g(x)$ which has also zero mean and unit variance. If $\{\psi_n(t)\}$ is the sequence of characteristic functions corresponding to $\{g_n(x)\}$ and $\psi(t)$ is the characteristic function of $g(x)$, then $\psi_n''(t)$ tends to $\psi''(t)$ as $n\to\infty$ uniformly for t.*

By this lemma, under the Lindeberg condition, $\theta_n''(t)\to\theta''(t)$ as $n\to\infty$ uniformly in any finite interval of t, and $I_1\to 0$, as $n\to\infty$. Since $|\theta''(t)|=|t^2-1|\exp\{-t^2/2\}$ is integrable, for any $\eta>0$ there exists an $A(\eta)$ such that $I_2<\eta$ for $A\geqq A(\eta)$.

Noticing that $|f_k''(\cdot)|\leqq\sigma_k^2$ and $|f_k'(t/s_n)|\leqq|t|s_n^{-1}\sigma_k^2$, we have

$$|\theta_n''(t)|\leqq(1+t^2)\max_{1\leqq j,i\leqq n}\prod_{\substack{k=1\\k\neq j\neq i}}^{n} f_k(t/s_n).$$

Using (2), we have, for $|t|<\varepsilon s_n$,

$$|\theta_n''(t)|\leqq(1+t^2)\exp\left\{(-1/8\,\varepsilon^2 s_n^2)\min_{1\leqq j,i\leqq n}\sum_{\substack{k=1\\k\neq j\neq i}}^{n}(1-c_k^2)t^2\right\}$$

$$\leqq(1+t^2)\exp\{(-1/8\,\varepsilon^2)(C^{-1}-2\,s_n^{-2})t^2\}.$$

Hence, the integrand of I_3 is integrable so that I_3 can be made as small as we desire for large n and for sufficiently large A. Further we consider

$$\theta_n''(t)=s_n^{-2}\Bigg[f_\nu''(t/s_n)\prod_{\substack{k=1\\k\neq\nu}}^{n}f_k(t/s_n)+f_\nu(t/s_n)\sum_{\substack{j=1\\j\neq\nu}}^{n}f_j''(t/s_n)\prod_{\substack{k=1\\k\neq j\neq\nu}}^{n}f_k(t/s_n)$$

$$+2f_\nu'(t/s_n)\sum_{\substack{j=1\\j\neq\nu}}^{n}f_j'(t/s_n)\prod_{\substack{k=1\\k\neq j\neq\nu}}^{n}f_k(t/s_n)$$

$$+f_\nu(t/s_n)\sum_{\substack{i=1\\i\neq\nu}}^{n}f_i'(t/s_n)\sum_{\substack{j=1\\j\neq i\neq\nu}}^{n}f_j'(t/s_n)\prod_{\substack{k=1\\k\neq j\neq i\neq\nu}}^{n}f_k(t/s_n)\Bigg],$$

and note that $|f_k'(\cdot)|\leqq\sigma_k$. By the condition (c), for $|t|\geqq\varepsilon s_n$,

$$|\theta_n''(t)|\leqq s_n^{-2}|f_\nu''(t/s_n)|+s_n^{-2}|f_\nu(t/s_n)|\sum_{j=1}^{n}\sigma_j^2\cdot\max_{1\leqq j\leqq n}\prod_{\substack{k=1\\k\neq j\neq\nu}}^{n}c_k$$

$$+2\,s_n^{-2}|f_\nu'(t/s_n)|\sum_{j=1}^{n}\sigma_j\cdot\max_{1\leqq j\leqq n}\prod_{\substack{k=1\\k\neq j\neq\nu}}^{n}c_k$$

$$+s_n^{-2}|f_\nu(t/s_n)|\left(\sum_{j=1}^{n}\sigma_j\right)^2\cdot\max_{1\leqq j,i\leqq n}\prod_{\substack{k=1\\k\neq j\neq i\neq\nu}}^{n}c_k.$$

Since $(\sum_{j=1}^{n}\sigma_j)^2\leqq ns_n^2$, and the Lindeberg condition implies $s_n^2\to\infty$, we have

$$\int_{|t|\geqq\varepsilon s_n}|\theta_n''(t)|dt\leqq s_n^{-1}\int|f_\nu''(t)|dt+s_n\cdot\max_{1\leqq j\leqq n}\prod_{\substack{k=1\\k\neq j\neq\nu}}^{n}c_k\cdot\int|f_\nu(t)|dt$$

$$+2\,n^{1/2}\cdot\max_{1\leqq j\leqq n}\prod_{\substack{k=1\\k\neq j\neq\nu}}^{n}c_k\cdot\int|f_\nu'(t)|dt$$

$$+ns_n\cdot\max_{1\leqq j,i\leqq n}\prod_{\substack{k=1\\k\neq j\neq i\neq\nu}}^{n}c_k\cdot\int|f_\nu(t)|dt,$$

and $I_4\to0$ as $n\to\infty$, because of the condition (b), (b′) and (c2). The theorem thus is proved.

3. A Renewal Density Theorem

Cox and SMITH (1953) showed the renewal density theorem from the local limit theorems. We state their result as a lemma.

Lemma 3. *Let $\{X_k,\ k=1,2,\cdots\}$ be a sequence of random variables with bounded means $EX_k=\mu_k$ and finite variances. Suppose that $s_n^2\sim Cn$ for $n\to\infty$, and that the distribution of $(S_n-\sum_{k=1}^{n}\mu_k)/s_n$ has the density $h_n(x)$ such as $\lim_{x\to\infty}h_n(x)=0$. If*

$$\lim_{n\to\infty}\sup_{x}x^\beta|h_n(x)-\phi(x)|=0$$

for $\beta=0,2$, then we have the renewal density theorem

$$\lim_{x\to\infty}\sum_{n=1}^{\infty}h_n(x)=\mu^{-1},\tag{3}$$

assuming $\mu=\lim_{n\to\infty}n^{-1}\sum_{k=1}^{n}\mu_k$ to exist and to be positive.

This is a slight modification of COX and SMITH's result. We, by the local limit theorems in the previous section, have the renewal density theorem for independent random variables. That is:

Theorem 2. *Suppose that a sequence $\{X_k\}$ of inependent random variable, with bounded means $EX_k=\mu_k$ and with variances such as $s_n^2\sim Cn$, satisfies the conditions in Theorem 1 except the condition $EX_k=0$. If $\mu=\lim_{n\to\infty}n^{-1}\sum_{k=1}^{n}\mu_k$ exists and is positive, then we have the renewal density theorem* (3).

It follows from the integrability of $\theta_n(t)$, by the Fourier inversion formula and the Riemann-Lebesgue lemma, that the density $h_n(x)$ tends to zero as $x \to \infty$.

4. Concluding Remarks

We give some remarks on the condition (c), which played an essential role in the non-identically distributed case.

First, we consider the relationship between our condition and SMITH's smoothing subsequence condition which is stated as follows (SMITH (1953), BASU (1974)). The sequence of random variables $\{X_k\}$ is said to contain a "smoothing subsequence" if there exists a subsequence of characteristic functions $\{f_{k_j}(t)\}$ which satisfy $|f_{k_j}(t)| \leqq A/|t|^{\alpha}$ for $|t| \geqq R$, for some positive number R, A, α. This definition is due to SMITH. Denote by n^* the number of members of the smoothing subsequence in $X_1, X_2, \cdots, X_n$. Then, SMITH and BASU have assumed in their paper that $\{X_k\}$ contains a smoothing subsequence and $\liminf_{n\to\infty} n^*/s_n^2 > 0$.

On the other hand, our condition (c) can be replaced by the following strengthened form: Suppose that the sequence of characteristic functions $\{f_k(t)\}$ contains a subsequence $\{f_{k_j}(t)\}$ with the property that for some positive constants ε and $c<1$, $|f_{k_j}(t)| \leqq c$ for $|t| \geqq \varepsilon$, and that $\liminf_{n\to\infty} \tilde{n}/s_n^2 > 0$ and, for some $\alpha > 1$, $\liminf_{n\to\infty} \tilde{n}^{\alpha}/ns_n > 0$, $\tilde{n}$ being the number of k_j with $|f_{k_j}(t)| \leqq c < 1$ among first n characteristic functions. We call this the condition (c′). Furthermore, we agree to call the conditions only for $\tilde{n}$ in here the condition (A).

The above two kinds of conditions have similar forms. As to the condition on the number of element k_j, the condition $\liminf_{n\to\infty} \tilde{n}^{\alpha}/ns_n > 0$ is added in ours. However, since α may be taken large, it holds automatically if $\tilde{n}$ is order n^{β} (β, however small). On the other hand, as to the condition on the subsequence of characteristic functions, $|f_{k_j}(t)| \leqq A/|t|^{\alpha}$ are required in SMITH's condition, while $|f_{k_j}(t)| \leqq c < 1$ is sufficient in ours.

In the next place we consider the case where X_k has the density function $q_k(x)$.

In connection with the condition that $q_k(x) < C_2$ for *all* k in SURVILA (1964), we remark the following. We now let $\mathcal{K}$ be a subclass of all positive integers, and suppose that $\tilde{n}$ satisfies the condition (A), where $\tilde{n}$ is the number of integers in $\mathcal{K} \cap \{i;\ 1 \leqq i \leqq n\}$. If we assume the existence and the boundness of $q_k(x)$ *only* for $k \in \mathcal{K}$ while we make a further assumption that $q_k(x)$, $k \in \mathcal{K}$, is of bounded variations in every finite interval, then the condition (c′) holds. Because, by the mean value theorem, for any $A < B$, there exists a ξ_k such that

$$\left|\int_A^B \cos tx\, q_k(x)dx\right| = \left|q_k(A)\int_A^{\xi_k} \cos tx\, dx\right| + \left|q_k(B)\int_{\xi_k}^B \cos tx\, dx\right|$$
$$\leqq 4\, C_2/t \to 0$$

as $t \to \infty$ uniformly with respect to $k \in \mathcal{K}$. As is known in this simple calculation, the subsequence of the characteristic function satisfies SMITH's condition as well.

Hence, the single condition $\liminf_{n\to\infty}\tilde{n}/s_n^2>0$ is sufficient as the requirement for $\tilde{n}$.

Finally, when the density functions are not necessarily bounded, the following uniformness condition is applied. If

$$\int_{-\infty}^{\infty}|q_k(x+h)-q_k(x)|dx\to 0 \tag{4}$$

as $h\to 0$ uniformly with respect to $k\in\mathcal{K}$, then the condition (c′) is satisfied. The reason is as follows: For $k\in\mathcal{K}$,

$$f_k(t)=\int_{-\infty}^{\infty}e^{itx}q_k(x)dx=-\int_{-\infty}^{\infty}e^{itx}q_k(x+(\pi/t))dx$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}e^{itx}[q_k(x)-q_k(x+(\pi/t))]dx,$$

from which

$$|f_k(t)|\leqq\frac{1}{2}\int_{-\infty}^{\infty}|q_k(x)-q_k(x+(\pi/t))|dx\to 0$$

as $t\to\infty$ uniformly with respect to $k\in\mathcal{K}$. We note here that the condition (4) holds always for *each k*, because of the mean continuity of the integrable function.

Acknowledgements

The author wishes to express his sincere appreciation to Professor TATSUO KAWATA of Keio University for his continuing guidances and encouragements.

REFERENCES

BASU, S. K. (1974): Density versions of the univariate central limit theorem, *Ann. Prob.*, **2**, 270–276.

COX, D. R. and SMITH, W. L. (1953): A direct proof of a fundamental theorem of renewal theory, *Skand. Akt.*, **1953**, 139–150.

CRAMÉR, H. (1937): *Random Variables and Probability Distributions*, Cambridge Univ. Press, Cambridge.

SMITH, W. L. (1953): A frequency-function form of the central limit theorem, *Proc. Camb. Phil. Soc.*, **49**. 462–472.

STATULYAVICHUS, V. A. (1965): Limit theorems for densities and the asymptotic expansions for distributions of sums of independent random variables, *Theory Prob. Applications*, **10**, 582–595.

SURVILA, P. (1964): A local limit theorem for density functions (in Russian), *Litovsk. Mat. Sb.*, **4**, 535–540.