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OPTIMAL ESTIMATION ALGORITHM FOR A DISTRIBUTED-PARAMETER SYSTEM WITH MULTI-POINTWISE SOURCES

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ABSTRACT

The optimal state estimation technique is presented for a discrete-time distributed-parameter system perturbed by unknown inputs from multi-pointwise sources. In the proposed algorithm, the estimation procedure for the unknown inputs is seperated from the estimation of the state so that the computational inaccuracy and the required memory capacity can remarkably be reduced.

1. Introduction

This paper describes the method for estimating the state of distribution of pollutant concentration in the atmospheric diffusion system perturbed by unknown pollutant emission rates at multi-pointwise sources. If we define a new state vector by adjoining the unknown emission rates to the state of the concentration, we can construct the optimal filter for the defined system by the aid of the state estimation theory for distributed-parameter systems. Many studies have been done on the optimal estimation filter for systems described by partial-differential equations (FALB 1967; TZAFESTAS et al. 1968a, b; THAU 1968; KUSHNER 1970; MATSUMOTO et al. 1970; MEDITCH 1971; SAKAWA 1972; KUMAR et al. 1972; ATRE 1972), however, in the conventional approaches, as the dimension of space coordinates and the number of the emission sources becomes large, the size of the covariance matrix of the defined state vector estimate increases excessively.

In this paper, we develope the Friedland's results (Friedland 1969) to distributed-parameter systems and present a new algorithm for avoiding the complexity in computing high-dimensional matrix eqnations and for obtaining the each element of the covariance separately. As a result, by computing the emission-free state estimate first, and correcting it by the quantity obtained from the emission-rate estimates, we can get the state estimate. In the presence of uncertainty in the emission term such as unknown source location, their maximum likelihood estimates can be computed by an optimization technique. Also in this case, the proposed algorithm effectively reduces the computational time, memory capacity and inaccuracy.

2. System Description

Consider a linear distributed-parameter system of diffusion type described by

$$\partial\theta(x, t)/\partial t = \sum_{i=1}^{q} \{\alpha_i \partial^2\theta(x, t)/x_i^2 - v_i \partial\theta(x, t)/\partial x_i\} + u(x, t), \quad x \in \Omega \tag{1}$$

defined for $t \geq t_0$ $(=0)$ on the domain Ω which is an open connected subset of a q-dimensional Euclidean space with coordinate vector $x=(x_1, \cdots, x_q)' \in \Omega$. Let the scalar function $\theta(x, t)$ be the state of pollutant concentration, and α_i and v_i be the diffusion coefficient and the wind velocity of the i-th direction respectively. The pollutant emission rate $u(x, t)$ is assumed to be of the form

$$u(x, t) = \sum_{l=1}^{N} u_l \prod_{p=1}^{q} \delta(x_p - r_{l_p}) = e'(x)u \tag{2}$$

where $e_l(x) = \prod_{p=1}^{q} \delta(x_p - r_{l_p})$, $e(x) = (e_1(x), \cdots, e_N(x))'$ and $u = (u_1, \cdots, u_N)'$, that is, the pollutant is emitted from multi-pointwise sources and the respective emission rates $\{u_l\}$ are assumed to be random constants.

The initial state $\theta(x, 0)$ and the emission rate u are Gaussian random variables with the mean and the covariance given as

$$E[\theta(x, 0)] = \bar{\theta}(x, 0), \qquad \mathrm{Cov}[\theta(x, 0)\theta(y, 0)] = \bar{P}_\theta(x, y, 0), \tag{3}$$

$$E[u] = \hat{u}(0), \qquad \mathrm{Cov}[uu'] = P_u(0), \tag{4}$$

$$\mathrm{Cov}[\theta(x, 0)u] = P_{\theta u}(x, 0) = 0. \tag{5}$$

The measurement equatiou is given by

$$z(x, k) = h(x, k)\theta(x, k) + w(x, k), \quad k = 0, 1, \cdots, \quad x \in \Omega_0 \tag{6}$$

where $z(x, k)$ denotes the measured data sampled at the instant $k\Delta$ at $x \in \Omega_0$ which is subset of Ω, and $w(x, k)$ is the white Gaussian noise independent of $\theta(x, 0)$ and u. The means and the covariance of $w(x, k)$ are given as

$$E[w(x, k)] = 0, \quad E[w(x, k)w(y, n)] = W(x, y, k)\delta_{kn}\ . \tag{7}$$

The solution of (1) and (2) can be represented for unbounded Ω as

$$\theta(x,t)=\left[\prod_{p=1}^{q}\alpha_p\{4\pi(t-t_0)\}^{\frac{1}{2}}\right]^{-1}\int\cdots\int_{-\infty}^{\infty}\theta(\xi,t_0)\exp\left[-\sum_{p=1}^{q}\frac{\{x_p-v_p(t-t_0)-\xi_p\}^2}{4\alpha_p(t-t_0)}\right]d\xi_1\cdots d\xi_q$$

$$+\left[\prod_{p=1}^{q}\alpha_p(4\pi)^{\frac{1}{2}}\right]^{-1}\sum_{l=1}^{N}u_l\int_{t_0}^{t}(t-\tau)^{-\frac{q}{2}}\exp\left[-\sum_{p=1}^{q}\frac{\{x_p-v_p(t-\tau)-r_{lp}\}^2}{4\alpha_p(t-\tau)}\right]d\tau\ , \tag{8}$$

then, replacing t with $\overline{k+1}\,\Delta$ and t_0 with $k\Delta$ respectively, we get the following discrete-time distributed-parameter system

$$\theta(x,k+1)=\mathcal{L}_x\theta(x,k)+c'(x)u(k) \tag{9}$$

$$u(k+1)=u(k),\qquad\qquad k=0,1,\cdots. \tag{10}$$

where $c(x)=(c_1(x),\cdots,c_N(x))'$ and the operator $\mathcal{L}_x(\cdot)$ are given from (8) by

$$\mathcal{L}_x(\cdot)=\left[\prod_{q=1}^{q}\alpha_p(4\pi\Delta)^{\frac{1}{2}}\right]\int^{-1}\cdots\int_{-\infty}^{\infty}\exp\left[-\sum_{p=1}^{q}\frac{\{x_p-v_p\Delta-\xi_p\}^2}{4\alpha_p\Delta}\right](\cdot)d\xi_1\cdots d\xi_q \tag{11}$$

$$c_l(x)=\left[\prod_{p=1}^{q}\alpha_p(4\pi)^{\frac{1}{2}}\right]^{-1}\int_{k\Delta}^{\overline{k+1}\Delta}(\overline{k+1}\,\Delta-\tau)^{-\frac{q}{2}}\exp\left[-\sum_{p=1}^{q}\frac{\{x_p-v_p(\overline{k+1}\,\Delta-\tau)-r_{lp}\}^2}{4\alpha_p(\overline{k+1}\,\Delta-\tau)}\right]d\tau\,. \tag{12}$$

Now, the problem is to estimate the state of concentration $\theta(x,k)$ and the emission rate u on the basis of the measured data $z(x,n)$, $x\in\Omega_0$, $n=0,\cdots,k$.

3. Recursive Form of Optimal Filter

By adjoining $u(k)$ to $\theta(x,k)$, we define the new state vector $\phi(x,k)$ by

$$\phi(x,k)=[\theta(x,k),\ u'(k)]'. \tag{13}$$

Then, equations (9), (10) and (6) can be rewritten as

$$\phi(x,\ k+1)=\mathcal{F}_x\phi(x,k),\qquad\qquad x\in\Omega \tag{14}$$

$$z(x,k)=H'(x,k)\phi(x,k)+w(x,k),\qquad x\in\Omega_0 \tag{15}$$

where

$$\mathcal{F}_x(\cdot)=\begin{bmatrix}\mathcal{L}_x(\cdot) & c'(x)\\ 0 & I\end{bmatrix},\quad H(x,k)=[h(x,k)\ \ 0]'.$$

We first show the optimal filter which gives an estimate $\hat{\phi}(x,k|k)$ of $\phi(x,k)$ based on the set of measured data $z(x,n)$; $n=0,1,\cdots,k$, $x\in\Omega_0$. The estimate $\hat{\phi}(x,k|k)$ is sought through a linear combination of the past set of measured data as follows

$$\hat{\phi}(x,k|k)=\sum_{n=0}^{k}\int_{\Omega_0}K(x,\xi\,;\ k,n)z(\xi,n)d\xi \tag{16}$$

where $K(x, \xi; k, n)$ is a vector kernel function. The estimate $\hat{\phi}(x, k|k)$ is defined to be optimal when it minimizes the variance $E||\phi(x, k)-\hat{\phi}(x, k|k)||^2$ where $||\cdot||^2$ denotes the Euclidean norm. A necessary and sufficient condition for the estimate to be optimal can be derived by use of calculus of variations.

Proposition 3. 1.

A necessary and sufficient condition for the estimate $\hat{\phi}(x, k|k)$ to be optimal is that

$$E[\{\phi(x, k)-\hat{\phi}(x, k|k)\}z(\xi, n)]=0, \quad 0\leq n\leq k, \ x\in\Omega; \ \xi\in\Omega_0 . \tag{17}$$

It is significant to derive the recursive form of the optical filter in order to get the estimate $\hat{\phi}(x, k|k)$ sequentially utilizing a digital computer. The recursive form is given in the following proposition.

Proposition 3. 2.

The recursive form of the optimal filter is given as

$$\hat{\phi}(x, k|k)=\hat{\phi}(x, k|k-1)+\int_{\Omega_0} K(x, \xi, k)[z(\xi, k)-H'(\xi, k)\hat{\phi}(\xi, k|k-1)]d\xi \tag{18a}$$

$$\hat{\phi}(x, k+1|k)=\mathcal{F}_x\hat{\phi}(x, k|k) \tag{18b}$$

where $K(x, \xi, k)$ is the optimal kernel and satisfies the Fredholm integral equation of the second type

$$\int_{\Omega_0} K(x, \xi, k)[H'(\xi, k)P(\xi, y, k)H(y, k)+W(\xi, y, k)]d\xi=P(x, y, k)H(y, k) \tag{19}$$

where $x, y\in\Omega_0$, and $P(x, y, k)$ denotes the covariance matrix of the estimate $\hat{\phi}(x,k|k-1)$ and recursively can be computed by

$$Q(x, y, k)=P(x, y, k)-\int_{\Omega_0} K(x, \xi, k)H(\xi, k)P(\xi, y, k)d\xi \tag{20a}$$

$$P(x, y, k+1)=\mathcal{F}_xQ(x, y, k)\mathcal{F}'_y . \tag{20b}$$

The proofs of Propositions 3.1 and 3.2 have been shown in reference (Sano 1975b).

4. Evaluation of Covariance Matrix

The covariance matrix of the combined state estimate $\hat{\phi}(x, k|k-1)$ is

$$P(x, y, k)=\begin{bmatrix} P_\theta(x, y, k) & P'_{\theta u}(x, k) \\ P_{\theta u}(y, k) & P_u(k) \end{bmatrix} \tag{21}$$

where $P_\theta(x, y, k)$, $P_{\theta u}(x, k)$ and $P_u(k)$ are the covariance of the respective elements. Then it will be noted that the elements of $P(x, y, k)$ of (20) are mutually coupled and must be simultaneously computed, therefore the rquired amount of memory and the round-off error are increased as the dimension of space coordinates and the number of emission sources become large. This section shows that the elements of the covariance matrix (21) can separately be computed. First we develope the Friedland's results (FRIEDLAND 1969) to distributed-parameter system and present useful propositions for the computation of the elements of the covariance.

Proposition 4.1.

Assume that $\bar{P}(x, y, k)$ is a solution of (20) with an initial condition $\bar{P}(x, y, 0)$. Then any other solution $P(x, y, k)$ of (20) with any initial conditions $P(x, y, 0) \neq \bar{P}(x, y, 0)$ can be described by

$$P(x, y, k)=\bar{P}(x, y, k)+R(x, k)M(k)R'(y, k) \tag{22}$$

where $R(x, k)$ and $M(k)$ denote $(N+1)\times N$ and $N\times N$ matrices respectively, and satisfy the equations

$$V(x, k)=R(x, k)-\int_{\Omega_0}\bar{K}(x, \xi, k)H'(\xi, k)R(\xi, k)d\xi \tag{23a}$$

$$\mathscr{R}(x, k+1)=\mathscr{F}_x V(x, k) \tag{23b}$$

$$M(k+1)=M(k)-M(k)\int_{\Omega_0}J(\xi, k)H'(\xi, k)R(\xi, k)d\xi M(k) \tag{24}$$

where $\bar{K}(x, \xi, k)$ and $J(\xi, k)$ are the solutions of the Fredholm integral equations

$$\int_{\Omega_0}\bar{K}(x, \xi, k)[H'(\xi, k)\bar{P}(\xi, y, k)H(y, k)+W(\xi, y, k)]d\xi=\bar{P}(x, y, k)H(y, k) \tag{25}$$

$$\int_{\Omega_0}J(\xi, k)[H'(\xi, k)P(\xi, y, k)H(y, k)+W(\xi, y, k)]d\xi=R'(y, k)H(y, k)\,. \tag{26}$$

This proposition will be proved in Appendix.

Let $\bar{P}(x, y, k)$ in Proposition 4.1 be a solution of (20) with the initial condition

$$\bar{P}(x, y, 0)=\begin{bmatrix}\bar{P}_\theta(x, y, 0) & 0\\ 0 & 0\end{bmatrix} \tag{27}$$

then, it can be obtained from (20) by

$$\bar{P}(x, y, k)=\begin{bmatrix}\bar{P}_\theta(x, y, k) & 0\\ 0 & 0\end{bmatrix}. \tag{28}$$

It can easily be seen that $\bar{P}_\theta(x, y, k)$ is the solution of (20) in the emission-free case $u(k)=0$ $(k=0, 1, \cdots)$ and then satisfies

$$\bar{Q}_\theta(x, y, k)=\bar{P}_\theta(x, y, k)-\int_{\Omega_0}\bar{K}_\theta(x, \xi, k)h(\xi, k)\bar{P}_\theta(\xi, y, k)d\xi \tag{29a}$$

$$\bar{P}_\theta(x, y, k+1) = \mathcal{L}_x \bar{Q}_\theta(x, y, k) \mathcal{L}_y \tag{29b}$$

where $\bar{K}_\theta(x, \xi, k)$ is obtained from the integral equation

$$\int_{\Omega_0} \bar{K}_\theta(x, \xi, k)[h(\xi, k)\bar{P}_\theta(\xi, y, k)h(y, k) + W(\xi. y, k)]d\xi = \bar{P}_\theta(x, y, k)h(y, k)\,. \tag{30}$$

which is the emission-free case of equation (19).

On the other hand, for any initial conditions which are different from (27)

$$P(x, y, 0) = \begin{bmatrix} P_\theta(x, y, 0) & P'_{\theta u}(x, 0) \\ P_{\theta u}(y, 0) & P_u(0) \end{bmatrix}, \tag{31}$$

the solution of (20) in the presence of the emission term can be obtained by the use of Proposition 4.1. Let the elements of $R(x, k)$, $V(x, k)$ and $\bar{K}(x, \xi, k)$ be denoted by

$$R(x, k) = \begin{bmatrix} R'_{\theta u}(x, k) \\ R_u(k) \end{bmatrix}, \quad V(x, k) = \begin{bmatrix} V'_{\theta u}(x, k) \\ V_u(k) \end{bmatrix}, \quad \bar{K}(x, \xi, k) = \begin{bmatrix} \bar{K}_\theta(x, \xi, k) \\ 0 \end{bmatrix}^*,$$

then the elements of the matrix $P(x, y, k)$ of (21) are expressed by use of Proposition 4.1 as

$$P_\theta(x, y, k) = \bar{P}_\theta(x, y, k) + R'_{\theta u}(x, k)M(k)R_{\theta u}(y, k) \tag{32a}$$

$$P'_{\theta u}(x, k) = R'_{\theta u}(x, k)M(k)R_u(k) \tag{32b}$$

$$P_u(k) = R_u(k)M(k)R_u(k). \tag{32c}$$

Furthermore, it follows from (23) that

$$V_{\theta u}(x, k) = R_{\theta u}(x, k) - \int_{\Omega_0} \bar{K}_\theta(x, \xi, k)h(\xi, k)R_{\theta u}(\xi, k)d\xi \tag{33a}$$

$$R_{\theta u}(x, k+1) = \mathcal{L}_x V_{\theta u}(x, k) + c'(x) V_u(k) \tag{33b}$$

$$V_u(k) = R_u(k) \tag{34a}$$

$$R_u(k+1) = V_u(k). \tag{34b}$$

At the initial time $k=0$, from (32) we get

$$P_\theta(x, y, 0) - \bar{P}_\theta(x, y, 0) = R'_{\theta u}(x, 0)M(0)R_{\theta u}(y, 0) = 0 \tag{35a}$$

$$P'_{\theta u}(x, 0) = R'_{\theta u}(x, 0)M(0)R_u(0) = 0 \tag{35b}$$

$$P_u(0) = R_u(0)M(0)R_u(0) \tag{35c}$$

where the assumptions (3) and (5) have been used in (35a) and (35b) respectively. It follows from (35a) and (35b) that $R_u(0) = I$ (unity matrix) and $R_{\theta u}(x, 0) = 0$, then from (34) we get $R_u(k) = V_u(k) = I$.

* It can be clarified in deriving (30) from (19) that the second elements of $\bar{K}(x, \xi, k)$ are all zero.

The above results may be summarized in the following propositions.

Proposition 4.2.

The covariance matrix $P(x, y, k)$ is given as

$$P_\theta(x, y, k) = \bar{P}_\theta(x, y, k) + R'_{\theta u}(x, k)M(k)R_{\theta u}(y, k) \tag{36a}$$

$$P'_{\theta u}(x, k) = R'_{\theta u}(x, k)M(k) \tag{36b}$$

$$P_u(k) = M(k) \tag{36c}$$

where $\bar{P}_\theta(x, y, k)$ is the solution of (29), and the equations for $R_{\theta u}(x, k)$ and $M(k)$ are simplified from (33) and (34) to

$$S(\xi, k) = h(\xi, k)R_{\theta u}(\xi, k) \tag{37}$$

$$V_{\theta u}(x, k) = R_{\theta u}(x, k) - \int_{\Omega_0} \bar{K}_\theta(x, \xi, k)S(\xi, k)d\xi \tag{38a}$$

$$R_{\theta u}(x, k+1) = \mathcal{L}_x V_{\theta u}(x, k) + c(x) \tag{38b}$$

$$M(k+1) = M(k) - M(k)\int_{\Omega_0} J(\xi, k)S'(\xi, k)d\xi M(k) \tag{39}$$

$$\int_{\Omega_0} J(\xi, k)[h(\xi, k)\bar{P}_\theta(\xi, y, k)h(y, k) + W(\xi, y, k) + S'(\xi, k)M(k)S(y, k)]d\xi = S(y, k) \tag{40}$$

where $R_{\theta u}(x, 0) = 0$ and $M(0) = P_u(0)$.

Proposition 4.3.

The elements of the matrix $Q(x, y, k)$ of (20) can also be expressed as

$$Q_\theta(x, y, k) = \bar{Q}_\theta(x, y, k) + V'_{\theta u}(x, k)M(k+1)V_{\theta u}(y, k) \tag{41a}$$

$$Q'_{\theta u}(x, k) = V'_{\theta u}(x, k)M(k+1) \tag{41b}$$

$$Q_u(k) = M(k+1). \qquad \text{(41b)}$$

This proposition can easily be proved by the use of Proposition 4.2.

5. Computation of Estimates

The estimates of the state $\hat{\theta}(x, k|k)$ and the emission rate $\hat{u}(k|k)$ have been given in Proposition 3.2 by

$$\hat{\theta}(x, k|k) = \hat{\theta}(x, k|k-1) + \int_{\Omega_0} K_\theta(x, \xi, k)[z(\xi, k) - h(\xi, k)\hat{\theta}(\xi, k|k-1)]d\xi \tag{42a}$$

$$\hat{\theta}(x, k|k-1) = \mathcal{L}_x \hat{\theta}(x, k-1|k-1) + c'(x)\hat{u}(k-1|k-1) \tag{42b}$$

$$\hat{u}(k|k) = \hat{u}(k|k-1) + \int_{\Omega_0} K_u(\xi, k)[z(\xi, k) - h(\xi, k)\hat{\theta}(\xi, k|k-1)]d\xi \tag{43a}$$

$$\hat{u}(k|k-1) = \hat{u}(k-1|k-1) \tag{43b}$$

where $K_\theta(x, \xi, k)$ and $K_u(\xi, k)$ are the elements of $K(x, \xi, k)$ in (19), and are related to $\bar{K}_\theta(x, \xi, k)$ of (30) as shown in the following.

Lemma 5.1.

$$K_\theta(x, \xi, k) = \bar{K}_\theta(x, \xi, k) + V'_{\theta u}(x, k)K_u(\xi, k) \tag{44}$$

Proof: From (19) and (20), we have

$$Q(x, y, k)H(y, k) = \int_{\Omega_0} K(x, \xi, k)\, W(\xi, y, k)d\xi. \tag{45}$$

By use of (41), we can express the respective elements of (45) as

$$\bar{Q}_\theta(x, y, k)h(y, k) + V'_{\theta u}(x, k)M(k+1)\, V_{\theta u}(y, k)h(y, k) = \int_{\Omega_0} K_\theta(x, \xi, k)\, W(\xi, y, k)d\xi \tag{46a}$$

$$M(k+1)\, V'_{\theta u}(y, k)h(y, k) = \int_{\Omega_0} K_u(\xi, k)\, W(\xi, y, k)d\xi. \tag{46b}$$

Furthermore, utilizing

$$Q_\theta(x, y, k)h(y, k) = \int_{\Omega_0} K_\theta(x, \xi, k)\, W(\xi, y, k)d\xi, \tag{47}$$

it follows from (46) that

$$\int_{\Omega_0} [\bar{K}_\theta(x, \xi, k) + V'_{\theta u}(x, k)K_u(\xi, k) - K_\theta(x, \xi, k)]\, W(\xi, y, k)d\xi = 0. \tag{48}$$

Since the cavariance $W(\xi, y, k)$ is assumed not to be identically zero, equation (44) has been proved.

In equations (42) and (43), the estimates $\hat{\theta}(x, k|k)$ and $\hat{u}(k|k)$ are mutually coupled. In the following we show that the emission estimate $\hat{u}(k|k)$ can be computed independently of the estimate $\hat{\theta}(x, k|k)$. We get the estimate $\hat{\theta}(x, k|k)$ by computing first the emission-free estimate $\bar{\theta}(x, k|k)$ separately and then correcting the latter by the quantity obtained from the emission estimate.

Proposition 5.2.

Let $\bar{\theta}(x, k|k)$ be the estimate of $\theta(x, k)$ in the emission-free case ($u(k) \equiv 0$), then the estimate $\hat{\theta}(x, k|k)$ in the presence of $u(k)$ is given by

$$\hat{\theta}(x, k|k)=\bar{\theta}(x, k|k)+V'_{\theta u}(x, k)\hat{u}(k|k) \tag{49a}$$

$$\hat{\theta}(x, k|k-1)=\bar{\theta}(x, k|k-1)+R'_{\theta u}(x, k)\hat{u}(k|k-1) \tag{49b}$$

where $\bar{\theta}(x, k|k)$, $\bar{\theta}(x, k|k-1)$, $\hat{u}(k|k)$ and $\hat{u}(k|k-1)$ are computed by

$$\bar{\theta}(x, k|k)=\bar{\theta}(x, k|k-1)+\int_{\Omega_0} \bar{K}_\theta(x, \xi, k)[z(\xi, k)-h(\xi, k)\bar{\theta}(\xi, k|k-1)]d\xi \tag{50a}$$

$$\bar{\theta}(x, k|k-1)=\mathcal{L}_x\bar{\theta}(x, k-1|k-1)$$

$$\hat{u}(k|k)=\hat{u}(k|k-1)+\int_{\Omega_0} K_u(\xi, k)[z(\xi, k)-h(\xi, k)\bar{\theta}(\xi, k|k-1)$$
$$-S'(\xi, k)\hat{u}(k|k-1)]d\xi \tag{51a}$$

$$\hat{u}(k|k-1)=\hat{u}(k-1|k-1) \tag{51b}$$

$$\int_{\Omega_0} K_u(\xi. k)[h(\xi, k)\bar{P}_\theta(\xi, y, k)h(y, k)+W(\xi, y, k)$$
$$+S'(\xi, k)M(k)S(y, k)]d\xi=M(k)S(y, k) \tag{52}$$

where $\bar{\theta}(x, 0|-1)=\bar{\theta}(x, 0)$ and $\hat{u}(0|-1)=\hat{u}(0)$, and $\bar{K}_\theta(x, \xi, k)$, $V_{\theta u}(x, k)$ and $R_{\theta u}(x, k)$ have been obtained in (30), (38a) and (38b) respectively.

Proof: The proposition is proved by means of mathematical induction. At the instant $k-1$, assume that

$$\bar{\theta}(x, k-1|k-1)=\bar{\theta}(x, k-1|k-1)+V'_{\theta u}(x, k-1)\hat{u}(k-1|k-1). \tag{53}$$

Let the innovation process be denoted by $\delta(x, k)$. Utilizing (42b), (53) and (38b), $\delta(x, k)$ can be rewritten as

$$\begin{aligned}\delta(\xi, k)&\equiv z(\xi, k)-h(\xi, k)\hat{\theta}(\xi, k|k-1)\\
&=z(\xi, k)-h(\xi, k)[\mathcal{L}_\xi\hat{\theta}(\xi, k-1|k-1)+c'(\xi)\hat{u}(k-1|k-1)]\\
&=z(\xi, k)-h(\xi, k)\mathcal{L}_\xi\bar{\theta}(\xi, k-1|k-1)-h(\xi, k)[\mathcal{L}_\xi V'_{\theta u}(\xi, k-1)+c'(\xi)]\hat{u}(k-1|k-1)\\
&=z(\xi, k)-h(\xi, k)\bar{\theta}(\xi, k|k-1)-S'(\xi, k)\hat{u}(k-1|k-1).\end{aligned} \tag{54}$$

By use of (53), (38b) and (44), we can rewrite (42) into

$$\begin{aligned}\hat{\theta}(x, k|k)&=\mathcal{L}_x\bar{\theta}(x, k-1|k-1)+[\mathcal{L}_x V'_{\theta u}(x, k-1)+c'(x)]\hat{u}(k-1|k-1)\\
&\quad+\int_{\Omega_0} [\bar{K}_\theta(x, \xi, k)+V_{\theta u}(x, k)K_u(\xi, k)]\delta(\xi, k)d\xi\\
&=\mathcal{L}_x\bar{\theta}(x, k-1|k-1)+R'_{\theta u}(x, k)u(k-1|k-1)\\
&\quad+\int_{\Omega_0} \bar{K}_\theta(x, \xi, k)[z(\xi, k)-h(\xi, k)\mathcal{L}_\xi\bar{\theta}(\xi, k-1|k-1)]d\xi\\
&\quad-\int_{\Omega_0} \bar{K}_\theta(x\,\xi, k)S'(\xi,k)\hat{u}(k-1|k-1)d\xi+\int_{\Omega_0} V'_{\theta u}(x, k)K_u(\xi, k)\delta(\xi, k)d\xi\end{aligned}$$

$$=\bar{\theta}(x, k|k)+V'_{\theta u}(x, k)\left[u(k-1|k-1)+\int_{\Omega_0} K_u(\xi, k)\delta(\xi, k)d\xi\right]$$
$$=\bar{\theta}(x, k|k)+V'_{\theta u}(x, k)u(k|k).$$

At the initial time $k=0$, it is satisfied that $\hat{\theta}(x, 0|-1)=\bar{\theta}(x, 0|-1)=\bar{\theta}(x, 0)$ and $R_{\theta u}(x, 0)=0$, then the proposition has been proved.

6. Discussion

The block diagram of the new estimation algorithm is shown in Fig. 1. In the block (A), the emission-free state estimate is given by use of (50), (30) and (29), and the computational procedure is independent of the emission-rate estimation. The estimate of the emission rate is obtained from (51), (52) and (37)~(40) as summarized in the block (B). The seperation of the estimation procedure becomes more effective when the location of the emission sources is unknown and to be identified in real time.

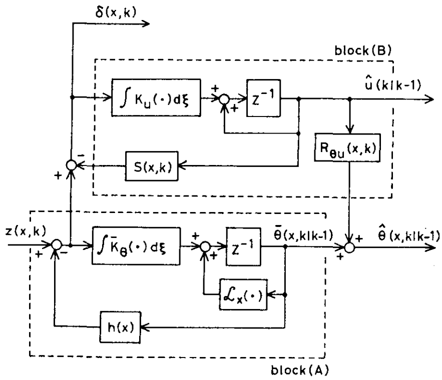

Fig. 1

We have presented a method for estimating the source location (Sano 1974a) by which the source location estimate $\hat{r}_l(k)$ is corrected so that the time-averaged innovation squares

$$I_k=\frac{1}{K}\sum_{n=k-K+1}^{k}\int_{\Omega_0}\delta^2(x, n)dx \tag{55}$$

may be minimized by use of an approximated gradient method where K denotes the moving averaging number. In order to obtain the gradient of I_k with respect to the source location estimates $\hat{r}_{l_p}(k)$, $l=1, \cdots, N$; $p=1, \cdots, q$, we must perturb the estimate $\hat{r}_{l_p}(k)$ to $\hat{r}_l(k)\pm\varepsilon_{l_p}$ where ε_{l_p} is a positive constant. Since the estimates $\hat{r}_{l_p}(k)$ is included only in (38b) through $c(x)$ of (12), the $2Nq$ blocks (B) associated with $\hat{r}_{l_p}(k)\pm\varepsilon_{l_p}$ are required to get the gradient of (55), however, the necessary block (A) is only one because it is independent of the emission term.

In the practice, by spatial discretization we have the discretized model which consists of an $L^q \times L^q$ matrix equation for $\bar{P}_\theta(x, y, k)$ in (29), an $L^q \times N$ matrix equation for $R_{\theta u}(x, k)$ in (38) and an L^q vector equation for $\bar{\theta}(x, k|k)$ in (50), where L denotes the number of the spatial discretization of each coordinate. Therefore, in the proposed algorithm we may compute the $L^q \times L^q$ matrix for $\bar{P}_\theta(x, y, k)$, the $L^q \times N$ matrix for $R_{\theta u}(x, k)$ and an $N \times N$ matrix for $M(k)$ in place of a higher-dimensional $(L^q+N)\times(L^q+N)$ matrix for $P(x, y, k)$ in (20), and then reduce the round-off error and the inaccuracy caused in the high-dimensional matrix manipulation of (19) and (20). Furthermore, in the identification of the source location, conventional techniques require the $2Nq(L^q+N)^2$ memory area while the presented algorithm uses the $2Nq(L^qN+N^2)+L^{2q}$, then saves $NL^q+(2Nq-1)L^{2q}$ memory area.

The detail descriptions for the identification technique of the emission sources have been shown in other references as for a one-dimensional diffusion system (Sano 1974a, b) and a two-dimensional air pollution diffusion model (Sano 1975a).

7. Conclusion

The optimal estimation algorithm has been proposed for a discrete-time distributed-parameter system subjected to unknown emission inputs from multi-pointwise sources. We get the state estimate by first computing the emission-free state estimate independently then correcting the latter by the quantity obtained from the emission input estimates, so that the computational complexity and inaccuracy and the required memory area can remarkably be reduced especially in identifying the uncertain source location.

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REFERENCES

Atra, S. R. (1972): Optimal Estimation in Distributed Process Using Innovation Approach, IEEE Trans. Automatic Control, **AC-17**, 710–712.

Falb, P. L., (1967): Infinite-Dimensional Filtering, Information and Control, **11**, 102–137.

Friedland, B., (1969): Treatment of Bias in Recursive Filtering, IEEE Trans. Automatic Control, **AC-14**, 359–367.

Kumar, G.R.V. and Sage, A. R., (1972): The Innovation Approach to Space-Time Filtering and Smoothing, Information Science, **4**, 193–215.

Kushner, H. J., (1970): Filtering for Linear Distributed Parameter systems, SIAM J. Control, **8**, 346–359.

MATSUMOTO, J. and ITO, K., (1970): Feedback Control of Distributed-Parameter Systems with Spatially Concentrated Controls, Int. J. Control, **12**, 401–419.

MEDITCH, J. S., (1971): Least-Squares Filtering and Smoothing for Linear Distributed Parameter Systems, Automatica, **7**, 315–322.

SAKAWA, Y., (1972): Optimal Filtering in Linear Distributed-Parameter Systems, Int. J. Control, **16**, 115–127.

SANO, A., (1974a): State Estimation for Distributed-Parameter Systems with Unknown Multi-Pointwise Sources, Proceedings of the Symposium on " Modelling for Prediction and Control of Air Pollution ", Kyoto, 45–52.

SANO, A., (1974b): A Method for Identifying Uncertain Pointwise Sources in Distributed-Parameter Systems, Proceedings of the 3rd SICE Symposium on " Control Theory ", Nagoya, 109–114.

SANO, A., (1975a): Filtering for Diffusion Systems with Uncertain Source Location, Proceedings of the 3rd Symposium on " Environmental Pollution Control", Tokyo, 163–170.

SANO, A., (1975b): Adaptive Algorithm for Measurement Optimization in a Distributed Parameter System, Trans. SICE, **11**, 221–228.

THAU, F. E., (1968): On Optimum Filtering for a Class of Linear Distributed Parameter Systems, Trans. ASME, J. Basic Eng., **91**, 173–198.

TZAFESTAS, S. G. and NIGHTINGALE, J. M., (1968a): Optimal Filtering, Smoothing and Prodiction in Linear Distributed-Parameter Systems, Proc. IEE, **115**, 1207–1212.

TZAFESTAS, S. G. and NIGHTINGALE, J. M., (1968): Concerning Optimal Filtering of Linear Distributed-Parameter Systems, Proc. IEE, **115**, 1737–1742.

APPENDIX

Proof of Proposition 4.1.

As $\bar{P}(x, y, k)$ is a solution of equation (20), it satisfies

$$\bar{P}(x, y, k+1) = \mathscr{F}_x\left[\bar{P}(x, y, k) - \int \bar{K}(x, \xi, k) H'(\xi, k) \bar{P}(\xi, y, k) d\xi\right] \mathscr{F}'_y \tag{A1}$$

where for simplicity the symbol Ω_0 is dropped. In the following, for the simplicity of notations we shall designate all quantities with the time index $k+1$ by a subscript $+$ and all quantities with the index k with an unsubscripted symbol, i.e. $P_+(x, y) \equiv P(x, y, k+1)$ and $P(x, y) \equiv P(x, y, k)$.

The proposition is proved by means of the mathematical induction. Assume that at the instant k

$$P(x, y) = \bar{P}(x, y) + R(x) M R'(y), \tag{A2}$$

then it is to be proved that at the $(k+1)$ instant under the conditions (23)~(26)

$$F(x, y) \equiv P_+(x, y) - \bar{P}_+(x, y) - R_+(x) M_+ R'_+(y) = 0. \tag{A3}$$

First of all, from (20), (A1) and (A2) we have

$$P_+(x, y)-\bar{P}_+(x, y)$$
$$=\mathcal{F}_x\left[P(x, y)-\int K(x, \xi)H'(\xi)P(\xi, y)d\xi-\bar{P}(x, y)+\int \bar{K}(x, \xi)H'(\xi)\bar{P}(\xi, y)d\xi\right]\mathcal{F}'_y$$
$$=\mathcal{F}_x\left[R(x)MR'(y)-\int K(x, \xi)H'(\xi)P(\xi, y)d\xi+\int \bar{K}(x, \xi)H'(\xi)\bar{P}(\xi, y)d\xi\right]\mathcal{F}'_y \tag{A4}$$

Then, from (23) and (24) it follows that

$$R_+(x)M_+R'_+(y)$$
$$=\mathcal{F}_x\left[R(x)MR'(y)-R(x)M\int J(\xi)H'(\xi)R(\xi)d\xi MR'(y)-R(x)M\int R'(\xi)H(\xi)\bar{K}'(y, \xi)d\xi\right.$$
$$+R(x)M\int J(\xi)H'(\xi)R(\xi)d\xi M\int R'(\eta)H(\eta)\bar{K}'(y, \eta)d\eta-\int \bar{K}(x, \xi)H'(\xi)R(\xi)d\xi MR'(y)$$
$$+\int \bar{K}(x, \xi)H'(\xi)R(\xi)d\xi M\int J(\eta)H'(\eta)R(\eta)d\eta MR'(y)$$
$$+\int \bar{K}(x, \xi)H'(\xi)R(\xi)d\xi M\int R'(\eta)H(\eta)\bar{K}'(y. \eta)d\eta$$
$$\left.-\int \bar{K}(x, \xi)H'(\xi)R(\xi)d\xi M\int J(\zeta)H'(\zeta)R(\zeta)d\zeta\, M\int R'(\eta)H(\eta)\bar{K}'(y, \eta)d\eta\right]\mathcal{F}'_y \tag{A5}$$

The third and fourth terms in the right side of (A5) can be rewritten by use of (26) and (A2) as follows;

$$=-R(x)M\int R'(\xi)H(\xi)\bar{K}'(y, \xi)d\xi$$
$$+R(x)M\iint J(\xi)H'(\xi)[P(\xi, \eta)-\bar{P}(\xi, \eta)]H(\eta)\bar{K}'(y, \eta)d\xi d\eta$$
$$=-R(x)M\int R'(\xi)H(\xi)\bar{K}'(y, \xi)d\xi$$
$$+R(x)M\int\left[R'(\eta)H(\eta)-\int J(\xi)\{H'(\xi)\bar{P}(\xi, \eta)H(\eta)+W(\xi, \eta)\}d\xi\right]\bar{K}'(y, \eta)d\eta$$
$$=-R(x)M\iint J(\xi)[H'(\xi)\bar{P})\xi, \eta)H(\eta)+W(\xi, \eta)]\bar{K}'(y, \eta)d\xi d\eta$$
$$=-R(x)M\int J(\xi)H'(\xi)\bar{P}'(y, \xi)d\xi \qquad \text{(by use of (25)).} \tag{A6}$$

In the same manner, the seventh and eighth terms of (A5) can also be rewritten as follows;

$$=\bar{K}(x, \eta)H'(\eta)R(\eta)d\eta M\int J(\xi)H'(\xi)\bar{P}'(y, \xi)d\xi. \tag{A7}$$

Substituting (A6) and (A7) into (A5) and utilizing (A2), we can rewrite (A5) as

$$R_+(x)MR'_+(y)=\mathcal{F}_x\Big[R(x)MR'(y)-R(x)M\int J(\xi)H'(\xi)P(\xi,y)d\xi$$

$$-\int\bar{K}(x,\xi)H'(\xi)R(\xi)d\xi MR'(y)$$

$$+\int\bar{K}(x,\xi)H'(\xi)R(\xi)d\xi M\int J(\xi)H'(\xi)P(y,\xi)d\xi\Big]\mathcal{F}'_y. \tag{A8}$$

Substituting (A4) and (A8) into (A3) yields

$$F(x,y)=\mathcal{F}_x\Big[\int K(x,\xi)H'(\xi)P(\xi,y)d\xi-\int\bar{K}(x,\xi)H'(\xi)P(\xi,y)d\xi$$

$$-R(x)M\int J(\xi)H'(\xi)P(\xi,y)d\xi$$

$$+\int\bar{K}(x,\eta)H'(\eta)R(\eta)d\eta M\int J(\xi)H'(\xi)P(\xi,y)d\xi\Big]\mathcal{F}'_y. \tag{A9}$$

The third term of the right hand side of (A9) can be rewritten by use of (19) as follows;

$$=-R(x)M\int J(\xi)\int[H'(\xi)P(\xi,\eta)H(\eta)+W(\xi,\eta)]K'(y,\eta)d\xi d\eta$$

$$=-R(x)M\int R'(\eta)H(\eta)K'(y,\eta)d\eta \qquad \text{(by use of (26))}$$

$$=-\int P(x,\xi)H(\xi)K'(y,\xi)d\xi+\int\bar{P}(x,\xi)H(\xi)K'(y,\xi)d\xi. \tag{A10}$$

The fourth term of (A9) can similarly be rewritten as follows;

$$=\int\bar{K}(x,\xi)H'(\xi)R(\xi)d\xi M\int R'(\eta)H(\eta)K'(y\ \eta)d\eta$$

$$=\int\int\bar{K}(x,\xi)H'(\xi)[P(\xi,\eta)-\bar{P}(\xi,\eta)]H(\eta)K'(y,\eta)d\xi d\eta$$

$$=\int\int\bar{K}(x,\xi)H'(\xi)P(\xi,\eta)H(\eta)K'(y,\eta)d\xi d\eta$$

$$-\int\bar{P}(x,\xi)H(\xi)K'(y,\xi)\,d\xi+\int\int\bar{K}(x,\xi)\,W(\xi,\eta)K'(y,\eta)d\xi d\eta$$

$$=\int\bar{K}(x,\xi)H'(\xi)P(y,\xi)\,d\xi-\int\bar{P}(x,\xi)H(\xi)K'(y,\xi)d\xi. \tag{A11}$$

Then, substituting (A10) and (A11) into (A9), we have

$$F(x,y)=\mathcal{F}_x\Big[\int K(x,\xi)H'(\xi)P(\xi,y)d\xi-\int\bar{K}(x,\xi)H'(\xi)P(\xi,y)d\xi$$

$$-\int P(x,\xi)H(\xi)K'(y,\xi)d\xi+\int\bar{P}(x,\xi)H(\xi)K'(y,\xi)d\xi$$

$$+\int \bar{K}(x, \xi)H'(\xi)P(y, \xi)d\xi - \int \bar{P}(x, \xi)H(\xi)K'(y, \xi)d\xi \Big] \mathcal{F}_y'$$

$$=0.$$

So, equation (A3) has been proved. At the initial time $k=0$, we may choose $R(x, 0)$ and $M(0)$ so that the equation

$$P(x, y, 0) - \bar{P}(x, y, 0) = R(x, 0)M(0)R'(y, 0)$$

may be satisfied. Therefore the Proposition has been proved.