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FUNCTIONAL CENTRAL LIMIT THEOREM FOR STATIONARY PROCESSES

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ABSTRACT

In this paper we shall deal with a functional central limit theorem for stationary processes.

We shall show by using Gordin's method that the functional central limit theorem (the version of Donsker's invariance principle) holds for a class of stationary sequences. Furthermore, we give various functional central limit theorem.

1. Introduction

In order to prove a central limit theorem for stationary processes, M. I. GORDIN (1969) gave a new method, which differs from the methods of S. N. BERNSTEIN (used in Chapters 18 and 19 of IBRAGIMOV and LINNIK (1971)). GORDIN's method is at first to approximate the stationary process under investigation by a sequence of martingale differences, and then to use the central limit theorem for martingale differences with finite variances which was proved independently by P. BILLINGSLEY (1961) and I. A. IBRAGIMOV (1963).

P. BILLINGSLEY (1968) proved the functional central limit theorem for a stationary ergodic sequence of martingale differences with finite variances.

In this paper we shall show by GORDIN's method that the functional central limit theorem holds for a class of stationary processes. Furthermore, we give various functional central limit theorem. Some theorems of BILLINGSLEY (1968, § 20, § 21) are obtained as corollaries of the results obtained in this paper.

Suppose that there exists a probability measure P defined on a Borel field $\mathcal{M}$ of sets of some space X. Space L_p corresponds to measure P; $|f|_p$ denotes norm of a function f in L_p. If a Borel field $\mathcal{N}$ is contained in $\mathcal{M}$, then $H(\mathcal{N})$ denotes Hilbert space of those function in L_2, which is measurable with respect to $\mathcal{N}$. P_G

denotes orthogonal projection onto closed subspace $G \subset H = L_2$.

Let T be a 1-1 measure-preserving point transformation on X and $\mathcal{M}_0$ be a Borel field such that $T^{-1}(\mathcal{M}_0) \subset \mathcal{M}_0$. Relation $Uf(x) = f(Tx)$ defines a unitary transformation U on H.

Spaces $\mathcal{M}_k$, H_k, S_k and $\mathcal{F}$ are defined by the following relations;

$$\mathcal{M}_k = T^{-k}(\mathcal{M}_0),$$

$$H_k = H(\mathcal{M}_k),$$

$$S_k = H_k \ominus H_{k+1},$$

$$\mathcal{F} = \left\{ \begin{array}{ll} f; & \text{measurable with respect to } \mathcal{M}_k \\ & \text{for some } k, \ -\infty < k < \infty. \end{array} \right\}.$$

Finally, for $f \in L_2$, let $C_n(f) = \sum_{k=0}^{n-1} U^K f$ be the partial sums and $X_n(f)$ be the random element of Skorohod space $D_{[0,1]}$ whose value at t is

$$X_n(f, \mathrm{t}) = \frac{1}{\sigma\sqrt{n}} C_{[nt]}(f), \tag{1.1}$$

where $X_n(f, \mathrm{t}) = 0$ if $[nt] < 1$, σ is a suitable positive constant and $0 \leqslant t \leqslant 1$.

In Section 2 we shall investigate the problem of finding the conditions which the relation

$$X_n \xrightarrow{D} W \tag{1.2}$$

holds, where W denotes Wiener process and $X_n \xrightarrow{D} W$ means that X_n converges in distribution to W.

In Section 3 we shall extend the results in Section 2 to the case of stationary processes with a continuous time parameter.

In Section 4, by using the preliminary lemma given by RÉNYI (1958), we shall show that the results in Section 2 remain true whenever P is replaced by an arbitrary probability measure P_0 on $(X, \mathcal{M})$ dominated by (absolutely continuous with respect to) P.

In Section 5 we shall prove Donsker's invariance principle for randomly selected partial sums by the same manner as P. BILLINGSLEY (1968, § 17).

In Section 6 the results obtained in this paper are applied to renewal theory.

2. Main Results

We first show the following theorem which implies Theorems 20.1 and 21.1 of BILLINGSLEY (1968).

Theorem 1.* Let T be ergodic and $f \in L_2$ such that

$$\sum_{a \geq 0} (|P_{H_a} f|_2 + |f - P_{H_{-a}} f|_2) < \infty.$$

* D. J. SCOTT (1973) Proved this theorem by using a Skorohod representation approach.

Then

$$\lim_{n\to\infty} |C_n(f)|_2/\sqrt{n} = \sigma; \tag{2.1}$$

if $\sigma > 0$ and X_n is defined by (1.1), then

$$X_n \xrightarrow{D} W. \tag{2.2}$$

Proof. For non-negative integer a and $f \in L_2$, define

$$f_1^{(a)} = P_{H_a} f, \qquad f_2^{(a)} = f - P_{H_{-a}} f,$$

$$f_a = f - f_1^{(a)} - f_2^{(a)}, \qquad h_a = \sum_{n=-\infty}^{\infty} U^{-n} P_{S_n} f_a,$$

$$g_a = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{-n-1} U^m P_{S_n} f_a,$$

then we have

$$f = h_a + g_a - U g_a + f - f_a. \tag{2.3}$$

From Theorems 1 and 2 of GORDIN (1969), we have relation (2.1), and from the proof of Theorem 1 of GORDIN (1969), it follows that

$$\sigma_a \longrightarrow \sigma, \quad \text{as} \quad a \longrightarrow \infty, \tag{2.4}$$

where $\sigma_a = |h_a|_2$. Since $\sigma > 0$, there is no loss of generality in assuming $\sigma_a > 0$ for each a.

Since $h_a \in S_0$, it follows that sequence $\{U^{-n} h_a\}$ is ergodic sequence of martingale differences. Therefore, from Theorem 23.1 of BILLINGSLEY (1968), we have

$$X_n(h_a) \xrightarrow{D} \frac{\sigma_a}{\sigma} W, \quad \text{as} \quad n \longrightarrow \infty \tag{2.5}$$

for each a, and from (2.4), we have also

$$\frac{\sigma_a}{\sigma} W \xrightarrow{D} W, \quad \text{as} \quad a \longrightarrow \infty. \tag{2.6}$$

Then, because of (2.5) and (2.6), the relation (2.2) will follow by Theorem 4.2 of BILLINGSLEY (1968) if we show that

$$\lim_{a\to\infty} \limsup_{n\to\infty} P\{d_0(X_n(f), X_n(h_a)) > \varepsilon\} = 0 \tag{2.7}$$

for each positive ε, where d_0 denotes the Skorohod topology on $D_{[0,1]}$. Since Skorohod topology is dominated by uniform topology, (2.7) will follow if we show that for each positive ε

$$\lim_{a\to\infty} \limsup_{n\to\infty} P\left\{\max_{1\leq i\leq n} \left|\frac{1}{\sqrt{n}} \sum_{k=0}^{i-1} U^k(f - h_a)\right| > 8\varepsilon\right\} = 0. \tag{2.8}$$

From (2.3), we have

$$P\left\{\max_{1\leq i\leq n}\left|\frac{1}{\sqrt{n}}\sum_{k=0}^{i-1}U^k(f-h_a)\right|>8\varepsilon\right\}$$

$$\leq P\left\{\max_{1\leq i\leq n}\left|\frac{1}{\sqrt{n}}\sum_{k=0}^{i-1}U^k(f-f_a)\right|>4\varepsilon\right\}$$

$$+P\left\{\max_{1\leq i\leq n}\left|\frac{1}{\sqrt{n}}\sum_{k=0}^{i-1}U^k(g_a-Ug_a)\right|>4\varepsilon\right\}$$

$$=J_1+J_2. \tag{2.9}$$

First of all, we shall prove that $\lim_{a\to\infty}\limsup_{n\to\infty} J_1=0$. From the definition of f_a, we have

$$J_1\leq\sum_{j=1}^{2}P\left\{\max_{1\leq i\leq n}\left|\frac{1}{\sqrt{n}}\sum_{k=0}^{i-1}U^k f_j^{(a)}\right|>2\varepsilon\right\}=\sum_{j=1}^{2}J_{1,j}. \tag{2.10}$$

Define

$$\xi_i=U^{i-1}E\{f+\cdots+fT^{-i+1}||\mathcal{M}_a\}$$
$$-U^iE\{fT^{-1}+\cdots+fT^{-i}||\mathcal{M}_a\},\qquad i=1,2,\cdots\cdots,$$

then

$$J_{1,1}\leq P\left\{\max_{1\leq i\leq n}\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{i}\xi_k\right|>\varepsilon\right\}$$

$$+P\left\{\max_{1\leq i\leq n}\left|\frac{1}{\sqrt{n}}U^i\sum_{k=1}^{i}E\{fT^{-k}||\mathcal{M}_a\}\right|>\varepsilon\right\}. \tag{2.11}$$

Since $\xi_i\in H_{a+i-1}\ominus H_{a+i}$, it follows by martingale-inequality (DOOB, 1953, p317) that

$$P\left\{\max_{1\leq i\leq n}\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{i}\xi_k\right|>\varepsilon\right\}\leq\frac{4}{\varepsilon^2 n}E\left|\sum_{k=1}^{n}\xi_k\right|^2\leq\frac{4}{\varepsilon^2 n}E\left|\sum_{k=0}^{n-1}U^kf_1^{(a)}\right|^2. \tag{2.12}$$

Next, we shall estimate the second term of (2.11).

$$P\left\{\max_{1\leq i\leq n}\left|\frac{1}{\sqrt{n}}U^i\sum_{k=1}^{i}E\{fT^{-k}||\mathcal{M}_a\}\right|>\varepsilon\right\}$$

$$\leq\sum_{i=1}^{n}P\left\{\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{i}E\{fT^{-k}||\mathcal{M}_a\}\right|>\varepsilon\right\}$$

$$\leq\sum_{i=1}^{n}\frac{1}{\varepsilon^2 n}E\left|\sum_{k=1}^{i}E\{fT^{-k}||\mathcal{M}_a\}\right|^2.$$

Furthermore, from Minkowski's inequality

$$\left|\sum_{k=1}^{i}E\{fT^{-k}||\mathcal{M}_a\}\right|_2\leq\sum_{k=1}^{i}\left|E\{fT^{-k}||\mathcal{M}_a\}\right|_2$$

$$=\sum_{k=1}^{i}\left|E\{f||\mathcal{M}_{a+k}\}\right|_2\leq\sum_{k\geq a}\left|P_{H_k}f\right|_2.$$

Then we have

$$P\left\{\max_{1\leqslant i\leqslant n}\left|\frac{1}{\sqrt{n}}U^{i}\sum_{k=1}^{i}E\{fT^{-k}||\mathcal{M}_{a}\}\right|>\varepsilon\right\}\leqslant\frac{1}{\varepsilon^{2}}\left(\sum_{k\geq a}\left|P_{H_k}f\right|_2\right)^2. \tag{2.13}$$

From (2.11), (2.12) and (2.13), it follows that

$$J_{1,1}\leqslant\frac{4}{\varepsilon^{2}n}E\left|\sum_{k=0}^{n-1}U^{k}f_{1}^{(a)}\right|^{2}+\frac{1}{\varepsilon^{2}}\left(\sum_{k\geq a}\left|P_{H_k}f\right|_2\right)^2. \tag{2.14}$$

Now, we shall estimate $J_{1,2}$ by the same manner as $J_{1,1}$.

Define

$$\eta_i=U^{-i+1}\sum_{k=0}^{i-1}(fT^{k}-E\{fT^{k}||\mathcal{M}_{-a}\})$$

$$-U^{-i}\sum_{k=1}^{i}(fT^{k}-E\{fT^{k}||\mathcal{M}_{-a}\}),\qquad i=1,2,\cdots,$$

then

$$J_{1,2}\leqslant P\left\{\max_{1\leqslant i\leqslant n}\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{i}\eta_k\right|>\varepsilon\right\}$$

$$+P\left\{\max_{1\leqslant i\leqslant n}\left|\frac{1}{\sqrt{n}}U^{-i}\sum_{k=1}^{i}(fT^{k}-E\{fT^{k}||\mathcal{M}_{-a}\})\right|>\varepsilon\right\}. \tag{2.15}$$

Since $\eta_i\in H_{-a-i}\ominus H_{-a-i+1}$, it follows by martingale-inequality (Doob, 1953, p317) that

$$P\left\{\max_{1\leqslant i\leqslant n}\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{i}\eta_k\right|>\varepsilon\right\}\leqslant\frac{4}{\varepsilon^{2}n}E\left|\sum_{k=0}^{n-1}U^{k}f_{2}^{(a)}\right|^{2}, \tag{2.16}$$

and we have by the same manner as (2.13)

$$P\left\{\max_{1\leqslant i\leqslant n}\left|\frac{1}{\sqrt{n}}U^{-i}\sum_{k=1}^{i}(fT^{k}-E\{fT^{k}||\mathcal{M}_{-a}\})\right|>\varepsilon\right\}$$

$$\leqslant\frac{1}{\varepsilon^{2}}\left(\sum_{k\geq a}\left|f-P_{H_{-k}}f\right|_2\right)^2. \tag{2.17}$$

Therefore, from (2.15), (2.16) and (2.17), we have

$$J_{1,2}\leqslant\frac{4}{\varepsilon^{2}n}E\left|\sum_{k=0}^{n-1}U^{k}f_{2}^{(a)}\right|^{2}+\frac{1}{\varepsilon^{2}}\left(\sum_{k\geq a}\left|f-P_{H_{-k}}f\right|_2\right)^2 \tag{2.18}$$

From Theorem 2 of Gordin (1969), it follows that

$$\lim_{a\to\infty}\limsup_{n\to\infty}\left|\sum_{k=0}^{n-1}U^{k}(f_{1}^{(a)}+f_{2}^{(a)})\right|_2\Big/\sqrt{n}=0,$$

so that, from (2.10), (2.14) and (2.18), we have

$$\lim_{a\to\infty}\limsup_{n\to\infty} J_1=0.$$

We turn now to the proof of $\lim_{a\to\infty}\limsup_{n\to\infty} J_2=0.$

$$J_2=P\left\{\max_{1\leqslant i\leqslant n}\left|\frac{1}{\sqrt{n}}\sum_{k=0}^{i-1}U^k(g_a-Ug_a)\right|>4\varepsilon\right\}$$

$$\leqslant P\left\{\frac{1}{\sqrt{n}}\left|g_a\right|>\varepsilon\right\}+P\left\{\max_{1\leqslant i\leqslant n}\left|\frac{1}{\sqrt{n}}U^i g_a\right|>\varepsilon\right\}$$

$$=J_{2,1}+J_{2,2}. \tag{2.19}$$

Since $g_a\in L_2$, we have

$$\lim_{n\to\infty} J_{2,1}=0 \tag{2.20}$$

for each a, and from absolutely continuity of Lebesgue integral, there exists a positive number $\rho(\mu)$ for each positive μ such that

$$\int_A |g_a|^2 dP<\mu \tag{2.21}$$

for every measurable set A for which $P(A)<\rho$. Furthermore, there exists a positive number $n_0(\varepsilon,\rho)$ such that $n\geq n_0(\varepsilon,\rho)$ implies

$$P\left\{\frac{1}{\sqrt{n}}\left|g_a\right|>\varepsilon\right\}\leqslant\rho, \tag{2.22}$$

By Chebyshev's inequality

$$J_{2,2}\leqslant\sum_{i=1}^{n}P\left\{\frac{1}{\sqrt{n}}U^i\left|g_a\right|>\varepsilon\right\}$$

$$\leqslant\sum_{i=1}^{n}\frac{1}{\varepsilon^2 n}\int_{T^{-i}\left\{\frac{1}{\sqrt{n}}|g_a|>\varepsilon\right\}}|g_a|^2 dP,$$

it follows from (2.21) and (2.22) that $J_{2,2}\leqslant\mu/\varepsilon^2$ holds for $n\geq n_0$ (ε,μ). Since μ is arbitrary small, we have for each a

$$\lim_{n\to\infty} J_{2,2}=0,$$

so that, from (2.19) and (2.20), we have

$$\lim_{a\to\infty}\limsup_{n\to\infty} J_2=0.$$

This completes the proof of Theorem 1.

If T is a Kolmogorov automorphism and $f\in\mathcal{F}$, then the condition of Theorem 1 is weakened as one in Theorem 2. In the next theorem, $\mathcal{M}_0$ is a Borel field determined by the definition of Kolmogorov automorphism.

Theorem 2. Let T be a Kolmogorov automorphism and $f\in L_{2+\delta}$ for some $0\leqslant\delta\leqslant\infty$ such that $f\in\mathcal{F}$ and

$$\sum_{a\geq 0}\left|P_{H_a}f\right|_{\frac{2+\delta}{1+\delta}}<\infty.$$

Then

$$\lim_{n\to\infty}|C_n(f)|_2/\sqrt{n}=\sigma;$$

if $\sigma>0$ and X_n is defined by (1.1), then

$$X_n \xrightarrow{D} W.$$

Proof. Since we can estimate all the terms except the second term of (2.11) by the same manner as the proof of Theorem 1, Theorem 2 is proved if we show that

$$\lim_{a\to\infty}\limsup_{n\to\infty}P\left\{\max_{1\leq i\leq n}\left|\frac{1}{\sqrt{n}}U^i\sum_{k=1}^{i}E\{fT^{-k}||\mathcal{M}_a\}\right|>\varepsilon\right\}=0 \tag{2.23}$$

for each positive ε.

Since T is a Kolmogorov auto., it follows that for any $B\in\mathcal{M}$

$$\lim_{k\to\infty}\alpha(k|B)=0, \tag{2.24}$$

where

$$\begin{aligned}\alpha(k|B)&=\sup_{A\in\mathcal{M}_k}|P(AB)-P(A)P(B)|\\ &=\sup_{A\in\mathcal{M}_0}|P(A\cap T^kB)-P(A)P(B)|.\end{aligned}$$

To prove (2.23), consider the sets

$$A_{a,n}=\left\{\max_{1\leq i\leq n}\left|\frac{1}{\sqrt{n}}U^i\sum_{k=1}^{i}E\{fT^{-k}||\mathcal{M}_a\}\right|>\varepsilon\right\},$$

$$A_{a,n,j}=\left\{\begin{array}{l}\max_{1\leq i<j}\left|\frac{1}{\sqrt{n}}U^i\sum_{k-1}^{i}E\{fT^{-k}||\mathcal{M}_a\}\right|\leq\varepsilon\\ \text{and }\left|\frac{1}{\sqrt{n}}U^j\sum_{k=1}^{j}E\{fT^{-k}||\mathcal{M}_a\}\right|>\varepsilon\end{array}\right\},$$

$$B_a=\left\{\sum_{k=1}^{\infty}\left|E\{fT^{-k}||\mathcal{M}_a\}\right|>\varepsilon\right\}.$$

First of all, we shall estimate $P(A_{a,n})$ for a such that $P(B_a)=0$. By Chebyshev's inequality,

$$\begin{aligned}P(A_{a,n})&=\sum_{j=1}^{n}P(A_{a,n,j})\\ &\leq\sum_{j=1}^{n}\frac{1}{\varepsilon^2 n}\int\left|U^j\sum_{k=1}^{j}E\{fT^{-k}||\mathcal{M}_a\}\right|^2dP\\ &=\frac{1}{\varepsilon^2}\int_{B_a^c}\left(\sum_{k=1}^{n}\left|E\{fT^{-k}||\mathcal{M}_a\}\right|\right)^2dP,\end{aligned}$$

so that, from the definition of B_a and MINKOWSKI's inequality, we have

$$P(A_{a,n}) \leqslant \frac{1}{\varepsilon} \int \sum_{k=1}^{n} \left| E\{fT^{-k} \| \mathcal{M}_a\} \right| dP$$

$$\leqslant \frac{1}{\varepsilon} \sum_{k \geq a} \left| P_{H_k} f \right|_{\frac{2+\delta}{1+\delta}}. \tag{2.25}$$

Next, we shall estimate $P(A_{a,n})$ for a such that $P(B_a)>0$. By Fubini's theorem

$$P(A_{a,n} \cap T^n B_a) = \sum_{j=1}^{n} P(A_{a,n,j} \cap T^n B_a)$$

$$\leqslant \sum_{j=1}^{n} \sum_{k=1}^{\infty} \frac{1}{\varepsilon} \int_{A_{a,n,j}} U^{-n} \left| E\{fT^{-k} \| \mathcal{M}_a\} \right| dP.$$

From well-known properties of conditional expectation, it follows that for any square integrable functions f and g, and for any integer n,

$$\int E\{f \| \mathcal{M}_a\} \cdot U^{-n} E\{g \| \mathcal{M}_a\}\, dP$$

$$= \int E\{f \| \mathcal{M}_a\} \cdot U^{-n} E\{g \| \mathcal{M}_{a+n}\}\, dP,$$

so that, we have

$$\int_{A_{a,n,j}} U^{-n} \left| E\{fT^{-k} \| \mathcal{M}_a\} \right| dP$$

$$= \int_{A_{a,n,j}} U^{-n} \left| E\{fT^{-k} \| \mathcal{M}_{a+n}\} \right| dP.$$

Therefore, we have

$$P(A_{a,n} \cap T^n B_a)$$

$$\leqslant \sum_{j=1}^{n} \sum_{k=1}^{\infty} \frac{1}{\varepsilon} \int_{A_{a,n,j}} U^{-n} \left| E\{fT^{-k} \| \mathcal{M}_{a+n}\} \right| dP$$

$$= \sum_{j=1}^{n} \frac{1}{\varepsilon} \int_{A_{a,n,j}} U^{-n} \sum_{k=1}^{\infty} \left| E\{fT^{-k} \| \mathcal{M}_{a+n}\} \right| dP$$

$$\leqslant \frac{1}{\varepsilon} \sum_{k \geq a+n} \left| P_{H_k} f \right|_{\frac{2+\delta}{1+\delta}}. \tag{2.26}$$

Since $A_{a,n}$ lies in $\mathcal{M}_0$ for any non-negative a, we have

$$P(A_{a,n} \cap T^n B_a^c) \leqslant P(A_{a,n}) P(B_a^c) + \alpha(n | B_a^c). \tag{2.27}$$

Since $P(B_a)>0$, it follows by (2.26) and (2.27) that

$$P(A_{a,n}) \leqslant \frac{1}{P(B_a)} \left\{ \frac{1}{\varepsilon} \sum_{k \geq a+n} \left| P_{H_k} f \right|_{\frac{2+\delta}{1+\delta}} + \alpha(n | B_a^c) \right\},$$

so that, from (2.24) and assumption, we have

$$\lim_{n\to\infty} P(A_{a,n})=0 \tag{2.28}$$

for a such that $P(B_a)>0$.

Therefore, from (2.25) and (2.28), it follows that for any a

$$\limsup_{n\to\infty} P(A_{a,n})\leqslant\frac{1}{\varepsilon}\sum_{k\ a}\left| P_{H_k}f\right|_{\frac{2+\delta}{1+\delta}},$$

so that we have (2.23). This completes the proof of Theorem 2.

3. Functional Central Limit Theorem in Continuous Time

Theorem 1 has a natural formulation with measure-preserving transformation T replaced by a flow $\{T_t;\ t\in R^1\}$. Let $\{T_t;\ t\in R^1\}$ be a flow defined on $(X, \mathcal{M}, P)$ and $\mathcal{M}_0$ be a Borel field such that $T_s\mathcal{M}_0\subset T_t\mathcal{M}_0$ for every $s<t$. Relation $U_tf(x)=f(T_t x)$ defines a group of unitary transformation $\{U_t;\ t\in R^1\}$ on H.

Spaces $\mathcal{M}_t$, H_t and $\mathcal{F}$ are defined by the same manner as §1, i.e.

$$\mathcal{M}_t=T_{-t}\mathcal{M}_0,$$

$$H_t=H(\mathcal{M}_t),$$

$$\mathcal{F}=\left\{\begin{array}{ll} f; & \text{measurable with respect to } \mathcal{M}_t \\ & \text{for some } t\in R^1. \end{array}\right\}.$$

As a corollary of Theorem 1 we have the following:

Corollary 1. Let $\{T_t;\ t\in R^1\}$ be a weakly mixing flow and $f\in L_2$ such that

$$\int_0^\infty (|P_{H_t}f|_2+|f-P_{H_{-t}}f|_2)dt<\infty. \tag{3.1}$$

Then

$$\lim_{t\to\infty}\left|\frac{1}{\sqrt{t}}\int_0^t U_sf(x)ds\right|_2=\sigma;$$

if $\sigma>0$ and Y_n is defined by

$$Y_n(f, t)=\frac{1}{\sigma\sqrt{n}}\int_0^{nt} U_sf(x)ds,\quad 0\leqslant t\leqslant 1,$$

then

$$Y_n(f)\xrightarrow{D} W. \tag{3.2}$$

Proof. If

$$f_0(x)=\int_0^1 U_s f(x)ds,$$

then, by Fubini's theorem

$$E|f_0|^2 \leqslant \int_X \int_0^1 \left| U_s f(x) \right|^2 ds dP = E|f|^2 < \infty. \tag{3.3}$$

Since $E\left\{ \int_0^1 U_s f\, ds \,\middle\|\, \mathcal{M}_a \right\} = \int_0^1 E\{U_s f \,\|\, \mathcal{M}_a\}\, ds$ with probability 1, we have

$$E|P_{H_a} f_0|^2 = E\left| \int_0^1 E\{U_s f \,\|\, \mathcal{M}_a\}\, ds \right|^2$$

$$= \int_0^1 \int_0^1 E(E\{U_s f \,\|\, \mathcal{M}_a\} \cdot E\{U_t f \,\|\, \mathcal{M}_a\}) ds dt$$

$$\leqslant \int_0^1 E|E\{U_s f \,\|\, \mathcal{M}_a\}|^2 ds$$

$$\leqslant E|E\{f \,\|\, \mathcal{M}_a\}|^2 = E|P_{H_a} f|^2.$$

Similarly,

$$E|f_0 - P_{H_{-a}} f_0|^2 \leqslant E|f - P_{H_{-a}} f|^2.$$

Since $E|P_{H_t} f|^2$ and $E|f - P_{H_{-t}} f|^2$ are non-increasing functions of t, condition (3.1) shows that

$$\sum_{a \geq 0} (|P_{H_a} f_0|_2 + |f_0 - P_{H_{-a}} f_0|_2) < \infty. \tag{3.4}$$

Because of (3.3) and (3.4), it follows by Theorem 1 that

$$\lim_{n\to\infty} \left| \sum_{k=0}^{n-1} U_1^k f_0 \right|_2 / \sqrt{n} = \sigma_0,$$

but it is easily proved that $\sigma_0 = \sigma$. Furthermore, we have

$$X_n(f_0) \xrightarrow{D} W,$$

where $X_n(f_0)$ is defined by (1.1). Therefore, relation (3.2) will follow by Theorem 4.1 of Billingsley (1968) if we show that

$$\delta_n \xrightarrow{P} 0,$$

where

$$\delta_n = \sup_t |X_n(f_0, t) - Y_n(f, t)|.$$

Now

$$\delta_n \leqslant \frac{1}{\sigma\sqrt{n}} \max_{1 \leqslant i \leqslant n} \int_{i-1}^{i} |U_s f|\, ds,$$

so that

$$P\{\delta_n \geq \varepsilon\} \leqslant \frac{1}{\sigma^2\varepsilon^2}\int_{\{|f_0(x)| > \sigma\varepsilon\sqrt{n}\}} |f_0(x)|^2 dP.$$

Since $E|f_0|^2 < \infty$, we have $\delta_n \xrightarrow{P} 0$. This completes the proof of Corollary 1.

Remark. (1) The relation (3.2) persists if n goes to infinity in a continuous manner.
(2) Gordin's Theorem will be extended by the same manner as Corollary 1.

4. Dominated Measures, Nonstationarity

We shall return to the case of discrete time. In this section, we shall show that Theorem 2 remain true if P is replaced by an arbitrary probability measure P_0 on $(X, \mathcal{M})$ dominated by P. Under P_0 transformation T need not be measure-preserving, i. e. process $\{U^n f;\ n=0, \pm 1, \cdots\}$ need not be stationary.

We shall need the following preliminary result given by Rényi (1958) (in which $\sigma(\mathcal{M}^*)$ denotes the Borel field generated by $\mathcal{M}^*$).

Lemma (Rényi). Let $E_1, E_2, \cdots$ be measurable sets in a probability space $(X, \mathcal{M}, P)$. Suppose there exist a constant α and a subfield $\mathcal{M}^*$ of $\mathcal{M}$ such that

$$P(E_n \cap E) \longrightarrow \alpha P(E)$$

for every E in $\mathcal{M}^*$. Suppose further that all the E_n lie in $\sigma(\mathcal{M}^*)$. If P dominates P_0, a second probability measure on $\mathcal{M}$, then

$$P_0(E_n) \longrightarrow \alpha.$$

In this section, symbol $|f|_{P,q}$ denotes norm of a function f in $L_q(P)$.

Theorem 3. Let T be a Kolmogorov automorphism, P_0 be a probabity measure on $\mathcal{M}$ dominated by P and $f \in L_{2+\delta}(P)$ for some $0 \leqslant \delta \leqslant \infty$ such that $f \in \mathcal{F}$ and

$$\sum_{a \geq 0} \left| P_{H_a} f \right|_{\frac{2+\delta}{1+\delta}} < \infty.$$

If $\sigma > 0$ and X_n is defined by (1.1), then

$$P_0\{X_n \in A\} \longrightarrow W(A) \tag{4.1}$$

for every W-continuity set A in $D_{[0,1]}$, where

$$\sigma = \lim_{n\to\infty} \left| \sum_{k=0}^{n-1} U^k f \right|_{P,2} \Big/ \sqrt{n}\,.$$

Remark. (1) In (4.1), symbol W denotes Wiener measure.
(2) Of cause, (4.1) implies

$$P_0\left\{\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}U^k f\leqslant x\right\}\longrightarrow\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{u^2}{2\sigma^2}}du.$$

Now, we shall prove Theorem 3 by the same manner as the proof of Theorem 16.3 of BILLINGSLEY (1968).

Proof. Define X'_n by

$$X'_n(f,t)=\frac{1}{\sigma\sqrt{n}}\sum_{k=p_n}^{[nt]-1}U^k f,$$

where $\{p_n\}$ is a sequence of integers going to infinity slowly enough that $p_n/\sqrt{n}\longrightarrow 0$ ($X'_n(f,t)=0$ if $[nt]<p_n+1$). If

$$\delta_n=\sup_t|X'_n(t)-X_n(t)|,$$

then

$$\delta_n\leqslant\frac{1}{\sigma\sqrt{n}}\sum_{i=0}^{p_n-1}\left|U^k f\right|.$$

By Minkowski's inequality and the fact that $p_n/\sqrt{n}\longrightarrow 0$,

$$\begin{aligned}|\delta_n|_{P,2}&\leqslant\frac{1}{\sigma\sqrt{n}}\sum_{i=0}^{p_n-1}\left|U^k f\right|_{P,2}\\&=\frac{p_n}{\sigma\sqrt{n}}\left|f\right|_{P,2}\longrightarrow 0,\end{aligned}$$

so that, we have

$$\delta_n\xrightarrow{P}0,\tag{4.2}$$

where (4.2) is interpreted in the sense of P.

By Theorem 2, we have

$$X_n\xrightarrow{D}W,\tag{4.3}$$

where again the relation is interpreted in the sense of P. Since Skorohod topology is dominated by uniform topology, it follows by (4.2), (4.3) and Theorem 4.1 of BILLINGSLEY (1968) that

$$X'_n\xrightarrow{D}W\tag{4.4}$$

in the sense of P.

Let A be a W-continuity set in $D_{[0,1]}$, temporarily fixed; (4.4) implies

$$P\{X'_n\in A\}\longrightarrow W(A).\tag{4.5}$$

If $E\in\mathcal{M}$, then, since $f\in\mathcal{F}$, $p_n\longrightarrow\infty$ and T is a Kolmogorov auto., we have

$$|P(\{X'_n\in A\}\cap E)-P\{X'_n\in A\}P(E)|\longrightarrow 0,$$

so that, it follows by (4.5) that

$$P(\{X'_n \in A\} \cap E) \longrightarrow W(A)P(E).$$

Therefore, it follows by Rényi's lemma that

$$P_0\{X'_n \in A\} \longrightarrow W(A). \tag{4.6}$$

Since (4.6) holds for every W-continuity set A, (4.4) holds when interpreted in the sense of P_0. Since P_0 is absolutely continuous with respect to P, it follows by (4.2) that for any positive ε

$$P_0\{\delta_n \geq \varepsilon\} \longrightarrow 0.$$

Therefore, applying Theorem 4.1 of BILLINGSLEY (1968) once more, we see that (4.3) holds in the sense of P_0, which completes the proof.

5. Randomly Selected Partial Sums

Sometimes one require an approximate distribution for a partial sum $C_\nu(f) = f + Uf + \cdots + U^{\nu-1}f$, where the index ν is itself a random variable. Here we shall prove several functional central limit theorems for such randomly selected partial sums.

To formulate a limit theorem, consider a sequence $\{\nu_n\}$ of positive-integer-valued random variables defined on $(X, \mathcal{M}, P)$. We seek conditions under which

$$Y_n \xrightarrow{D} W, \quad \text{as} \quad n \longrightarrow \infty, \tag{5.1}$$

where

$$Y_n(f, t) = \frac{1}{\sigma\sqrt{\nu_n}} C_{[\nu_n t]}(f), \qquad 0 \leqslant t \leqslant 1. \tag{5.2}$$

Now BILLINGSLEY (1968) proved the interesting result as follows.

Lemma (BILLINGSLEY, 1968, Theorem 17.1). If

$$\frac{\nu_n}{a_n} \xrightarrow{P} \theta,$$

where θ is a positive constant and the a_n are constants going to infinity, then

$$X_n(f) \xrightarrow{D} W$$

implies (5.1), where $X_n(f)$ is defined by (1.1).

As a corollary of Theorems 1 and 2, we have the following results by using this BILLINGSLEY's lemma.

Corollary 2. Let T be ergodic and $f \in L_2$ such that

$$\sum_{a\geq 0}(|P_{H_a}f|_2+|f-P_{H_{-a}}f|_2)<\infty.$$

If

$$\frac{\nu_n}{a_n}\xrightarrow{P}\theta,$$

where θ is a positive constant and the a_n are constants going to infinity, then (if $\sigma>0$)

$$Y_n\xrightarrow{D}W,$$

where Y_n is defined by (5.2) and

$$\sigma=\lim_{n\to\infty}\left|\sum_{k=0}^{n-1}U^k f\right|_2\Big/\sqrt{n}\,.$$

When T is a Kolmogorov automorphism, we have the following result. In the remainder of this section, $\mathcal{M}_0$ is a Borel field determined by the definition of Kolmogorov auto.

Corollary 3. Let T be a Kolmogorov auto. and $f\in L_{2+\delta}$ for some $0\leqslant\delta\leqslant\infty$ such that $f\in\mathcal{F}$ and

$$\sum_{a\geq 0}\left|P_{H_a}f\right|_{\frac{2+\delta}{1+\delta}}<\infty.$$

If

$$\frac{\nu_n}{a_n}\xrightarrow{P}\theta,$$

where θ is a positive constant and the a_n are constants going to infinity, then (if $\sigma>0$)

$$Y_n\xrightarrow{D}W,$$

where Y_n, σ are the same as those in Coro. 2.

Next, we shall show that Corollary 3 remains valid even if θ is not constant.

Theorem 4. Let T be a Kolmogorov auto. and $f\in L_{2+\delta}$ for some $0\leqslant\delta\leqslant\infty$ such that $f\in\mathcal{F}$ and

$$\sum_{a\geq 0}\left|P_{H_a}f\right|_{\frac{2+\delta}{1+\delta}}<\infty.$$

If

$$\frac{\nu_n}{a_n}\xrightarrow{P}\theta,$$

where θ is a positive random variable and the a_n are constants going to infinity, then (if $\sigma>0$)

$$Y_n \xrightarrow{D} W,$$

where Y_n, σ are the same as those in Coro. 2.

Proof. Define $X_n(f)$ by (1.1) and $X'_n(f)$ by

$$X'_n(f, t) = \frac{1}{\sigma\sqrt{n}} \sum_{k=p_n}^{[nt]-1} U^k f,$$

where $\{p_n\}$ is a sequence of positive integers going to infinity slowly enough that $p_n/\sqrt{n} \longrightarrow 0$ ($X'_n(f, t)=0$ if $[nt]<p_n+1$). Then, as in § 4, we have

$$\sup_t | X_n(f, t) - X'_n(f, t) | \xrightarrow{P} 0$$

and

$$P(\{X'_n \in A\} \cap E) \longrightarrow W(A)P(E)$$

for every W-continuity set A and for every E in $\mathcal{M}$. The proof now goes through precisely as the proof of Theorem 17.2 of BILLINGSLEY (1968).

6. Application to Renewal Theory

BILLINGSLEY (1968) proved a interesting result for functional central limit theorem connected with renewal theory. Here we shall state several properties which are immediate consequences from BILLINGSLEY's result.

Let $f(x)$, lies in L_2, be a positive function and define

$$\nu_\tau(x) = \max\left\{k;\ \sum_{i=0}^{k-1} U^i f(x) \leqslant \tau\right\}, \quad \tau \geq 0,$$

where $\nu_\tau(x)=0$ if $f(x)>\tau$. Furthermore, we shall define two random elements in $D_{[0,1]}$ by

$$X_n(f, t) = \frac{1}{\sigma\sqrt{n}} \sum_{i=0}^{[nt]-1} (U^i f - \mu),$$

$$Z_n(f, t) = (\nu_{nt} - nt/\mu)/(\sigma\mu^{-3/2}\sqrt{n}).$$

where $0 \leqslant t \leqslant 1$ and μ, σ are suitable positive numbers.

BILLINGSLEY (1968) proves that $X_n \xrightarrow{D} W$ implies $Z_n \xrightarrow{D} W$. Therefore, we have the following properties from the results in Section 2 and 4.

Corollary 4. Let T be ergodic and $f \in L_2$ such that $f>0$ and for some $\mu>0$,

$$\sum_{a\geq 0} \left(\left| P_{H_a} f - \mu \right|_2 + \left| f - P_{H_{-a}} f \right|_2 \right) < \infty.$$

Then (if $\sigma>0$)

$$Z_n \xrightarrow{D} W,$$

where $\sigma=\lim_{n\to\infty}\left|\sum_{k=0}^{n-1}(U^k f-\mu)\right|_2 \Big/ \sqrt{n}$.

When T is a Kolmogorov auto., we have the following corollary. In the remainder of this section, $\mathcal{M}_0$ is a Borel field determined by the definition of Kolmogorov auto..

Corollary 5. Let T be a Kolmogorov auto. and $f\in L_{2+\delta}$ for some $0\leqslant\delta\leqslant\infty$ such that $f>0$, $f\in\mathcal{F}$ and

$$\sum_{a\geq 0}\left|P_{H_a}f-\mu\right|_{\frac{2+\delta}{1+\delta}}<\infty$$

for some $\mu>0$. Then (if $\sigma>0$)

$$Z_n \xrightarrow{D} W,$$

where $\sigma=\lim_{n\to\infty}\left|\sum_{k=0}^{n-1}(U^k f-\mu)\right|_2 \Big/ \sqrt{n}$.

The following corollary shows that Corollary 5 remains true if P is replaced by an arbitrary probability measure P_0 on $\mathcal{M}$ dominated by P. Symbols in the following corollary are the same as those in Section 4.

Corollary 6. Let T be a Kolmogorov auto., P_0 be a probability measure on $\mathcal{M}$ dominated by P and $f\in L_{2+\delta}(P)$ for some $0\leqslant\delta\leqslant\infty$ such that $f>0$, $f\in\mathcal{F}$ and

$$\sum_{a\geq 0}\left|P_{H_a}f-\mu\right|_{P,\frac{2+\delta}{1+\delta}}<\infty$$

for some $\mu>0$. Then (if $\sigma>0$)

$$P_0\{Z_n\in A\} \longrightarrow W(A) \tag{6.1}$$

for every W-continuity set A in $D_{[0,1]}$, where

$$\sigma=\lim_{n\to\infty}\left|\sum_{k=0}^{n-1}(U^k f-\mu)\right|_{P,2} \Big/ \sqrt{n}.$$

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