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By

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CONSISTENCY OF A PAIR OF MATRIX EQUATIONS WITH AN APPLICATION

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ABSTRACT

Some necessary and sufficient conditions for a pair of consistent linear matrix equations to have a common solution are given. A simpler algebraic proof of Mitra's theorem on the rank of sum of non-negative definite matrices is also given.

1. Introduction

The first purpose of this note is to show some necessary and sufficient conditions for a pair of matrix equations

(1.1) $$A_1XB_1=C_1$$

and

(1.2) $$A_2XB_2=C_2,$$

each of them is consistent, to have a common solution (Theorem 1). One of the conditions was obtained by Mitra (1973a). Our new ones and the proof are more rudimental. The problem considered here is different from the one treated by Morris and Odell (1968) who have not assumed given matrix equations to be consistent.

Studying the algebraic network theory Mitra (1973b) proved a theorem on the rank of sum of n.n.d. (non-negative definite) matrices based on statistical considerations. The second purpose of this note is to give a simpler algebraic

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proof of Mitra's theorem (Theorem 2). The proof is partly based on Theorem 1 and motivated this note.

Capital letters denote matrices with complex elements throughout the paper. For a complex matrix A, A^* denotes its transposed conjugate and A^- a generalized inverse of A, that is a matrix such that $AA^-A=A$. $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are the range space and the null space of A respectively. O denotes the zero matrix of suitable size.

2. Main Results

Lemma. Let A_1, A_2, A; B_1, B_2, and B be matrices such that

(2) $$\mathcal{R}(A^*)=\mathcal{R}(A_1^*)\cap\mathcal{R}(A_2^*) \quad\text{and}\quad \mathcal{R}(B)=\mathcal{R}(B_1)\cap\mathcal{R}(B_2).$$

A matrix Z satisfies

(3) $$AZB=O$$

if and only if there exist Y_1 and Y_2 such that $Z=Y_1+Y_2$ and

(4) $$A_iY_iB_i=O, \qquad i=1,2.$$

Proof. Necessity. Let $\mathcal{S}$ and $\mathcal{T}$ be subspaces such that

$$\mathcal{R}(A_2^*)=\mathcal{R}(A^*)\oplus\mathcal{S}$$

and

$$\mathcal{R}(B_2)=\mathcal{R}(B)\oplus\mathcal{T}.$$

There exist projectors (idempotent matrices) P and Q such that $\mathcal{R}(P^*)=\mathcal{R}(A_1^*)$ and $\mathcal{N}(P^*)\supset\mathcal{S}$; $\mathcal{R}(Q)=\mathcal{R}(B_1)$ and $\mathcal{N}(Q)\supset\mathcal{T}$, since $\mathcal{R}(A_1^*)\cap\mathcal{S}=\{\mathbf{0}\}$ and $\mathcal{R}(B_1)\cap\mathcal{T}=\{\mathbf{0}\}$. Put $Y_1=Z-PZQ$ and $Y_2=PZQ$. Clearly these satisfy (4), since $A_1PZQB_1=A_1ZB_1$ and $A_2PZQB_2=O$.

Sufficiency. Because of (2) A and B can be written as $A=W_1A_1=W_2A_2$ and $B=B_1V_1=B_2V_2$. (4) implies $W_iA_iY_iB_iV_i=AY_iB=O$, $i=1,2$, and thus (3).

Theorem 1. The following conditions are equivalent. We assume A and B to to satisfy (2).

(i) Matrix equations (1.1) and (1.2) have a common solution.

(ii) There exist Y_1 and Y_2 which satisfy (4) and

$$Y_1+Y_2=X_1-X_2$$

for some (any) solution X_i of $(1.i)$, $i=1,2$.

(iii) $$A(X_1-X_2)B=O$$

for some (any) solution X_i of $(1,i)$, $i=1,2$.

(iv) $$AA_1^-C_1B_1^-B=AA_2^-C_2B_2^-B,$$

where g-inverses are some fixed ones or any ones.

(v) (Mitra 1973a)

$$A_1^*A_1(A_1^*A_1+A_2^*A_2)^-A_2^*C_2B_2^*(B_1B_1^*+B_2B_2^*)^-B_1B_1^*$$
$$=A_2^*A_2(A_1^*A_1+A_2^*A_2)^-A_1^*C_1B_1^*(B_1B_1^*+B_2B_2^*)^-B_2B_2^*.$$

Proof. (i)$\rightarrow$(iv). If X is a common solution of (1.1) and (1.2), then

$$AA_1^-C_1B_1^-B=AA_1^-A_1XB_1B_1^-B=AXB$$
$$=AA_2^-A_2XB_2B_2^-B=AA_2^-C_2B_2^-B.$$

(iv)$\rightarrow$(iii). Put $X_i=A_i^-C_iB_i^-$, $i=1,2$, and let Z_i be a solution of $(1.i)$, $i=1,2$. X_1 and X_2 satisfy (iii) and

$$A_iX_iB_i=A_iA_i^-C_iB_i^-B_i=A_iA_i^-A_iZ_iB_iB_i^-B_i$$
$$=A_iZ_iB_i=C_i.$$

Thus X_i is a solution of $(1.i)$, $i=1,2$.
(iii)$\longleftrightarrow$(ii). Just apply Lemma.
(ii)$\longleftrightarrow$(i). Let X be a common solution of (1.1) and (1.2). For any solution X_i of $(1.i)$, $i=1,2$, put $Y_1=X_1-X$ and $Y_2=X-X_2$.
(iv)$\longleftrightarrow$(v). The equation of (v) can be written as

$$A_1^*A_1(A_1^*A_1+A_2^*A_2)^-A_2^*A_2A_2^-C_2B_2^-B_2B_2^*(B_1B_1^*+B_2B_2^*)^-B_1B_1^*$$
$$=A_2^*A_2(A_1^*A_1+A_2^*A_2)^-A_1^*A_1A_1^-C_1B_1^-B_1B_1^*(B_1B_1^*+B_2B_2^*)^-B_2B_2^*,$$

where $A_1^*A_1(A_1^*A_1+A_2^*A_2)^-A_2^*A_2=A_2^*A_2(A_1^*A_1+A_2^*A_2)^-A_1^*A_1$ is a matrix with the property of the matrix A of (iv) and $B_2B_2^*(B_1B_1^*+B_2B_2^*)^-B_1B_1^*=B_1B_1^*(B_1B_1^*+B_2B_2^*)^-B_2B_2^*$ of B of (iv). (See Anderson and Duffin (1969) or Rao and Mitra (1971)).

Remark 1. The theorem is easily generalized as follows. Let

(5.1) $$A_{1i}XB_{1i}=C_{1i}, \qquad i=1,2,\cdots,m,$$

and

(5.2) $$A_{2j}XB_{2j}=C_{2j}, \qquad j=1,2,\cdots,n.$$

be sets of matrix equations, and each set be consistent. Corresponding to these consider sets of homogeneous equations

(6.1) $$A_{1i}YB_{1j}=O, \qquad i=1,2,\cdots,m,$$

and

(6.2) $$A_{2j}YB_{2j}=O, \qquad j=1,2,\cdots,n.$$

Let A and B be matrices such that

(7.1) $$\mathcal{R}(A^*)=\{\mathcal{R}(A_{11}^*)+\mathcal{R}(A_{12}^*)+\cdots+\mathcal{R}(A_{1m}^*)\}\cap \{\mathcal{R}(A_{21}^*)+\mathcal{R}(A_{22}^*)+\cdots+\mathcal{R}(A_{2n}^*)\}$$

and

(7.2) $$\mathcal{R}(B)=\{\mathcal{R}(B_{11})+\mathcal{R}(B_{12})+\cdots+\mathcal{R}(B_{1m})\}\cap \{\mathcal{R}(B_{21})+\mathcal{R}(B_{22})+\cdots+\mathcal{R}(B_{2n})\}.$$

Now the following conditions are equivalent.

(i) (5.1) and (5.2) have a common solution.

(ii) There exist solutions Y_i of $(6.i)$, $i=1,2$, such that $Y_1+Y_2=X_1-X_2$ for some (any) solutions X_i of $(5.i)$, $i=1,2$.

(iii) $A(X_1-X_2)B=O$ for some solutions X_i of $(5.i)$, $i=1,2$. (This statement is not always valid for arbitrary solutions.)

Remark 2. As a special case of the theorem consider a pair of matrix equations $A_iX=C_i$, $i=1,2$, each consistent. The following conditions are equivalent.

(i) $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ is consistent.

(ii) $\mathcal{R}(X_1-X_2)\subset \mathcal{N}(A_1)+\mathcal{N}(A_2)$ for some (any) X_1 and X_2 such that $A_1X_1=C_1$ and $A_2X_2=C_2$.

(ii*) $\mathcal{R}(A_1^-C_1-A_2^-C_2)\subset \mathcal{N}(A_1)+\mathcal{N}(A_2)$ for some (any) generalized inverses A_1^- and A_2^-.

(iii) $A_1^*A_1(A_1^*A_1+A_2^*A_2)^-A_2^*C_2=A_2^*A_2(A_1^*A_1+A_2^*A_2)^-A_1^*C_1$.

Remark 3. We generalize the theorem to another direction. Let matrix equations $A_iX=C_i$, $i=1,2,\cdots,k$, each be consistent. Then the following conditions are equivalent.

(i) $\begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix} X=\begin{bmatrix} C_1 \\ \vdots \\ C_k \end{bmatrix}$ is consistent.

(ii) $A_i^*A_i(\sum_{j=1}^{k} A_j^*A_j)^-(\sum_{j\neq i} A_j^*C_j)=(\sum_{j\neq i} A_j^*A_j)(\sum_{j=1}^{k} A_j^*A_j)^-A_i^*C_i, \qquad i=1,2,\cdots,k.$

(iii) $A_i^*A_i(\sum_{j=1}^{i} A_j^*A_j)^-(\sum_{j=1}^{i-1} A_j^*C_j)=(\sum_{j=1}^{i-1} A_j^*A_j)(\sum_{j=1}^{i} A_j^*A_j)^-A_i^*C_i, \qquad i=2,3,\cdots,k.$

(iv) $\mathcal{R}(X^{(i)}-X_{i+1})\subset \bigcap_{j=1}^{i}\mathcal{N}(A_j)+\mathcal{N}(A_{i+1})$, for some $X^{(i)}$ and X_{i+1} such that $(\sum_{j=1}^{i} A_j^*A_j)X^{(i)}=\sum_{j=1}^{i} A_j^*C_j$ and $A_{i+1}X_{i+1}=C_{i+1}$, for $i=1,2,\cdots,k-1$.

3. Rank of Sum of N.N.D. Matrices

In this section we give a simpler algebraic proof of Mitra's theorem on the rank of sum of two n.n.d. matrices.

Theorem 2. (Mitra 1973 b) Let A and B be n.n.d. matrices of the same order and conformably partitioned as

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where A_{11} and B_{11} are square matrices of the same order. If

$$\text{rank}\,(A+B)=\text{rank}\,(A_{11}+B_{11}), \tag{8}$$

then

$$\text{rank}\,A=\text{rank}\,A_{11} \quad \text{and} \quad \text{rank}\,B=\text{rank}\,B_{11}. \tag{9}$$

Conversely (9) implies (8) if and only if $A_{11}(A_{11}+B_{11})^{-}B_{12}=B_{11}(A_{11}+B_{11})^{-}A_{12}$, or equivalently $\mathcal{N}(X_1-X_2)\supset\mathcal{R}(A_{11})\cap\mathcal{R}(B_{11})$ for some X_1 and X_2 such that $A_{21}=X_1A_{11}$ and $B_{21}=X_2B_{11}$.

Proof. The n.n.d. matrices A and B can be expressed as

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}=\begin{bmatrix} A_1A_1^* & A_1A_2^* \\ A_2A_1^* & A_2A_2^* \end{bmatrix}=\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}[A_1^* \quad A_2^*],$$

$$B=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}=\begin{bmatrix} B_1B_1^* & B_1B_2^* \\ B_2B_1^* & B_2B_2^* \end{bmatrix}=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}[B_1^* \quad B_2^*].$$

Put

$$C=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix},$$

and

$$A+B=CC^* \quad \text{and} \quad A_{11}+B_{11}=C_1C_1^*.$$

Hence

$$\begin{aligned} &\text{rank}\,(A+B)=\text{rank}\,(A_{11}+B_{11}) \\ \longleftrightarrow\ &\text{rank}\,(C_1^*C_1+C_2^*C_2)=\text{rank}\,(C_1^*C_1) \\ \longleftrightarrow\ &\mathcal{R}(C_1^*)\supset\mathcal{R}(C_2^*) \\ \longleftrightarrow\ &\mathcal{R}\begin{bmatrix} A_1^* \\ B_1^* \end{bmatrix}\supset\mathcal{R}\begin{bmatrix} A_2^* \\ B_2^* \end{bmatrix} \end{aligned}$$

$$\rightarrow \mathcal{R}(A_1^*) \supset \mathcal{R}(A_2^*) \quad \text{and} \quad \mathcal{R}(B_1^*) \supset \mathcal{R}(B_2^*).$$

Therefore

$$\text{rank}\, A = \text{rank}\, [A_1^* \quad A_2^*] = \text{rank}\, A_1^* = \text{rank}\, A_{11},$$

and

$$\text{rank}\, B = \text{rank}\, [B_1^* \quad B_2^*] = \text{rank}\, B_1^* = \text{rank}\, B_{11}.$$

The second half of the theorem is shown easily from the above discussion and Remark 2 to Theorem 1.

Remark 1. We may have another indirect proof of the first half of the theorem by using Theorems 2 and 3 of Anderson (1971), or equivalently by using the corollary to Theorem 1 and Theorem 6 of Carlson and others (1974).

Remark 2. The theorem can be written as follows: Let A and B be n.n.d. matrices of the same order, and P any orthogonal projector. Then

$$\text{rank}\, (A+B) = \text{rank}\, P(A+B)P$$

$$\rightarrow \text{rank}\, A = \text{rank}\, PAP \quad \text{and} \quad \text{rank}\, B = \text{rank}\, PBP.$$

The reverse statement is true if and only if $PAP\{P(A+B)P\}^{-}PB = PBP\{P(A+B)P\}^{-}PA$. PAP restricted in $\mathcal{R}(P)$ is the compression of A into $\mathcal{R}(P)$ (Halmos 1967).

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