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FOURIER SERIES OF WEAKLY STATIONARY AND HARMONIZABLE STOCHASTIC PROCESSES

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ABSTRACT

Much mathematical attensions have not been paid to Fourier series of stochastic processes in the literature, while the growing importance of the topic is going to be gradually recognized as the investigations of analytical properties of processes develop in recent years. This paper provides an expository article exhibiting mostly the results obtained by the author and his colleagues particularly on the behaviors of Fourier coefficients of stochastic processes.

1. Introduction

In this paper we summarize the results on the Fourier series of stochastic processes. We shall mainly deal with weakly stationary processes for which the main tool of analysis with the series expansion has been the so-called the Karhunen-Loève expansion, (see LOÈVE (1963)) but the ordinary Fourier series expansion was also drew some attentions. Actually RICE (1944) has made an effective use of Fourier series in his celebrated paper on mathematical theory of random noise. See also KAWATA (1955). However papers on Fourier series of stochastic processes appeared in the literature rather scatterly and the author feels that it would be of some interest to present the summary of papers on Fourier series of stationary processes. We also include some results on the Fourier series of harmonizable processes.

The author, however, do not attempt to collect all results on Fourier series of stochastic processes but is interested in the following topics:

(i) the uncorrelatedness or the asymptotic uncorrelatedness of the Fourier coefficients

(ii) the behavior of the Fourier coefficients

(iii) the asymptotic joint distribution of the Fourier coefficients.

As a matter of fact, RICE (1944) used the Fourier series of a truncated random noise process assuming the Gaussianness of Fourier coefficients and also sometimes the independence of them. These assumptions hold in the approximate sense for the random noise process. Therefore it must be of interest to investigate when the Fourier coefficients are asymptotically uncorrelated for a weakly stationary process. It must be noted that the random noise is a strongly stationary process and the independence of Fourier coefficients is, of course, a consequence of uncorrelatedness of Gaussian processes. The random noise as defined by RICE is not generally Gaussian and it is important to know how the random noise process is close to a Gaussian process. (See PAPOULIS (1971)) This makes sense of the topic (iii).

2. Weakly stationary processes and Fourier series.

Suppose throughout this section that $X(t,\omega)$, $-\infty<t<\infty$ is a weakly stationary process, that is, a stochastic process of the second order with the following properties:

(1) (i) $$\mathrm{E}X(t,\omega)=0,\quad -\infty<t<\infty,$$

(2) (ii) $$\mathrm{E}X(s,\omega)\overline{X(t,\omega)}=\rho(s-t),\quad -\infty<s,t<\infty.$$

$\rho(u)$ is a continuous function of a single variable and is the covariance function.

(3) (iii) $$\int_a^b (\mathrm{E}|X(t,\omega)|^2)^{1/2}dt<\infty$$

for every pair $a, b(a<b)$ of finite real numbers. ω is an element of the probability space.

The representation

(4) $$\rho(u)=\int_{-\infty}^{\infty} e^{i\lambda u}dF(\lambda)$$

is possible where $F(\lambda)$ is the *spectral distribution function* bounded and nondecreasing, and $X(t,\omega)$ itself has the integral representation, in the L^2 sense: $\mathrm{E}\left|\int_b^a \cdot\right|^2\to 0$, $a, b\to\infty$, or $a, b\to-\infty$,

(5) $$X(t,\omega)=\int_{-\infty}^{\infty} e^{i\lambda t}\xi(d\lambda,\ \omega),$$

where $\xi(S,\omega)$, S being any Borel set, is an orthogonal random measure (See PROHOROV-ROZANOV (1969)) that is a random measure with $\mathrm{E}\xi(S_1)\overline{\xi(S_2)}=0$ for Borel sets S_1, S_2 with $S_1\cap S_2=\phi$.

Let $T>0$ be any positive number and form the Fourier series of $X(t,\omega)$ over $(-T/2, T/2)$:

(6) $$X(t, \omega) \sim \sum_{n=-\infty}^{\infty} C_n(\omega) e^{2n\pi it/T},$$

where

(7) $$C_n(T, \omega) = C_n(\omega) = \frac{1}{T}\int_{-T/2}^{T/2} X(t, \omega) e^{-2n\pi it/T} dt$$

which makes sense in view of the assumption (3). Then *for every t,* $-T/2 < t < T/2$,

$$E|S_n(t, \omega) - X(t, \omega)|^2 \to 0,$$

where $S_n(t, \omega)$ is the n-th partial sum of (6). (See MANN (1951), KAWATA (1960, 1966, 1969))

We begin with the following theorem of R. C. DAVIS (1953).

Theorem 1. *The necessary and sufficient condition that, for a pair* (n, m) *of unequal integers and for every* $T > 0$,

(8) $$EC_n(T, \omega)\overline{C_m(T, \omega)} = 0,$$

is that $F(\lambda)$ *is degenerated at the origin:*

$$F(\lambda) = c, \ \lambda < 0,$$
$$= d, \ \lambda > 0$$

c, d being constants such that $c < d$.

Let $X(t, \omega)$ be a weakly stationary process which is periodic with period $T (T > 0)$, that is $\rho(u)$ is a periodic function of period T. This is equivalent to

(9) $$E|X(t+T, \omega) - X(t, \omega)|^2 = 0, \quad -\infty < t < \infty.$$

It is easily shown that a weakly stationary process is periodic with period T if and only if $F(\lambda)$ is a step function whose point spectrum is contained in the sequence $\{2k\pi/T, \ k = 0, \pm 1, \pm 2, \cdots\}$ and hence the covariance function has the form

(10) $$\rho(u) = \sum_{k=-\infty}^{\infty} \alpha_k e^{2k\pi iu/T}, \quad \alpha_k \geqq 0.$$

With this terminology we have the following simple fact.

Therem 2. *In order that for the sequence* $\{C_n(\omega), \ n = 0, \pm 1, \cdots\}$ *of the Fourier coefficients of a weakly stationary process over* $(-T/2, T/2)$, *we have* $EC_m(\omega)C_n(\omega) = 0$ *for every pair m, n of unequal integers, it is necessary and sufficient that the process is periodic with period T.*

In this connection, we mention the following simple generalization of this theorem, which H. YU recently informed me.

We agree to call an almost periodic weakly stationary process a weakly stationary process for which $\rho(u)$ is an uniformly almost periodic function. In this case $X(t, \omega)$ is uniformly almost periodic in L^2 sense, namely for any $\varepsilon > 0$, the set of τ such that $E|X(t+\tau,\omega)-X(t)|^2<\varepsilon$ is relatively dense. Also we have thot there is an at most countably infinite set of λ for which $C(\lambda,\omega)=\lim_{T\to\infty}(1/2T)\int_{-T}^{T}X(t)e^{-i\lambda t}dt$ differs from zero. l.i.m. means the limit in the L^2 sense. If $\{\lambda_n\}$ is the sequence of λ of such a set, the sequence of the Fourier exponents, then $\rho(u)$ can be written by $\sum C_n e^{i\lambda_n u}$ and $X(t,\omega)$ by $\sum C_n(\omega)e^{i\lambda_n t}$, where $C_n(\omega)=\xi(\lambda_n+0)-\xi(\lambda_n-0)$. From this it is obvious that $EC_m(\omega)\overline{C_n(\omega)}=0$ for every pair (m,n) of different integers.

3. Asymptotic uncorrelatedness of Fourier coefficients of a weakly stationary process.

We also suppose throughout this section that $X(t,\omega)$ is a weakly stationary process stated in **1** and $C_n(\omega)=C_n(T,\omega)$ $n=0,\ \pm1,\ \cdots$ are Fourier coefficients over $(-T/2,\ T/2)$. We also consider the Fourier coefficients of real form:

$$A_n(\omega)=A_n(T,\omega)=\frac{2}{T}\int_{-T/2}^{T/2}X(t,\omega)\cos(2n\pi it/T)\,dt\quad n=0,1,2,\cdots \tag{1}$$

$$B_n(\omega)=B_n(T,\omega)=\frac{2}{T}\int_{-T/2}^{T/2}X(t,\omega)\sin(2n\pi it/T)\,dt\quad n=1,2,\cdots \tag{2}$$

About the asymptotic uncorrelatedness of Fourier coefficients, ROOT and PITCHER (1955) have first shown

Theorem 3. *If the covariance function $\rho(u)$ satisfies that $\rho(u)\varepsilon L^1(-\infty,\infty)$ and $\int_{-\infty}^{\infty}\rho(u)du\neq0$, then*

$$E|C_n(\omega)|^2=O(T^{-1}) \tag{3}$$

$$EC_m(\omega)C_n(\omega)/(E|C_m(\omega)|^2\cdot E|C_n(\omega)|^2)^{1/2}\to0 \tag{4}$$

for $m\neq n$, as $T\to\infty$.

This result may be said that under the stated conditions, the Fourier coefficients of a weakly stationary process are asymptotically uncorrelated.

BLACHMAN (1957) also discussed the similar problem for Fourier coefficients of a Gaussian noise.

The author (KAWATA (1965, 1966, 1969)) gave more precise estimations of the covariances of Fourier coefficients under some conditions on the spectral distribution of a weakly stationary process,

Theorem 4. *Suppose that the given weakly stationary process is real valued and that the spectral distribution has the density $f(\lambda)$ which is continuous at $\lambda=0$. Then*

$$\lim_{T\to\infty}T\cdot EA_n^2(\omega)=4\pi f(0),\ n=1,2,\cdots, \tag{5}$$

$$\lim_{T\to\infty}T\cdot EA_0^2(\omega)=8\pi f(0), \tag{6}$$

(7) $$\lim_{T\to\infty} T\cdot \mathrm{E}A_m(\omega)A_m(\omega)=0, \quad m\neq n,$$

(8) $$\lim_{T\to\infty} T\cdot \mathrm{E}B_m(\omega)B_n(\omega)=0, \; m\neq n,$$

(9) $$\mathrm{E}A_m(\omega)B_n(\omega)=0, \quad \text{for all} \quad (m, n).$$

Under the stronger condition of $f(\lambda)$, we have

Theorem 5. *If the spectral density $f(\lambda)$ of a real valued weakly stationary process has a continuous derivative in a neighborhood of the origin and*

$$\int_{0+}^{\infty} \frac{f(\lambda)-f(0)}{\lambda^2} d\lambda$$

exists, then

(10) $$\lim_{T\to\infty} T^2\mathrm{E}A_m(\omega)A_n(\omega)=16\int_{0+}^{\infty} \frac{f(\lambda)-f(0)}{\lambda^2} d\lambda, \; m\neq n,$$

(11) $$\lim_{T\to\infty} T^2\mathrm{E}B_m(\omega)B_n(\omega)=0, \; m\neq n.$$

4. Appoximate Fourier series.

As we stated in **2**, a periodic weakly stationary process has uncorrelated Fourier coefficients while it is not true for a nonperiodic process. However we may define, even for a nonperiodic weakly stationary process, a random trigonometric series which has uncorrelated coefficients and is very close to a given weakly stationary process.

Let us be given a weakly stationary process

(1) $$X(t, \omega)=\int_{-\infty}^{\infty} e^{it\lambda}\xi(d\lambda, \omega)$$

with mean 0 and covariance function $\rho(u)$ as in **2**. Let $T>0$ be any positive number.

Define the sequence $\{\xi_n(\omega), n=0, \pm 1, \cdots\}$ of random variables

(2) $$\xi_n(\omega)=\xi_n(\omega, T)=\int_{2n\pi/T}^{2(n+1)\pi/T} \xi(d\lambda, \omega)$$

and form

(3) $$\hat{X}_T(t, \omega)=\sum_{n=-\infty}^{\infty} \xi_n(\omega)e^{2n\pi it/T},$$

It is obvious that

(4) $$\mathrm{E}\xi_m(\omega)\overline{\xi_n(\omega)}=0, \quad m\neq n,$$

from which the convergence of the series (3) for every t in L^2 sense easily follows. $\hat{X}_T(t,\omega)$ then defines a periodic weakly stationary process and the covariance function is given by

$$\hat{\rho}(u)=\hat{\rho}(u,T)=\sum_{n=-\infty}^{\infty} e^{2n\pi iu/T}\int_{2n\pi/T}^{2(n+1)\pi/T} dF(x)$$

$F(x)$ being the spectral distribution function of $X(t,\omega)$.

We may show that

$$\text{(5)} \qquad \mathrm{E}|X(t,\omega)-\hat{X}_T(t,\omega)|^2 \leqq 4\sin^2\frac{\pi t}{T}\rho(0)$$

for $T\geqq 2|t|$. In other words, if t is an arbitrary but fixed number, $X(t,\omega)$ and $\hat{X}_T(t,\omega)$ are very close to each other when T is very large.

The series (3) is called the *approximate Fourier series* of $X(t,\omega)$. The definition is found in a slightly different form in PAPOULIS (1965) and in the form (3) with (2), in the paper of the author (KAWATA (1966, 1969)). In this paper a theorem on the absolute convergence of the approximate Fourier series was given [see later, Theorem 18] and it was used to improve a theorem on sample continuity of a weakly stationary process. See KAWATA (1966, 1969) KAWATA-KUBO (1970).

5. Behaviors of Fourier coefficients.

Let $X(t,\omega)$ be a weakly stationary process and its Fourier coefficients be $C_n(T,\omega)=C_n(\omega)$ over $(-T/2, T/2)$, as in **2**. We are now interested in the behaviour of C_n when $|n|\to\infty$, for fixed T. Actually we are going to deal with the magnitude of $\sum_{n>N}\mathrm{E}|C_n(\omega)|^2$ or $\sum_{n\leqq N}|C_n(\omega)|^2$ as $N\to\infty$.

We first mention the following result (KAWATA (1966, 1969)).

Theorem 6. (i) *If, for* $0\leqq\alpha<1$, *the spectral distribution function* $F(\lambda)$ *satisfies*

$$\text{(1)} \qquad \int_{-\infty}^{\infty}|\lambda|^\alpha dF(\lambda)<\infty,$$

then

$$\text{(2)} \qquad \sum_{|n|>N}\mathrm{E}|C_n|^2=o(N^{-\alpha})$$

as $N\to\infty$.

(ii) *If* $X(t,\omega)$ *is periodic with period* T, *then under the condition* (1) *with* $0\leqq\alpha<2$, (2) *holds, and for* $\alpha=2$, *with the condition* (1), *we have*

$$\text{(3)} \qquad \sum_{|n|>N}\mathrm{E}|C_n|^2=O(N^{-2}).$$

(iii) *If* $X(t,\omega)$ *is periodic with period* T, *and*

$$\text{(4)} \qquad \int_{-\infty}^{\infty}|\lambda|(\log^+|\lambda|)^\beta dF(\lambda)<\infty,\ \beta>0,$$

then

$$(5)\qquad \sum_{|n|>N} \mathrm{E}|C_n|^2 = o(1/(N\log^\beta N)).$$

As in this theorem, there are some differences in connection with the magnitude of $\sum \mathrm{E}|C_n|^2$, between periodic processes and nonperiodic ones.

The difference appears significantly in Parseval relation, that is, we have

Theorem 7. *If $X(t,\omega)$ is a weakly stutlonary process, then for any $T>0$, we have the relation*

$$(6)\qquad 4\sum_{N=-\infty}^{\infty} E|C_n|^2 \sin^2\frac{\pi nh}{T} = \left(1-\frac{h}{T}\right)[2\rho(0)-\rho(h)-\rho(-h)] + \frac{h}{T}[2\rho(0)-\rho(T-h)-\rho(-T+h)],$$

for $0\leqq h\leqq T$, where $\rho(u)$ is the covariance function.

The right hand side, for a periodic weakly stationary process with period T, reduced to $2\rho(0)-\rho(h)-\rho(-h)$.

This theorem enables us to show the following two theorems.

Theorem 8. *Suppose $\int_{-\infty}^{\infty}|\lambda|dF(\lambda)<\infty$. In order for a given weakly stationary process $X(t,\omega)$ to be periodic with period T, it is necessary and sufficient that, for the Fourier coefficients C_n over $(-T/2, T/2)$, we should have*

$$(7)\qquad \frac{1}{N}\sum_{|n|\leqq N} n^2\mathrm{E}|C_n|^2 \to 0.$$

The following theorem says more than this theorem.

Theorem 9. *If $\int_{-\infty}^{\infty}|\lambda|dF(\lambda)<\infty$, then for any weakly stationary process we have*

$$(8)\qquad \lim_{N\to\infty}\frac{\pi^2}{N}\sum_{|n|\leqq N} n^2\mathrm{E}|C_n|^2 = \rho(0)-\frac{1}{2}\rho(T)-\frac{1}{2}\rho(-T).$$

6. Limit joint distribution of Fourier coefficients.

In this section we assume that the weakly stationary process $X(t,\omega)$ is real valued. Consider the Fourier coefficients $A_n(\omega)$, $n=0,1,2,\cdots$, $B_n(\omega)$, $n=1,2,\cdots$ of $X(t,\omega)$ over $(-T/2, T/2)$. We are now interested in the joint limit distribution function of

$$(1)\qquad \left(\frac{1}{2}A_0(\omega),\ A_1(\omega),\ \cdots,\ A_n(\omega),\ B_1(\omega),\ \cdots,\ B_n(\omega)\right)$$

when $T\to\infty$ and consider the problem when they will become close to Gaussian distribution. If we have that they behave like independent Gaussian variables when T is very large, then the approximation of a weakly stationary process by a Fourier series with independent Gaussian variables as Fourier coefficients might be recognized as permissible. As to this question we know the following theorem. Before stating it we note that a weakly stationary process $X(t, \omega)$ with spectral density can be written by

$$X(t, \omega)=\int_{-\infty}^{\infty} \Phi(\lambda-t)\eta(d\lambda) \tag{9}$$

where $\Phi(\lambda)$ is a nonrandom function of $L^2(-\infty, \infty)$ and $\eta(S, \omega)=\eta(S)$, S being Borel sets, is an orthogonal random measure such that

$$\mathrm{E}|\eta(S)|^2=mS, \tag{10}$$

where mS is the Lebesgue measure of S. We write (10) also by $\mathrm{E}|y(d\lambda)|^2=d\lambda$. The covariance function of $X(t, \omega)$ is then given by

$$\rho(u)=\int_{-\infty}^{\infty} |\hat{\Phi}(\lambda)|^2 e^{it\lambda}d\lambda \tag{11}$$

where $\Phi(\lambda)$ is the Fourier transform in L^2, $\int_{-\infty}^{\infty} e^{i\lambda v}\Phi(v)dv$, of $\Phi(v)$. (See DOOB [1] p. 532)

We then have (KAWATA (1966, 1969))

Theorem 10. *Let $X(t, \omega)$ be a real valued weakly stationary process which has the spectral density and the form* (9). *Suppose that $\eta(S_1)$ and $\eta(S_2)$ are independent for any disjoint Borel sets S_1 and S_2, and $\mathrm{E}|y(d\lambda)|^3=o(d\lambda)$ for small $d\lambda$, and moreover that $\Phi(\lambda)\in L^1(-\infty, \infty)\cap L^3(-\infty, \infty)$. Then the joint distribution of*

$$((T/2)^{1/2}A_0(\omega),\ T^{1/2}A_1(\omega),\ \cdots,\ T^{1/2}A_n(\omega),\ T^{1/2}B_1(\omega),\ \cdots,\ T^{1/2}B_n(\omega))$$

converges to $N\left(0, \frac{1}{2}\sigma^2\right)*\Pi^{2n}*N(0, \sigma^2)$, as $T\to\infty$ *where* $\sigma=\left|\int_{-\infty}^{\infty} \Phi(\lambda)d\lambda\right|$ *and* $\Pi^{2n}*$ *means the* $2\,n$*-fold convolution.*

7. Fourier series of harmonizable process.

A harmonizable process $X(t, \omega)$ is a process of the second order, which has the representation

$$X(t, \omega)=\int_{-\infty}^{\infty} e^{it\lambda}Z(d\lambda, \omega), \tag{1}$$

where $Z(S)=Z(S, \omega)$ is a random measure (generalized measure) not necessarily orthogonal with

$$F(S_1, S_2)=F(S_1\times S_2)=EZ(S_1)\overline{Z(S_2)} \tag{2}$$

(S_1, S_2 being Borel sets), which defines a complex signed measure of bounded variation over the whole 2-dimensional Euclidean space R^2. (See PROHOROV-ROZANOV (1969))

Suppose for simplicity that $\mathrm{E}X(t,\omega)=0$, $-\infty<t<\infty$. If $X(t,\omega)$ is harmonizable with (1), then its covariance function $\rho(s,t)$ has the representation

$$(3) \qquad \rho(s,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{is\lambda-it\mu}d^2F(\lambda,\mu),$$

where $\mathrm{E}(\lambda,\mu)$ is a function of bounded variation over R^2, which is a point function corresponding to $F(S_1,S_2)$ in (2). Conversely if $\rho(s,t)$ is the covariance function of some process of the second order and has the representation (3), then the process is harmonizable. (See M. LOÈVE (1963))

We consider the Fourier series of a harmonizable process $X(t,\omega)$, $|t|<T/2$. It is shown that *the partial sum $S_n(t,\omega)$ of the Fourier series of $X(t,\omega)$ converges in mean to $X(t,\omega)$ for every $|t|<T/2$, that is*

$$\mathrm{E}|S_n(t,\omega)-X(t,\omega)|^2\to 0, \quad \textit{for every} \quad |t|<T/2.$$

H. YU extended Theorem 6 of the author to the case of harmonizable processes and showed the following result. (H. YU (1971))

Theorem 15. *Let $C_n=C_n(\omega)$ be the Fourier coefficients of a harmonizable process $X(t,\omega)$ with spectral distribution $F(S)$, S being Borel sets of R^2.*

(i) If

$$(4) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\lambda|^{\alpha}|d^2F(\lambda,\mu)|<\infty$$

for $0\leqq\alpha<1$, *then*

$$(5) \qquad \sum_{|n|>N}\mathrm{E}|C_n|^2=o(N^{-\alpha})$$

as $N\to\infty$.

(ii) *If $X(t,\omega)$ is periodic with period T, then under the condition* (4) *with* $0\leqq\alpha<2$, (5) *holds, and if* $\alpha=2$,

$$(6) \qquad \sum_{|n|>N}\mathrm{E}|C_n|^2=O(N^{-2})$$

as $N\to\infty$.

(iii) *If $X(t,\omega)$ is periodic with period T, and*

$$(7) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\lambda|(\log^+|\lambda|)^{\beta}|d^2F(\lambda,\mu)|<\infty,\ \beta>0,$$

then

$$(8) \qquad \sum_{|n|>N}\mathrm{E}|C_n(\omega)|^2=o\left(\frac{1}{N\log^{\beta}N}\right)$$

as $N\to\infty$.

The approximate Fourier series for a harmonizable process is also defined in just the same way as in 4.

8. Absolute convergence of Fourier series.

The problem of the absolute convergence of the Fourier series of a stochastic process seems an important topic and actually as was stated in 4 it provides an effective means for the study of the sample properties of stochastic processes. (See KAWATA (1966, 1969) and KAWATA-KUBO (1970))

Let $X(t, \omega)$ be a weakly stationary process with zero mean and spectral distribution function $F(x)$. The author showed (KAWATA (1966, 1969))

Theorem 16. *Let* $C_n=C_n(\omega)$, $n=0, \pm1, \pm2, \cdots$ *be the Fourier coefficient of* $X(t,\omega)$ *over* $|t|\leqq T/2$, $T>0$. *If*

$$\int_{-\infty}^{\infty}|x|(\log^+|x|)^\beta dF(x)<\infty,\ \beta>2, \tag{1}$$

then $\sum_{n=-\infty}^{\infty}|C_n|<\infty$ with probability one.

We also mention in this connection the following theorem.

Theorem 17. *If*

$$\int_{-\infty}^{\infty}\log^+|x|dF(x)<\infty, \tag{2}$$

then

$$\sum_{n=-\infty}^{\infty}\log(|n|+1)\mathrm{E}|C_n|^2<\infty. \tag{3}$$

This would be compared with Theorem 6. As a matter of fact, (3) implies

$$\sum_{|n|<N}\mathrm{E}|C_n|^2=o(\log N)^{-1} \tag{4}$$

as $N\to\infty$.

Moreover for studying the sample continuity of a weakly stationary process, it has been shown (KAWATA (1966, 1969)) that the following result is of particular interest

Theorem 18. *Let* $g(x)$ *be an even function which is nonnegative and nondecreasing for* $x>0$ *and is such that*

$$\sum_{n=1}^{\infty}\frac{1}{g(n)}<\infty. \tag{5}$$

Suppose the spectral distribution function $F(x)$ *of a weakly stationary process* $X(t,\omega)$

satisfies

$$(6) \qquad \int_{-\infty}^{\infty} g(x)dF(x)<\infty.$$

Then the Fourier series **4** (3) *defining the approximate Fourier series* $\hat{X}_T(t,\omega)$ *converges absolutely with probability one.*

Theorem 16 was extended to the case of harmonizable processes by YU (1971) who showed

Theorem 19. *If* $F(\lambda,\mu)$ *is the spectral distribution of a harmonizable process and*

$$(7) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\lambda|(\log^{+}|\lambda|)^{\beta}d^2F(\lambda,\mu)<\infty$$

for some $\beta>0$, *then* $\sum_{n=-\infty}^{\infty}|C_n|<\infty$ *with probability one.*

Finally we conclude giving the following theorem which is an analogue of Bary's theorem [see BARY (1964) or ZYGMUND (1968)] on the absolute convergence for an ordinary Fourier series.

Theorem 20. *Let* $X(t,\omega)$ *be a periodic weakly stationary process with period* T. *If, writing*

$$(8) \qquad \Phi(h)=\frac{1}{h}\int_0^h [2\rho(0)-\rho(u)-\rho(-u)]du$$

$\rho(u)$ *being the covariance function,*

$$(9) \qquad \sum_{n=1}^{\infty}\left[\Phi\left(\frac{1}{n}\right)\right]^{\alpha/2} n^{-(\alpha-2\beta)/2}<\infty$$

for $1\leqq\alpha\leqq 2$, $0\leqq\beta<1$, $\alpha>2\beta$, *holds, then*

$$\sum_{n=-\infty}^{\infty}|C_n|^{\alpha}|n|^{\beta}<\infty$$

with probability one.

The extension to a harmonizable process was made by YU (1971) when $\alpha=1$, $\beta=0$.

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