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ON FOURIER-TYPE EXPANSION OF FUNCTIONS WHICH RELATE TO ELASTIC CIRCULAR PLATES

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ABSTRACT

Let us consider the problem of free vibration of an elastic circular plate under a given condition (such as the case of fixed edge-lines). We arrive at an infinite set of eigen-values (k_m) and corresponding set of eigen-functions $Y_m(r)$ ($m=1, 2, 3, \cdots$). These functions are known to form complete set of orthogonal functions. Also, it is shown (by referring to theory of linear integral equation with symmetrical kernel, for example) that any given function $F(r)$ can be expanded into form of an infinite series of Fourier-type, provided that $F(r)$ satisfies certain required condition. In the present note, the author has given some considerations about the expansion of Fourier-type of this nature, when the given function $F(r)$ does not satisfy the above mentioned requirement.

1. A Set of Orthogonal Functions which relate to Vibration of a Circular Elastic Plate

In what follows, we shall treat the case of small free vibration of an elastic circular flat-plate of uniform thickness. Similar considerations can be made for other case such as elastic circular plate of non-uniform thickness, at least in the principle. (The general discussion about eigen-value problem has been given by the author.) (KITO 1972–1973)

The fundamental equation of free vibration of our problem, then, can be written in following form;

$$[\Delta\Delta - k^4]w = 0 \tag{1}$$

where we denote by Δ the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$. The transverse displacement ξ at any point (x, y) on middle plane of our elastic plate will be expressed by

$$\xi = w(x, y) \sin \omega t$$

ω being angular frequency of free vibration. The constant k in equation (1) is an eigen-value which corresponds to natural frequency ω. There may arise many cases of boundary conditions, such as free-, supported-, or built- in edge-lines. Here we restrict ourselves to the case of fixed (built-in) edge-lines, but similar accounts can be made about cases of other boundary conditions. The eigen-value k and corresponding eigen-function $w(x, y)$ must be determined in such way to satisfy the differential equation (1) together with the given boundary condition. Let us assume that we have two sets $[k_1, w_1(x, y)]$ and $[k_2, w_2(x, y)]$ of solutions of our problem. Then we shall have, from equation (1),

$$w_2 \Delta\Delta w_1 - w_1 \Delta\Delta w_2 = (k_1^4 - k_2^4) w_1 w_2 \tag{2}$$

Now, by actual calculation, we see that the expression

$$B = w_2 \Delta\Delta w_1 - w_1 \Delta\Delta w_2$$

can be given in form of

$$B = \partial M/\partial x + \partial N/\partial y$$

where we have put

$$\begin{aligned} M &= \left(w_2 \frac{\partial^3 w_1}{\partial x^3} - w_1 \frac{\partial^3 w_2}{\partial x^3} \right) - \left(\frac{\partial w_2}{\partial x} \frac{\partial^2 w_1}{\partial x^2} - \frac{\partial w_1}{\partial x} \frac{\partial^2 w_2}{\partial x^2} \right) \\ &\quad + 2\left(w_2 \frac{\partial^3 w_1}{\partial x \partial y^2} - w_1 \frac{\partial^3 w_2}{\partial x \partial y^2} \right), \\ N &= \left(w_2 \frac{\partial^3 w_1}{\partial y^3} - w_1 \frac{\partial^3 w_1}{\partial y^3} \right) - \left(\frac{\partial w_2}{\partial y} \frac{\partial^2 w_1}{\partial y^2} - \frac{\partial w_1}{\partial y} \frac{\partial^2 w_2}{\partial y^2} \right) \\ &\quad - 2\left(\frac{\partial w_2}{\partial x} \frac{\partial^2 w_1}{\partial x \partial y} - \frac{\partial w_1}{\partial x} \frac{\partial^2 w_2}{\partial x \partial y} \right). \end{aligned} \tag{3}$$

Next, let us put

$$A = B - (k_1^4 - k_2^4) w_1 w_2$$

and take integral about a domain S enclosed by one or several closed (regular) curves (Fig. 1). From the equation (2) we have

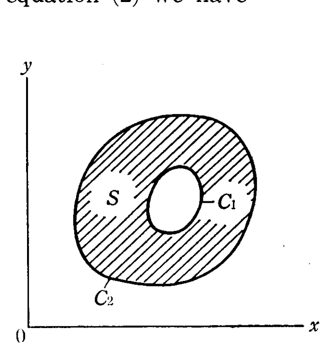

Fig. 1. Domain of Integration.

$$\iint_S A\,dx\,dy=0$$

The closed curves C_1, C_2, etc. being such regular closed curves so that we can apply Gauss' theorem, we have

$$\int_C [M\,dy-N\,dx]=(k_1^4-k_2^4)\iint_S w_1 w_2\,dx\,dy \tag{4}$$

where left hand side represents contour integrals along closed curves $C_1, C_2, \cdots$ (with suitable signs).

According to our boundary condition, we have

$$w=0, \qquad \partial w/\partial\nu=0$$

along the boundary curve, and we have, from equation (4),

$$\iint_S w_1(x,\,y)w_2(x,\,y)\,dx\,dy=0 \tag{5}$$

provided that $k_1 \neq k_2$, and also that w_1, w_2 and their derivatives up to third order are continuous functions.

For the case of a circular plate, we put

$$w(x,\,y)=W(r)\cos n\theta$$

where n is an integer. Then we have

$$\varDelta\varDelta w=\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)^2 W(r)\cos n\theta$$

and the equation (3) becomes

$$\int_{r=b}^{r=a}\int_{\theta=0}^{\theta=2\pi} Ar\,dr\,d\theta=0 \tag{6}$$

taking the case of circular plate of inner radius b and outer radius a. Also, the equation (5) is formed into

$$\int_b^a W_1(r)\,W_2(r)r\,dr=0 \tag{7}$$

Next, in order to connect our problem to theory of linear integral equation with symmetrical kernel, we consider two functions $W(r)$ and $K(r,\,\xi)$ defined in following manner.

(a) For $w=W(r)\cos n\theta$ the function $W(r)$, together with its derivatives W', W'' and W''' are continuous functions in the domain $b \leq r \leq a$, and satisfy the differential equation

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)^2 W-k^4 W=0 \tag{8}$$

and such that at $r=b$ and $b=a$, we have $W=0$, $W'=0$. In the case of whole

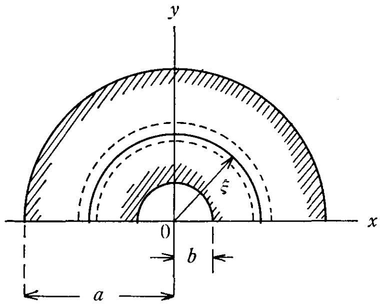

Fig. 2. Annular Domain of Integration.

circle in which $b=0$, the boundary condition at $r=b$ is to be replaced by the condition that at $r=0$, W and W' are regular.

(b) For $U=K(r, \xi)\cos n\theta$, the function $K(r, \xi)$ as function of r is, together with its derivatives $U'_r, \cdots U'''_r$ are continuous in two annular domains $b\leq r\leq \xi-0$ and $\xi+0\leq r\leq a$ (see Fig. 2). Moreover, we set conditions at $r=b$ and $r=a$ that $K(r, \xi)=0$, $K'_r(r, \xi)=0$.

(c) We introduce discontinuity of $K_r^{(3)}(r, \xi)$ at $r=\xi\pm 0$ given by the equation

$$\left| K_r^{(3)}(r, \xi) \right|_{r=\xi+0}^{r=\xi-0} = -1 \tag{10}$$

As a next step, we replace w_1 by w given in above item (a), and w_2 by U given in above item (b), in the equation (4). The domain S of integration in eq. (4) is considered to consist of two annular regions $b\leq r\leq \xi-0$ and $\xi+0\leq r\leq a$.

Thus we arrive at the equation

$$W(r)=k^4\int_b^a K(r, \xi)\, W(\xi)\xi\, d\xi \tag{11}$$

which is a homogeneous linear integral equation in $W(r)$. Furthermore, let us put in eq. (4), $U_1=K(r, \xi)$ in place of w_1, and $U_2=K(r, \eta)$ in place of w_2. To fix ideas we assume that $b<\xi<\eta<a$. The domain of integration in eq. (4) is, here taken to consist of three annular regions represented by $b\leq r\leq \xi-0$, $\xi+0\leq r\leq \eta-0$, $\eta+0\leq r\leq a$ respectively. Thus we obtain, from eq. (4), the following relation

$$K(\xi, \eta)=K(\eta, \xi)$$

which shows us that $K(r, \xi)$ in integral equation (11) is a symmetrical kernel.

In what follows, we shall confine ourselves to the case of whole circle $0\leq r\leq a$ this being most common case. But similar treatment can also be made for the case of annular ring region $b\leq r\leq a$. In the case of whole circle, the condition at $r=0$ is that $W(r)$ and $W'(r)$ are regular in the neibourhood of $r=0$.

According to theory of linear integral equation with symmetrical kernel by KNESER (1922), the equation (11) has an infinite set of eigen-values and eigen-functions $[k_m, W_m(r)]$ $(m=1, 2, \cdots)$. Functions $W_m(r)$ and $W_s(r)$ $(m\neq s)$ among them are orthogonal each other. A given function $F(r)$ can be expanded into an infinite series of Fourier type, using these orthogonal functions $W_m(r)$. It is known that for any function $F(r)$ which can be expressed in source-like form of

$$F(r)=\int_0^a K(r,\xi)\phi(\xi)\xi\,d\xi \tag{12}$$

this Fourier type expansion is valid (converges) (KNESER 1922; COURANT, HILBERT 1931). Also it can be shown that any function which is finite and continuous together with its derivatives F', F'', F''' in the interval and which is such that $F(a)=0$, $F'(a)=0$, can be expressed in form of (12). This is done by putting

$$w_1=\varphi(r)\cos n\theta=\Delta\Delta[F(r)\cos n\theta]$$

$$w_2=K(r,\xi)\cos n\theta$$

into the expression B, and integrating over the domain S of Fig. 2.

2. Set of Orthogonal Functions which are related to Problem of Circular Plates

In what follows, in considering elastic circular plate $(0\leq r\leq a)$ we shall put $a=1$, which can be done without loss of generality. The solution of equation (8), which satisfies above-metioned condition of fixed boundary is given by

$$Y(r)=AJ_n(kr)+BJ_n(ikr) \tag{13}$$

wherein constants A and B are so chosen that we have

$$\left.\begin{aligned} AJ_n(k)+BJ_n(ik)=0 \\ AJ'_n(k)+iBJ'_n(ik)=0 \end{aligned}\right\} \tag{13 a}$$

whence we have

$$J'_n(k)/J_n(k)=iJ'_n(ik)/J_n(ik) \tag{14}$$

The eigen-value k^4 must be a solution of the equation (14). It will be seen that there exist an infinite number of solutions of this eq. (14). The fact that all these eigen-values k^4 are real numbers will be seen from the theory of linear integral equation (11). Furthermore, it will be seen that all the eigen-values may be taken to be positive real numbers. In order to see it, we remark following identity

$$w\Delta\Delta w=\frac{\partial M_1}{\partial x}+\frac{\partial N_1}{\partial y}+\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]$$

where we have put

$$M_1=w\frac{\partial^3 w}{\partial x^3}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+w\frac{\partial^3 w}{\partial x\partial y^2}$$

$$N_1=w\frac{\partial^3 w}{\partial y^3}-\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}$$

Therefore, by integrating both sides of the relation

$$w\Delta\Delta w = k^4(w)^2$$

which is derived from equation (1), over the domain S, and inserting the boundary condition, we arrive at a relation of form of

$$(\text{positive value}) = k^4\ (\text{positive value})$$

Thus we infer that k^4 is a positive number. Putting then $k^4 = c^2$ we have $k^2 = \pm c$. Thus there are following four cases

$$k = \sqrt{c}, \quad -\sqrt{c}, \quad +i\sqrt{c}, \quad -i\sqrt{c}$$

As we see from the solution (13) and (14), practically the same function is obtained, when any one of above four choices is made. Therefore, we may for our estimation take only one case of $k = \sqrt{c}$, that is positive real eigen-value. The constants A and B must be so chosen that the eigen-function $Y(r)$ is normalized, that is

$$\int_0^1 [Y(r)]^2 r\, dr = 1$$

This is done by using the known formula

$$\int_0^r J_n(kr)J_n(lr)r\, dr = \frac{r}{k^2 - l^2}[lJ_n(kr)J_n'(lr) - kJ_n(r)J_n'(kr)]$$

$$\int_0^r [J_n(kr)]^2 r\, dr = \frac{1}{2} r^2 \left[\{J_n'(kr)\}^2 + \left(1 - \frac{n^2}{k^2 r^2}\right)\{J_n(kr)\}^2 \right]$$

Thus we obtain

$$Y_m(r) = \frac{J_n(k_m r)}{J_n(k_m)} - \frac{J_n(ik_m r)}{J_n(ik_m)} \tag{15}$$

where k_m $(m = 1, 2, \cdots)$ are positive roots of equation (14).

3. Property of Orthogonal Functions $Y_m(r)$

It is known that for a value of real argument z, which is very large in comparison with unity, we have approximately

$$J_n(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

$$J_n'(z) = -\sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - \frac{n\pi}{2}\right) - \frac{1}{2}\sqrt{\frac{2}{nz}} \cdot \frac{1}{z} \cos\left(z - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

$$I_n(z) = \frac{e^z}{\sqrt{2\pi z}}$$

$$I_n'(z) = \frac{e^z}{\sqrt{2\pi z}}\left[1 - \frac{1}{2z}\right]$$

We note also that (McLachlan 1934)

$$\frac{J_n'(ik)}{J_n(ik)} = \frac{(i)^{n-1} I_n'(k)}{(i)^n I_n(k)}$$

Therefore, for a very large value of integer m, k_m will approximately be given by roots of equation

$$\tan\left(k_m - \frac{\pi}{4} - \frac{n\pi}{2}\right) = -1 \tag{16}$$

So that we have approximately

$$\left.\begin{aligned} k_m &= (n+2m)\frac{\pi}{2} \\ \psi &= \frac{2m-1}{2}\pi + \frac{\pi}{4} = k_m - \frac{\pi}{4} - \frac{n\pi}{2} \end{aligned}\right\} \tag{17}$$

On the other hand, when n is a positive integer, we know the maximum value of $|J_n(z)|$ for real positive argument of z. Let us denote this maximum value by $J_n(M)$. Also we have, approximately

$$J_n(k_m) = \sqrt{\frac{2}{\pi k_m}} \cdot \frac{1}{\sqrt{2}} = \sqrt{\frac{1}{\pi k_m}}$$

when m is very large. Thus we obtain an inequality

$$\left|\frac{J_n(k_m r)}{J_n(k_m)}\right| > J_n(M)\sqrt{\pi k_m}\,A \tag{18}$$

where A is a positive constant. On the other hand, we have

$$\frac{J_n(ik_m r)}{J_n(ik_m)} = \frac{I_n(k_m r)}{I_n(k_m)}$$

For positive value of z, the function $I_n(z)$ is an increasing function, and for very large value of m (that is, for very large value of k_m), there exist two constants B, C such that

$$I_n(k_m r) < B\frac{\exp(k_m \eta)}{\sqrt{2\pi k_m \eta}}$$

$$I_n(k_m) > C\frac{\exp(k_m)}{\sqrt{2\pi k_m}}$$

so long as we have $0 < r < \eta < 1$. Hence we have an inequality

$$\left|\frac{I_n(k_m r)}{I_n(k_m)}\right| < \frac{B}{C}\frac{1}{\sqrt{\eta}}\exp\{-(1-\eta)k_m\} < E\exp(-\lambda k_m)$$

Thus we deduce from eq. (15) that when $m\,(k_m)$ is very large there exist a positive constant A_1 such that for any value of r lying in range of $0<r<\eta<1$ we have

$$|Y_m(r)<J_n(M)\sqrt{\pi k_m}A_1 \tag{20}$$

4. Expansion of Fourier-type of Function which does not satisfy the Boundary Condition

As was already remarked, any function $F(r)$ which satisfy following three conditions, can be expanded into an infinite series of Fourier type in form of

$$F(r)=a_1Y_1(r)+a_2Y_2(r)\cdots+a_mY_m(r)+\cdots \tag{21}$$

(α) F, F', F'' and F''' are finite and continuous in the interval $0\leq r\leq 1$

(β) at $r=0$, F, F' are finite and regular

(γ) at $r=1$, $F(1)=0$ and $F'(1)=0$

Even if the given function $F(r)$ does not satisfy these conditions (α), (β), (γ), we can obtain formal expansion in form of (21), but we can say nothing about validity (convergence) of it. In what follows, we shall consider about some cases of functions $F(r)$, which do not satisfy these conditions (α), (β), and (γ).

Let us take up the case of a function $F(r)$ which satisfies condition (α) and (β), but not the condition (γ). For that case we form a new function $G(r)$, in following manner

$$G(r)=\begin{cases} F(r) & 0\leq r\leq 1-c \\ \varphi(\xi) & 0\leq\xi\leq c \qquad (\xi=1-r)\end{cases}$$

where c is a positive constant which we may take as small as we please. Suitably choosing the function $\varphi(\xi)$ we can make this new function $G(r)$ to satisfy all conditions (α), (β), and (γ). One way to accomplish this purpose is to put

$$\varphi(\xi)=a_2\xi^2+a_3\xi^3+a_4\xi^4+a_5\xi^5$$

here we have (KITO 1973)

$$a_2=10\varphi(c)-\ 6\varphi'(c)+\frac{2}{3}\varphi''(c)-\frac{1}{6}\varphi'''(c)$$

$$-a_3=20\varphi(c)-14\varphi'(c)+\ 4\,\varphi''(c)-\frac{1}{2}\varphi'''(c)$$

$$a_4=15\varphi(c)-11\varphi'(c)+\frac{7}{2}\varphi''(c)-\frac{1}{2}\varphi'''(c)$$

$$-a_5=\ 4\varphi(c)-\ 3\varphi'(c)+1\,\varphi''(c)-\frac{1}{6}\varphi'''(c)$$

For this modified function $G(r)$, we can make Fourier type expansion in form of (21), which converges in our region $0\leq r\leq 1$. Similar consideration can be made, for example, for a given function $F(r)$ which satisfies the conditions (α), (β), and

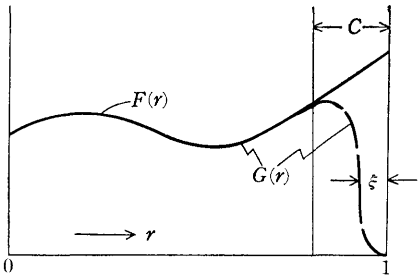

Fig. 3. Modified Function $G(r)$.

(γ) except that $F'(r)$ has a finite discontinuity at $r=e$ $(0<e<1)$.

Next, let us show, by actual example, that there exist many functions which do not satisfy the condition (γ) but whose expansion in form of (21) is absolutely convergent in the region $0\leq r\leq 1$. For this purpose, we remark a formula relating to integral involving Bessel functions namely (k, l being two positive constants).

$$\int_0^r \frac{J_n(kr)}{J_n(k)} J_n(lr) r dr = \frac{r}{k^2-l^2}\left[l\frac{J_n(kr)}{J_n(k)} J_n'(lr) - k\frac{J_n'(kr)}{J_n(k)} J_n(lr)\right]$$

Denoting by $J(k, l)$ value of above integral for $r=1$, we have

$$J(k, l) = \frac{1}{k^2-l^2}\left[lJ_n'(l) - k\frac{J_n'(k)}{J_n(k)} J_n(l)\right]$$

When we make k very large in comparison with unity, while l remain fixed, we have approximately

$$J(k, l) = \frac{1}{k^2}[lJ_n'(l) + kJ_n(l)\tan\psi] = \frac{1}{k}\left[+J_n(l)\tan\psi + \frac{a}{k} + \cdots\right]$$

Next, we have

$$\begin{aligned} I(k, l) &= \int_0^1 \frac{J_n(ikr)}{J_n(ik)} J_n(lr) r\, dr \\ &= \frac{1}{(ik)^2-l^2}\left[\frac{lJ_n(ik)}{J_n(ik)} J_n'(l) - ikJ_n(l)\frac{J_n'(ik)}{J_n(ik)}\right] \\ &= \frac{-1}{k^2+l^2}\left[lJ_n'(l) - kJ_n(l)\frac{I_n'(k)}{I_n(k)}\right] \end{aligned}$$

which, again when k is very large, can be written approximately

$$I(k, l) = \frac{J_n(l)}{k}\left[1 + \frac{b}{k} + \cdots\right]$$

Combining these approximate estimates we obtain following approximate formula for coefficients a_m in expression of (21), for any large value of m, as follows;

$$a_m=\int_0^1 J_n(lr)Y_m(r)r\,dr=\frac{J_n(l)}{k_m}\left[-2+\frac{b}{k_m}+\cdots\right]$$

Let us then, consider a function $Y(r)$, represented by the expression

$$Y(r)=\alpha_1 J_n(l_1 r)+\cdots+\alpha_s J_n(l_s r)$$

wherein $\alpha_1, l_1, \cdots, \alpha_s, l_s$ are constants. When these constants are chosen such that

$$\alpha_1 J_n(l_1)+\cdots+\alpha_s J_n(l_s)=0$$

the coefficient a_m of expansion in form of (21), of this function will be such that

$$|a_m|<\frac{A}{k_m^2}$$

where A is a fixed positive constant.

Thus we see that the infinite series (21) is absolutely convergent for $0\leq r\leq 1$, showing us that there actually exist infinitely many functions $F(r)$, which admit expansion in form of (21), even if $Y(r)$ do not satisfy the condition (γ). What will be represented by this expansion, for the case of these function $F(r)$? The answer is given by Fischer-Riesz theorem (WIENER 1933) in functional analysis. Putting

$$S_M=a_1 Y_1(r)+\cdots+a_M Y_M(r)$$

$$S_N=a_1 Y_1(r)+\cdots+a_N Y_N(r)$$

the value of integral

$$A_{MN}=\int_0^1 [S_M-S_N]^2 r\,dr$$

tends to zero as $M\to\infty$, $N\to\infty$, M and N being made to infinity independently of each other. This means that in this case the infinite series (21) represents function $F(r)$ almost everywhere in the range of $0\leq r\leq 1$, in the terminology of functional analysis.

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