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By

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COMPUTATIONAL METHODS FOR SOLVING THE NONLINEAR COMPLEMENTARITY PROBLEM

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Abstract

In the present paper, two combinatorial methods which approximate solutions of the nonlinear complementarity problem are given, and then one of them is applied to several problems in fixed point theory, mathematical programming and game theory.

Introduction

For a given continuous function f from the β-dimensional Euclidean space, R^β into itself, the problem of finding an $x \in R^\beta$ such that

$$x \geqq 0, \quad f(x) \geqq 0 \qquad \text{and} \qquad x_i \cdot f_i(x) = 0 \quad (i = 1, 2, \cdots, \beta)$$

is known as the complementarity problem (abbreviated by the CP), where x_i and $f_i(x)$ denote the i-th components of x and $f(x)$, respectively. If f is linear, piecewise linear or nonlinear, the problem is said to be linear, piecewise linear or nonlinear, respectively. The CP is closely connected with mathematical programming, game theory, economic equilibrium theory and fixed point theory; the studies of the CP brought about many fruitful results in such theories.

The linear CP was first studied by DANTZIG and COTTLE (1967) in the field of mathematical programming. They called attention to the fact that the KUHN-TUCKER stationary condition corresponding to a quadratic programming problem becomes the linear CP, and provided an algorithm for solving the linear CP, which was referred to as the principal pivoting method; the algorithm solves convex quadratic programming problems.

Another algorithm for the linear CP is LEMKE's algorithm which was originally developed for finding Nash equilibrium points for bimatrix games (LEMKE and

HOWSON (1964)), and later extended to general linear complementarity problems (LEMKE (1965)). Under certain conditions (see, for example, COTTLE and DANTZIG (1968), EAVES (1971-b) and SAIGAL (1972)), LEMKE's algorithm either computes a solution of the CP or shows that the problem has no solution. LEMKE's algorithm has several applications in linear programming (RAVINDRAN (1970)), quadratic programming (EAVES (1971-c)), noncooperative games (WILSON (1971)), ROSENMÜLLER (1971)), etc.

Since LEMKE's algorithm appeared, a lot of combinatorial methods have been developed. By combining LEMKE's idea and the notion of primitive sets, SCARF (1967-a) established an algorithm for finding an equilibrium point of the balanced cooperative game. His algorithm was later applied to mathematical programming, fixed point theory and economic equilibrium theory (SCARF (1967-b) and SCARF and HANSEN (1969)). It is noteworthy that he first gave a constructive proof of the BROUWER fixed point theorem by using his algorithm. KUHN (1968) made the essential structure of SCARF's algorithm clear, and introduce the notion of simplicial subdivisions into the combinatorial methods. The notion was succeeded by EAVES. He showed an algorithm for approximating KAKUTANI fixed points (1971-a). Roughly speaking, these fixed point algorithms work as follows. First, the original problem is approximated by a fixed point problem with a continuous piecewise linear function. The simplicial subdivision plays a main role in the approximation process. Then a combinatorial iteration procedure computes in a finite number of steps a fixed point of the approximate problem. One defect of the above process is that to get a better approximate fixed point of the original problem the combinatorial iteration procedure to the approximate problem with higher accuracy is entirely repeated. In other words, the information which is obtained in computation of an approximate fixed point is not used to compute approximate fixed point with higher accuracy. EAVES (1972-b) solved this difficulty by introducing a simplicial subdivision with a special structure; the algorithm provided in EAVES (1972-b) generates an infinite sequence of approximate fixed points which, by itself, converges to an exact fixed point. Further, EAVES and SAIGAL (1972) extend the result to fixed point problems on unbounded regions.

There are some extentions of LEMKE's algorithm to certain classes of piecewise linear functions. The first one is due to SCARF (1966). He gave an algorithm for a class of concave piecewise liner functions. COTTLE and DANTZIG (1970) dealt with a class of convex piecewise linear functions. These two classes were unified by SEKINE, NISHINO and the author (1973). They showed that LEMKE's algorithm can be so extended as to be applicable to a general class of piecewise linear functions. The class includes the above two classes as a special case. Theoretically, any continuous function can be approximated by some functions of the class. This implies that their algorithm can be used to compute approximate solutions of the nonlinear CP. The practical application of their algorithm to the nonlinear CP, however, has not been done yet.

Some sufficient conditions for the nonlinear CP to have a solution were given by some authors. COTTLE (1966) showed the existence of a solution when f is continuously differentiable and has positively bounded JACOBIAN matrix. This fact was proved by using the idea of the principal pivoting method; but solutions can not be calculated by the method unless f is linear.

KARAMARDIAN (1969 and 1972) pointed out that under certain restrictions on f the nonlinear CP is replaced by the KAKUTANI fixed point problem. EAVES (1971-d) established a basic theorem on the nonlinear CP, and derived a sufficient condition for the CP to have a solution from it. The condition is weaker than some of the restrictions which KARAMARDIAN imposed on f. The basic theorem was proved by the BROWDER fixed point theorem. The above results suggest that the algorithms developed for fixed point problems can also be used for computing solutions of the nonlinear CP.

The main purpose of this paper is to provide two kinds of computational methods for the nonlinear CP. Under a modification of the sufficient condition assumed in EAVES (1971-d), each of the methods generates an infinite sequence of approximate solution of the nonlinear CP.

This paper consists of five chapters. Chap. 1 furnishes the mathematical background for the subsequent chapters. This is done by pointing out a common structure of the combinatorial methods and presenting some mathematical notions.

In Chap. 2 we deal with relations between the CP and the fixed point theorem. First, the basic theorem by EAVES (1971-d) is briefly explained. Then a computational method for solving the nonlinear CP is given. The method is a modification of the fixed point algorithm developed by EAVES and SAIGAL (1972).

Chap. 3 is devoted to another computational method, which is based on the simplicial subdivision with the special structure used in EAVES and SAIGAL and the extended LEMKE's method by SEKINE, NISHINO and the author (1973).

In Chap. 4, the algorithm provided in Chap. 3 is applied to several problems arising in fixed point theory, mathematical programming and game theory. The most important application is the one for a balanced cooperative game. The author's method for approximating points of the core of the game is more efficient than the existing method which was provided by SCARF (1967-a).

In Chap. 5, several computational methods for solving the nonlinear CP are compared. It is shown that the method given in Chap. 3 solves the nonlinear CP for which the other methods are applicable, and that it is characterized by some features which increase computational efficiency; a numerical example illustrates the effectiveness of the features.

Notations

R denotes the set of real numbers, and $R^{m\times n}$ the set of $m\times n$ matrices with real components. We use $R^{1\times n}$ and $R^{m\times 1}$ to denote the sets of row and column vectors, respectively. For $A\in R^{m\times n}$, A^T denote the transpose of A. By $A_{i\cdot}$ and $A_{\cdot j}$ we denote the i-th row and j-th column of $A\in R^{m\times n}$, respectively. If it is unnecessary to distinguish between a row and a column vector, we use R^m. For $x, x'\in R^m$, x_i denotes the i-th component of x, $\langle x, x'\rangle$ the Euclidean inner product of x and x', and $||x||$ the Euclidean norm of x. By E and e, we denote any identity matrix and any vector with all components unity, respectively; their size follows from the context. R^m_+ denotes the nonnegative orthant of R^m, $\{x\in R^m : x\geqq 0\}$. For each $C\subset R^m$, $\operatorname{co} C$ denotes the convex hull of the set C, $|C|$ the number of elements contained in the set C, and $\operatorname{diam} C$ the nonnegative number $\sup\{||x-x'|| : x, x'\in C\}$.

Chap. 1. Mathematical background

We can convert the CP into a minimization problem (abbreviated by the MP in the following): Minimize $g(x)=\sum_{i=1}^{\beta} x_i f_i(x)$ subject to $x \geqq 0$ and $f(x) \geqq 0$. If the CP has a solution $\hat{x}$, it is a global minimum solution of the MP such that $g(\hat{x})=0$. Conversely, if the CP has no solution then the MP has either no global minimum solution or a global minimum solution $\tilde{x}$ such that $g(\tilde{x})>0$. However, any local (but not global) minimum solution of the MP is of no value to the CP. This is the reason why usual iteration procedure for nonlinear programming such as the gradient projection method and the feasible direction method are not suitable for the CP. A common property of them is that they intend to decrease the value of the objective function at each iteration step. When we apply them to nonlinear programming, it happens that though the value of the objective function decreases at each step the global minimum can not be obtained.

While, the algorithms which will be described in Chap. 2 and Chap. 3 are essentially different from such iteration techniques. They have the same structure as the combinatorial methods which have been mentioned in Introduction. The outline of our algorithms is as follows. First, we construct a nonlinear system corresponding to the CP; each solution with a specified property to the nonlinear system is a solution to the CP. Second, by using a simplicial subdivision, we approximate the nonlinear system by a linear inequality system. In this process, lexicographical ordering is introduced into the linear inequality system to avoid degeneracy; such a linear system is said to be a lexico inequality system. Third, we make a graph with a special structure. Each node of the graph is associated with a basic solution to the lexico inequality system. Fourth, we choose an initial node of the graph. Fifth, starting from the initial node of the graph, we obtain a sequence of nodes along edges of the graph. The sequence corresponds to a sequence of approximate solutions of the CP, which converges to a solution of the CP.

In Chap. 1 we give some definitions and properties about lexico inequality systems, simplicial subdivisions and graphs. Proofs of lemmas are omitted here.

§1. Lexico inequality systems

We say that $c \in R^{1\times m}$ is lexico positive if $c_{j^*}>0$ and lexico nonnegative if $c_{j^*} \geqq 0$, where $j^*=\min\{j : c_j \neq 0 \text{ or } j=m\}$. We write these as $c \succ 0$ and $c \succsim 0$, respectively. By $C \succsim 0$, where $C \in R^{n\times m}$, we mean $C_{i\cdot} \succsim 0$ for $i=1, 2, \cdots, n$. Let K be a set of countable elements. Let $L(\cdot)$ be a fixed function from K into $R^{n\times 1}$, and $Q \in R^{n\times m}$ a fixed matrix. Consider the following lexico inequality system:

(1-1) $$\sum_{i\in K} L(i)X(i)=Q \quad \text{and} \quad X(i) \succsim 0 \ (i \in K),$$

where $X(\cdot)$ is an unknown function from K into $R^{1\times m}$. For each solution X of (1-1), let $I(X)$ denote the set $\{i \in K : X(i) \succ 0\}$.

(1-2) **Definition:** A solution X of (1-1) uses a column $L(i)$ if $i \in I(X)$. A solution

X is basic if $\{L(i): i \in I(X)\}$ is linearly independent, and nondegenerate basic if $\{L(i): i \in I(X)\}$ forms a basis of $R^{n\times 1}$. The lexico inequality system (1-1) is nondegenerate if every solution uses at least n columns; if rank $Q=n$ then (1-1) is trivially nondegenerate.

Let X be a nondegenerate basic solution of (1-1), $I(X)=\{i_1, i_2, \cdots, i_n\}$ and $r \in K-I(X)$. Since $\{L(i_1), L(i_2), \cdots, L(i_n)\}$ forms a basis of $R^{n\times 1}$, there exist some real numbers $\lambda(i_j)$ $(j=1, 2, \cdots, n)$ which are uniquely determined by the linear equation

$$\sum_{j=1}^{n} \lambda(i_j) L(i_j) = L(r). \tag{1-3}$$

(1-4) **Lemma:** There is a basic solution which uses the column $L(r)$ and a subset of $\{L(i_1), L(i_2), \cdots, L(i_n)\}$ if and only if $\lambda(i_j)>0$ for some j in (1-3).

§2. Simplicial subdivisions

Let $\xi^0, \xi^1, \cdots, \xi^k$ be points of the m-dimensional Euclidean space R^m, and $\sigma=\{\xi^0, \xi^1, \cdots, \xi^k\}$. If the set $\left\{\begin{pmatrix}\xi^i\\1\end{pmatrix} \in R^{m+1} : i=0, 1, \cdots, k\right\}$ is linearly independent, σ is said to be (k-)simplex. The nonnegative number k is the dimension of σ, and we write $\dim \sigma = k$. Let Ξ be a collection of simplices satisfying the following:

(2-1) **Condition:**
(a) If $\sigma \in \Xi$ and $\phi \neq \sigma' \subset \sigma$ then $\sigma' \in \Xi$.
(b) If $\sigma, \sigma' \in \Xi$ then $\mathrm{co}\,\sigma \cap \mathrm{co}\,\sigma' = \mathrm{co}\,\{\sigma \cap \sigma'\}$.
(c) For every bounded set $U \subset R^m$, the number of $\sigma \in \Xi$ such that $U \cap \mathrm{co}\,\sigma \neq \phi$ is finite.

Such a Ξ is said to be a simplicial complex. The dimension of Ξ, denoted by $\dim \Xi$, is the maximal dimension of $\sigma \in \Xi$. For $k=0, 1, \cdots, \dim \Xi$, let $\Xi_k = \{\sigma \in \Xi : \dim \sigma = k\}$. If $C = \cup \{\mathrm{co}\,\sigma : \sigma \in \Xi\}$, where C is a subset of R^m, then Ξ is said to be a simplicial subdivision of C. For the remainder of this paper, Ξ denotes a simplicial subdivision of $R^\alpha \times [0, \infty)$, where α is an integer included in $\{1, 2, \cdots, \beta\}$. Let $\Delta = \sigma \in \Xi : \sigma \subset R^\alpha \times \{0\}\}$.

(2-2) **Lemma:**
(a) $\dim \Xi = \alpha + 1$.
(b) Δ is a simplicial complex, and a simplicial subdivision of $R^\alpha \times \{0\}$. The dimension of Δ is α.
(c) Suppose that $\bar{\sigma} \in \Xi$. Then the collection $\{\sigma \in \Xi_{\alpha+1} : \bar{\sigma} \subset \sigma\}$ consists of two $(\alpha+1)$-simplices if $\bar{\sigma} \notin \Delta$, and one $(\alpha+1)$-simplex if $\bar{\sigma} \in \Delta$.
(d) If $\bar{\sigma} \in \Delta_{\alpha-1}$ then the collection $\{\sigma \in \Delta_\alpha : \bar{\sigma} \subset \sigma\}$ consists of two α-simplices.

Furthermore, we impose the following condition on Ξ.

(2-3) **Condition:**
(a) diam $\psi_1(\sigma) < \varepsilon^*$ for every $\sigma \in \Xi$, where $\psi_1(\xi_1, \xi_2, \cdots, \xi_\alpha, \xi_{\alpha+1}) = (\xi_1, \xi_2, \cdots, \xi_\alpha)$ for every $(\xi_1, \xi_2, \cdots, \xi_\alpha, \xi_{\alpha+1}) \in R^{\alpha+1}$ and ε^* is a given positive number.

(b) For each infinite sequence $\{\sigma^p : p=1,2,\cdots\}$, if $\lim_{p\to\infty} \min \psi_2(\sigma^p)=+\infty$ then $\lim_{p\to\infty}$ diam $\psi_1(\sigma^p)=0$, where $\psi_2(\xi_1,\cdots,\xi_\alpha,\xi_{\alpha+1})=\xi_{\alpha+1}$ for every $(\xi_1,\cdots,\xi_\alpha,\xi_{\alpha+1})\in R^{\alpha+1}$.

EAVES (1972-b) provided two kinds of simplicial subdivisions, K_1 and K_2 of $\{x\in R_+^{\alpha+1}: \sum_{i=1}^{\alpha+1} x_i=1\}\times[0,\infty)$ which satisfy the above condition. They were later extended by EAVES and SAIGAL (1972) to the simplicial subdivision of $R^\alpha\times[0,\infty)$; one of them uses a structure of K_1, and another K_2.

Since Δ is a simplicial subdivision of $R^\alpha\times\{0\}$, there is an α-simplex σ^0 such that $(0,0)\in\text{co}\,\sigma^0$. There exist nonnegative numbers $\lambda(\xi)$ $(\xi\in\sigma^0)$ which are uniquely determined by

$$0=\sum_{\xi\in\sigma^0}\psi_1(\xi)\lambda(\xi) \qquad \text{and} \qquad 1=\sum_{\xi\in\sigma^0}\lambda(\xi) \tag{2-4}$$

Choose $\xi^0\in\sigma^0$ such that $\lambda(\xi^0)>0$, and let $\{\xi^1,\xi^2,\cdots,\xi^\alpha\}=\sigma^0-\{\xi^0\}$. Then the $\alpha\times\alpha$ matrix

$$\Phi=-[\psi_1(\xi^1),\psi_1(\xi^2),\cdots,\psi_1(\xi^\alpha)] \tag{2-5-α}$$

in nonsingular. Now define

$$\tilde{\Lambda}(\xi^0)=(\lambda(\xi^0),0)\in R^{1\times(1+\alpha)} \quad \text{and} \quad \tilde{\Lambda}(\xi^i)=(\lambda(\xi^i),E_{i\cdot})\in R^{1\times(1+\alpha)} \quad (i=1,2,\cdots,\alpha) \tag{2-6-α}$$

Then for $\sigma=\sigma^0$, $[\tilde{\Lambda}(\xi)(\xi\in\sigma^0)]$ is a solution of the lexice inequality system

$$\sum_{\xi\in\sigma}\begin{bmatrix}-\psi_1(\xi)\\ 1\end{bmatrix}\Lambda(\xi)=\begin{bmatrix}0 & \Phi\\ 1 & e\end{bmatrix} \qquad \text{and} \qquad \Lambda(\xi)\succsim 0 \quad (\xi\in\sigma) \tag{2-7-σ}$$

It follows from Condition (2-1) that for any $\sigma\in\Delta$ such that $\sigma\neq\sigma^0$ (2-7-σ) has no solution.

§3. The graph principle

A graph G is a pair of two sets, $N(G)$ and $E(G)$ satisfying the following condition:

(3-1) **Condition:**
(a) $N(G)$ is a nonempty set of at most countable elements.
(b) $E(G)$ is a subset of $\{\{\omega,\omega'\}:\omega,\omega'\in N(G)$ and $\omega\neq\omega'\}$.

$\omega\in N(G)$ is said to be a node of G, and $\{\omega,\omega'\}\in E(G)$ an edge of G. When $N(G)$ is finite, we call G a finite graph, and otherwise an infinite graph. If $\{\omega,\omega'\}\in E(G)$, two nodes ω and ω' are said to be adjacent. The number of nodes which are adjacent to a node ω is said to be the degree of ω, and is denoted by $\deg\omega$. The following iteration procedure can be applied to any pair (G,ω^0), where G is a graph and $\omega^0\in N(G)$.

(3-2) **Iteration procedure:**
(i) Let $\omega^1=\omega^0$ and $p=1$.

(ii) If there is a node ω different from ω^{p-1} and adjacent to ω^p, let $\omega^{p+1}=\omega$. Otherwise stop.
(iii) Increase p by 1 and return to (ii).

But the generated sequence $\omega^1, \omega^2, \cdots$ by the iteration is not necessarily unique. Now we consider the class of graphs satisfying the following condition:

(3–3) **Condition:**
(a) There is a node $\omega^0 \in N(G)$ such that $\deg \omega^0=0$ or 1.
(b) $\deg \omega=0, 1$ or 2 for each $\omega \in N(G)$.

(3–4) **Graph principle:** When a graph G satisfies Condition (3–3) Iteration procedure (3–2) uniquely generates a finite sequence of distinct nodes $\omega^1, \omega^2, \cdots$. Suppose that the sequence is a finite sequence $\omega^1, \omega^2, \cdots, \omega^r$. If $\omega^1=\omega^r$ then $\deg \omega^0=0$, and otherwise $\deg \omega^0=\deg \omega^r=1$.

Chap. 2. Relations between the CP and the fixed point theorem

This chapter consists of §4, §5 and §6. In §4 we briefly review the basic theorem of the CP (Eaves (1971–d)), from which the author derives a sufficient condition for the CP to have a solution. This condition is slightly weaker than that which Eaves himself provided as a corollary of the basic theorem. In §5 and §6 we describe a computational method for the nonlinear CP.

§4 Eaves's basic theorem of the complementarity problem

Let C be a convex subset of R^β. If $x \in R^\beta$ satisfies

(4–1) $$x \in C \quad \text{and} \quad \langle x'-x, f(x)\rangle \geqq 0 \quad \text{for every} \quad x' \in C,$$

x is said to be a stationary point of the pair (f, C). Let $D_\rho=\{x \in R^\beta_+ : \langle d, x\rangle \leqq \rho\}$, where $d \in R^\beta$ is positive and ρ is a nonnegative number. Define the upper semicontinuous point to set function $\Gamma(\cdot\,;\rho)$, with a parameter $\rho \in R_+$, from R^β into the class of compact convex polyhedra of R^β_+ as follows:

(4–2) $$\Gamma(x;\rho)=\{u \in D_\rho : \langle u, f(x)\rangle=\min_{v \in D_\rho}\langle v, f(x)\rangle\} \quad \text{for each} \quad x \in R^\beta.$$

Then the following three are equivalent:
(i) $x \in R^\beta$ is a stationary point of the pair (f, D_ρ).
(ii) $x \in \Gamma(x;\rho)$.
(iii) There is an $x_0 \in R$ such that

(4–3) $$x_0 \geqq 0,\ x \geqq 0, f(x)+dx_0 \geqq 0,\ \rho-\langle d, x\rangle \geqq 0,$$
$$\langle x, f(x)+dx_0\rangle=0 \quad \text{and} \quad x_0(\rho-\langle d, x\rangle)=0.$$

Eaves (1971–d) established the following basic theorem by using the Browder fixed point theorem, and gave a sufficient condition for the CP to have a solution as an application of it. We say that a bounded set $B \subset R^{\beta}_{+} - D$ separates D from ∞ if each unbounded closed connected set in R^{β}_{+} that meets D also meets B.

(4-4) **Theorem:** (EAVES (1971-d)). There exists a closed connected subset S of R^{β}_{+} such that
(a) $x \in \Gamma(x;\rho)$ for each $x \in S$ and $\rho = \langle x, d \rangle$,
(b) for each $\rho \geqq 0$ there is an $x \in S \cap \Gamma(x;\rho)$.

(4-5) **Corollary:** (EAVES (1971-d)). Suppose that $B \subset R^{\beta}_{+} - D_{\rho}$ separates D_{ρ} (for some $\rho \geqq 0$) from ∞ and that for each $\bar{x} \in B$ there is an $x \in D_{\rho}$ for which $\langle x - \bar{x}, f(\bar{x}) \rangle \leqq 0$. Then the complementarity problem has a solution.

Now the author shows that the CP has a solution under a weaker sufficient condition than that of Corollary (4-5). The idea for the proof is the same as Corollary (4-5), and the proof is omitted.

(4-6) **Corollary:** Suppose that $B \subset R^{\beta}_{+} - \{0\}$ separates the origin $\{0\}$ from ∞ and that for each $\bar{x} \in B$ there is an $x \in R^{\beta}_{+}$ for which $\langle x - \bar{x}, d \rangle < 0$ and $\langle x - \bar{x}, f(\bar{x}) \rangle \leqq 0$. Then the CP has a solution.

§5. The graph F

The remainder of this chapter is devoted to giving a computational method for solving the nonlinear CP, which is based on the fixed point algorithm developed by EAVES and SAIGAL (1972). For this purpose, we impose the following condition on the function f.

(5-1) **Condition:** Let $d \in R^{\beta}$ be a positive vector. A compact set $B \subset R^{\beta}_{+} - \{0\}$ separates the origin from ∞, and for each $\bar{x} \in B$ there is an $x \in R^{\beta}_{+}$ such that $\langle x - \bar{x}, d \rangle < 0$ and $\langle x - \bar{x}, f(\bar{x}) \rangle < 0$.

This condition is stronger than that of Corollary (4-6). Hence the CP has a solution. The following lemma is easily verified.

(5-2) **Lemma:** $\Gamma(x;\rho) = \rho \cdot \Gamma(x;1)$ for all $\rho \geqq 0$. For each $x \in R^{\beta}$ there is an $\varepsilon > 0$ such that $\Gamma(x';\rho) \subset \Gamma(x;\rho)$ for all $x' \in B_{\varepsilon}(x)$ and all $\rho \geqq 0$.

(5-3) **Lemma:** There is an $\varepsilon^* > 0$ such that if $C \subset R^{\beta}, C \cap B \neq \phi$ and diam $C < \varepsilon^*$ then $\{\mathrm{co}\, C\} \cap [\mathrm{co}\, \{\Gamma(x;\rho) : \rho \geqq 0, x \in C\}] = \phi$.

Proof. Let $\bar{x} \in B$. Assume that $\bar{x} \in \Gamma(\bar{x};\rho)$ for some $\rho \geqq 0$. Choose $x \in R^{\beta}_{+}$ such that $\bar{x}$ and x satisfy the two inequalities of Condition (5-1). Since (4-3) holds for the $\bar{x}$, we see $0 \leqq \langle x, f(\bar{x}) + d\bar{x}_0 \rangle = \langle x - \bar{x}, f(\bar{x}) + d\bar{x}_0 \rangle < \langle x - \bar{x}, d \rangle \bar{x}_0 \leqq 0$. This is a contradiction. Thus we have shown $\bar{x} \notin \Gamma(\bar{x};\rho)$ for any $\rho \geqq 0$, or equivalently $\bar{x} \notin \bigcup_{\rho \geqq 0} \Gamma(\bar{x};\rho) = \bigcup_{\rho \geqq 0} \rho \cdot \Gamma(\bar{x};1)$ (the last equality follows from Lemma (5-2)). Since $\bigcup_{\rho \geqq 0} \rho \cdot \Gamma(\bar{x};1)$ is a closed convex set, there is an ε_1 satisfying $B_{\varepsilon_1}(\bar{x}) \cap [\mathrm{co}\{\Gamma(\bar{x};\rho) : \rho \geqq 0\}] = \phi$. By Lemma (5-2), we also have an $\varepsilon_2 > 0$ such that

$$\mathrm{co}\, \{\Gamma(x';\rho) : \rho \geqq 0\} \subset \mathrm{co}\, \{\Gamma(\bar{x};\rho) : \rho \geqq 0\} \qquad \text{for all} \quad x' \in B_{\varepsilon_2}(\bar{x}).$$

Hence, by defining $\varepsilon=\min\{\varepsilon_1, \varepsilon_2\}$, we obtain

$$B_\varepsilon(\bar{x})\cap[\mathrm{co}\{\Gamma(x';\rho):\rho\geqq 0, x'\in B_\varepsilon(\bar{x})\}]=\phi.$$

By the above argument, for each $x\in B$ there is an $\varepsilon(x)$ such that

$$B_{\varepsilon(x)}(x)\cap[\mathrm{co}\{\Gamma(x';\rho):\rho\geqq 0, x'\in B_{\varepsilon(x)}(x)\}]=\phi.$$

Since every covering of a compact set by open sets contains a finite covering (Borel-Lebesgue Theorem), we can find $x^1, x^2, \cdots, x^k\in B$ such that

$$B\subset\cup\{B_{\frac{\varepsilon(x^j)}{2}}(x^j):j=1,2,\cdots,k\}.$$

Let $\varepsilon^*=\min\left\{\dfrac{\varepsilon(x^j)}{2}:j=1,2,\cdots,k\right\}$. Suppose that $C\cap B\neq\phi$ and $\mathrm{diam}\, C<\varepsilon^*$. Then $C\subset B_{\varepsilon(x^j)}(x^j)$ for some x^j, hence $C\cap[\mathrm{co}\{\Gamma(x';\rho):\rho\geqq 0, x'\in C\}]=\phi$. Q.E.D.

In what follows, we shall assume that the simplicial subdivision Ξ of $R^\beta\times[0,\infty)$ satisfies Condition (2-3) for the ε^* whose existence has been ensured by the above lemma. Let $\rho^+=\max\{\langle x, d\rangle: x\in B\}$. For a given fixed number $\rho^*>\rho^+$, define $\theta(\cdot)$: $[0,\infty)\to[0,\rho^*]$ such that $\theta(\rho)=\min\{\rho,\rho^*\}$ for each $\rho\geqq 0$. Furthermore, define $u(\cdot)$: $\Xi_0\to R^\beta_+$ as follows. For each $\xi\in\Xi_0$, we first find the smallest index i^* such that

$$\frac{f_i(\psi_1(\xi))}{d_i}\geqq\frac{f_{i^*}(\psi_1(\xi))}{d_{i^*}}\qquad\text{for}\quad i=1,2,\cdots,\beta.$$

If $f_{i^*}(\psi_1(\xi))/d_{i^*}\geqq 0$ then let $u(\xi)=0$, and otherwise let $u(\xi)_{i^*}=\theta(\psi_2(\xi))/d_{i^*}$ and $u(\xi)_i=0$ for any $i\neq i^*$. It is obvious that $u(\xi)\in\Gamma(\psi_1(\xi);\theta(\psi_2(\xi)))$. Let $g(\cdot)$ be a function from Ξ_0 into $R^{(\beta+1)\times 1}$ of the form

$$g(\xi)=\begin{bmatrix}u(\xi)-\psi_1(\xi)\\ 1\end{bmatrix}\qquad\text{for each}\quad\xi\in\Xi_0.$$

Let $\Lambda^0(\xi)=\tilde{\Lambda}(\xi)$ for $\xi\in\sigma^0$ and $\Lambda^0(\xi)=0$ for $\xi\notin\sigma^0$, where $\tilde{\Lambda}(\xi)(\xi\in\sigma^0)$ are defined by (2-6-β). Then Λ^0 is a solution of the following lexico inequality system:

(5-4) $$\sum_{\xi\in\Xi_0}g(\xi)\Lambda(\xi)=Q\qquad\text{and}\qquad\Lambda(\xi)\succsim 0\qquad(\xi\in\Xi_0),$$

where

$$Q=\begin{bmatrix}0, & \Phi\\ 1, & e\end{bmatrix}\in R^{(\beta+1)\times(1+\beta)},$$

and Φ is defined by (2-5-β). For a solution Λ of (5-4), let $\tau(\Lambda)$ denote the set $\{\xi\in\Xi_0:\Lambda(\xi)\succ 0\}$. Now we are in position to construct a graph F; $N(F)$ consists of basic solutions of (5-4) such that $\tau(\Lambda)\in\Xi_\beta$, and $E(F)=\{\{\Lambda,\Lambda'\}:\Lambda\neq\Lambda', \tau(\Lambda)\cup\tau(\Lambda')\in\Xi_{\beta+1}\}$. Then the following lemma holds.

(5-5) **Lemma:**
(a) If $\Lambda\in N(F)$ and $\tau(\Lambda)\in\Delta$ then $\Lambda=\Lambda^0\in N(F)$.
(b) $\deg\Lambda^0=1$. If $\Lambda\in N(F)$ and $\tau(\Lambda)\notin\Delta$ then $\deg\Lambda=2$.

Proof. (a) Suppose that $\Lambda\in N(F)$ and $\tau(\Lambda)\in\Delta$. Then $\psi_2(\xi)=0$ for each $\xi\in\tau(\Lambda)$, which implies $\Gamma(\psi_1(\xi);\theta(\psi_2(\xi)))=\{0\}$ for each $\xi\in\tau(A)$. Hence A satisfies (2-7-$\tau(A)$), and $\tau(A)=\sigma^0$. Thus $A=A^0$.

(b) Let $A\in N(F)$ and $\{\tau(A),\bar{\xi}\}\in\Xi_{\beta+1}$. Since $\{g(\xi):\xi\in\tau(A)\}$ forms a basis of $R^{(\beta+1)\times 1}$, there are $\mu(\xi)$ $(\xi\in\tau(A))$ which are uniquely determined by the equation $\sum_{\xi\in\tau(A)} g(\xi)\mu(\xi)=g(\bar{\xi})$. It follows from the $(\beta+1)$-th row of the equation that at least one $\mu(\xi)$ must be positive. Hence, by Lemma (1-4), there is a basic solution $\bar{A}$ which uses $g(\bar{\xi})$ and a subset of $\{g(\xi):\xi\in\tau(A)\}$. Since $\tau(\bar{A})\in\Xi_\beta$ and $\tau(A)\cup\tau(\bar{A})=\{\tau(A),\bar{\xi}\}\in\Xi_{\beta+1}$, $\bar{A}\in N(F)$ is adjacent to A. While, if $\tau(A)\notin\Delta$ then there are exactly two $\bar{\xi}$ such that $\{\tau(A),\bar{\xi}\}\in\Xi_{\beta+1}$, and otherwise one. Thus the desired result holds.

§6. A computational method based on the fixed point algorithm

Now we describe one of the computational methods which we propose in this thesis.

(6-1) **Algorithm:**
(i) Let $p=1$ and $A^1(\xi)=A^0(\xi)$ $(\xi\in\Xi)$.
(ii) Find $\bar{\xi}\in\Xi_0$ such that $\{\tau(A^p),\bar{\xi}\}\in\Xi_{\beta+1}$ and $\bar{\xi}\notin\tau(A^{p-1})$. Compute $g(\bar{\xi})$. Let A^{p+1} be a basic solution of (5-4) which uses $g(\bar{\xi})$ and a subset of $\{g(\xi):\xi\in\tau(A^p)\}$.
(iii) Increase p by 1 and return to (ii).

The algorithm is a unique application of Iteration procedure (3-2) to the pair (F,A^0); the validity of the application has been ensured by the discussion in the preceding section. As the consequence of the above algorithm, an infinite sequence of distinct basic solutions $A^1, A^2, \cdots$ is obtained. For $p=1,2,\cdots$, let

(6-2) $$\lambda^p(\xi)=A^p(\xi)_{\cdot 1}\ (\xi\in\tau(A^p)) \quad\text{and}\quad x^p=\sum_{\xi\in\tau(A)^p}\psi_1(\xi)\lambda^p(\xi).$$

Then, for $p=1,2,\cdots$,

(6-3) $$x^p\in\mathrm{co}\,\{\psi_1(\tau(A^p))\}\cap\mathrm{co}\,\{u(\tau(A^p))\} \quad\text{and}\quad u(\xi)\in\Gamma(\psi_1(\xi);\theta(\psi_2(\xi)))\ (\xi\in\Xi_0).$$

(6-4) **Theorem:** Let S^* be the set of all cluster points of the sequence $\{x^p\}$, then
(a) $\{x^p\}\subset D_{\rho+}$,
(b) S^* is a nonempty subset of solutions of the CP.

Proof. (a) Assume to the contrary that $x^r\notin D_{\rho+}$ for some r. Let

$$T=\left[\bigcup_{p=1}^{r-1}\mathrm{co}\,\{x^p,x^{p+1}\}\right]\cup\{x^r+td:t\geqq 0\}.$$

Then T is unbounded, closed and connected, so that there is an $\bar{x}\in T\cap B$. But $\{x^r+td:t\geqq 0\}\cap B=\phi$. Hence $\bar{x}\in\mathrm{co}\,\{x^q,x^{q+1}\}\cap B$ for some q. Let $\bar{\sigma}=\tau(A^q)\cup\tau(A^{q+1})$. Then $\bar{x}\in\mathrm{co}\,\{\psi_1(\bar{\sigma})\}\cap B$. We also see that $\mathrm{diam}\,\psi_1(\bar{\sigma})<\varepsilon^*$. We can therefore apply Lemma (5-3) so that

$$\mathrm{co}\,\{\psi_1(\bar{\sigma})\}\cap[\mathrm{co}\,\{\Gamma(\psi_1(\xi);\theta(\psi_2(\xi))):\xi\in\bar{\sigma}\}]=\phi\,;$$

hence

$$\mathrm{co}\,\{\psi_1(\tau(A^q))\}\cap[\mathrm{co}\,\{\Gamma(\psi_1(\xi);\theta(\psi_2(\xi))):\xi\in\tau(A^q)\}]=\phi.$$

This contradicts (6–3). Thus we have shown (a).
(b) Let $\hat{x}\in S^*$. Then a subsequence of $\{x^p\}$ converges to $\hat{x}$. For simplicity of notations, we assume that the sequence $\{x^p\}$ itself converges to $\hat{x}$. It follows from $x^p\in\psi_1(\tau(\Lambda^p))\cap D_{\rho^+}\neq\phi$ for $p=1,2,\cdots$ and Condition (2–1–c) that

(6–5) $$\lim_{p\to\infty}\min\psi_2(\tau(\Lambda^p))=+\infty.$$

Hence, by Condition (2–3–b),

(6–6) $$\lim_{p\to\infty}\operatorname{diam}\psi_1(\tau(\Lambda^p))=0.$$

(6–5) implies that $\min\psi_2(\tau(\Lambda^p))>\rho^*$ for all $p\geqq r$ and some r. Hence, we have from (6–3) that

(6–7) $$u(\xi)\in\Gamma(\psi_1(\xi);\rho^*)\qquad\text{for all}\quad \xi\in\cup\{\tau(\Lambda^p):p\geqq r\}.$$

On the other hand, by Lemma (5–2), we can find a positive number ε such that

(6–8) $$\Gamma(x';\rho^*)\subset\Gamma(\hat{x};\rho^*)\qquad\text{for all}\quad x'\in B_\varepsilon(\hat{x}).$$

The relation (6–6) together with $\lim_{p\to\infty}x^p=\hat{x}$ implies that there is a positive integer $q\geqq r$ for which $\psi_1(\xi)\in B_\varepsilon(\hat{x})$ for all $\xi\in\tau(\Lambda^p)$ and all $p\geqq q$. By (6–7) and (6–8) we obtain $u(\xi)\in\Gamma(\hat{x};\rho^*)$ for all $\xi\in\tau(\Lambda^p)$ and all $p\geqq q$. Since $\Gamma(\hat{x};\rho^*)$ is convex, we see $\operatorname{co}\{u(\tau(\Lambda^p))\}\subset\Gamma(\hat{x};\rho^*)$ for all $p\geqq q$. By recalling (6–3), we see $x^p\in\Gamma(\hat{x};\rho^*)$ for all $p\geqq q$. Since $\Gamma(\hat{x};\rho^*)$ is closed, we obtain $\hat{x}\in\Gamma(\hat{x};\rho^*)$. But $\langle\hat{x},d\rangle<\rho^*$. Thus $\hat{x}$ is a solution of the CP (see (i), (ii) and (iii) in §4). Q.E.D.

Finally, we evaluate the accuracy of an approximate solution x^p in terms of $\operatorname{diam}\psi_1(\tau(\Lambda^p))$. Suppose that the sequence $\{x^p\}$ defined by (6–2) converges to a solution of the CP. As stated in the above proof, there is a positive integer r satisfying $\psi_2(\tau(\Lambda^p))>\rho^*$ for all $p\geqq r$; hence (6–7) holds. Let $p\geqq r$ be fixed. Now we show that $f(\psi_1(\xi))\geqq 0$ for some $\xi\in\tau(\Lambda^p)$. Assume to the contrary that $f(\psi_1(\xi))\ngeqq 0$ for any $\xi\in\tau(\Lambda^p)$. Then, by the construction of $u(\xi)$, we see that $\langle u(\xi),d\rangle=\rho^*$ for all $\xi\in\tau(\Lambda^p)$. It follows from (6–3) that $\langle x^p,d\rangle=\rho^*$, which contradicts Theorem (6–4). Thus we have obtained that

(6–9) $$f(\psi_1(\bar{\xi}))\geqq 0\qquad\text{for some}\quad \bar{\xi}\in\tau(\Lambda^p).$$

Suppose that $\xi_k>0$ for some $k\in\{1,2,\cdots,\beta\}$ and all $\xi\in\tau(\Lambda^p)$. Then, by (6–3), we can find $\tilde{\xi}\in\tau(\Lambda^p)$ such that $u(\tilde{\xi})_k>0$. By the construction of $u(\tilde{\xi})$, $f_k(\psi_1(\xi))$ must be negative. Thus we have shown that

(6–10) $$f_k(\psi_1(\tilde{\xi}))<0\qquad\text{for some}\quad \tilde{\xi}\in\tau(\Lambda^p),\quad\text{if}\quad \bar{\xi}_k>0\quad\text{for all}\quad \xi\in\tau(\Lambda^p)$$

It follows from (6–9), (6–10) and $x^p\in\psi_1(\tau(\Lambda^p))\cap R^\beta_+\neq\phi$ that for each $x\in\psi_1(\tau(\Lambda^p))\cap R^\beta_+$ and $i=1,2,\cdots,\beta$

(6–11)
$$f_i(x)\geqq f_i(x)-f_i(\psi_1(\bar{\xi}))\qquad\text{for some}\quad \bar{\xi}\in\tau(\Lambda^p)$$
and
$$f_i(x)\leqq f_i(x)-f_i(\psi_1(\tilde{\xi}))\qquad\text{for some}\quad \tilde{\xi}\in\tau(\Lambda^p),\quad\text{if}\quad x_i>\operatorname{diam}\psi_1(\tau(\Lambda^p)$$

The right hand side of the last two inequalities is approximately zero if diam $\psi_1(\tau(\Lambda^p))$ is sufficiently small. Thus each $x \in \psi_1(\tau(\Lambda^p)) \cap R^\beta_+$ can be regarded as an approximate solution of the CP. Furthermore, if f is continuously differentiable each $x \in \psi_1(\tau(\Lambda^p)) \cap R^\beta_+$ satisfies that

$$f_i(x) > -\nu_i \operatorname{diam} \psi_1(\tau(\Lambda^p)) \quad (i=1, 2, \cdots, \beta)$$

(6-12) and

$$|f_i(x)| < \nu_i \operatorname{diam} \psi_1(\tau(\Lambda^p)) \quad \text{if} \quad x_i > \operatorname{diam} \psi_1(\tau(\Lambda^p)),$$

where $\nu_i = \max\{\langle \nabla f_i(x'), y' \rangle : x' \in D_{\rho^*}, \|y'\|=1\}$. The above relation gives an accuracy of the approximate solution x in terms of diam $\psi_1(\tau(\Lambda^p))$.

Chap. 3. An extention of Lemke's method

Throughout this part we deal with the CP which has a function of the form

$$h(x_1, x_2, \cdots, x_\alpha) + \sum_{i=\alpha+1}^{\beta} A_{\cdot i} x_i + b$$

where $h(\cdot) : R^\alpha \to R^{\beta \times 1}$ is a nonlinear function, $A_{\cdot i} \in R^{\beta \times 1}$ $(i=\alpha+1, \cdots, \beta)$ and $b \in R^{\beta \times 1}$. In the case $\alpha=0$ the CP under consideration is linear and the algorithm given in §7 and §8 coincides with Lemke's algorithm for the linear CP. For any $\gamma \in \{\alpha, \alpha+1, \cdots, \beta\}$, we can use a simplicial subdivision of $R^\gamma \times [0, \infty)$ in the algorithm. But in view of practical computation, $\gamma = \alpha$ is the best.

§7. The graph H

Let Ξ be a simplicial subdivision of $R^\alpha \times [0, \infty)$ satisfying Condition (2-3). Let

$$F = (0, \Phi, 0) \in R^{\alpha \times (1+\alpha+\beta)},\ G = (1, e, 0) \in R^{1 \times (1+\alpha+\beta)},$$

$$H = (b, 0, E) \in R^{\beta \times (1+\alpha+\beta)},\ W^0(\xi) = (\tilde{A}(\xi), 0) \in R^{1 \times (1+\alpha+\beta)} \quad (\xi \in \sigma^0)$$

and

$$W^0(\xi) = (0, 0) \in R^{1 \times (1+\alpha+\beta)} \quad (\xi \notin \sigma^0),$$

where Φ is defined by (2-5-α) and $\tilde{A}(\xi)$ by (2-6-α). Let $d \in R^\beta_+$ satisfy

(7-1) $$d_i > 0 \quad \text{if} \quad [H + \sum_{\xi \in \Xi_0} h(\psi_1(\xi)) W^0(\xi)]_{i\cdot} \precsim 0.$$

Define

$$L(\xi) = \begin{bmatrix} -\psi_1(\xi) \\ 1 \\ -h(\psi_1(\xi)) \end{bmatrix} \in R^{(\alpha+1+\beta) \times 1} \qquad (\xi \in \Xi_0)$$

$$M_{\cdot 0} = \begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix} \in R^{(\alpha+1+\beta) \times 1},$$

$$M_{\cdot i}=\begin{bmatrix} E_{\cdot i} \\ 0 \\ 0 \end{bmatrix} \in R^{(\alpha+1+\beta)\times 1} \qquad (i=1, 2, \cdots, \alpha),$$

$$M_{\cdot i}=\begin{bmatrix} 0 \\ 0 \\ -A_{\cdot i} \end{bmatrix} \in R^{(\alpha+1+\beta)\times 1} \qquad (i=\alpha+1, \cdots, \beta),$$

$$N_{\cdot j}=\begin{bmatrix} 0 \\ 0 \\ E_{\cdot j} \end{bmatrix} \in R^{(\alpha+1+\beta)\times 1} \qquad (j=1, 2, \cdots, \beta),$$

$$M=(M_{\cdot 0}, M_{\cdot 1}, \cdots, M_{\cdot \beta}) \in R^{(\alpha+1+\beta)\times(1+\beta)},$$

$$N=(N_{\cdot 1}, N_{\cdot 2}, \cdots, N_{\cdot \beta}) \in R^{(\alpha+1+\beta)\times \beta},$$

$$Q=\begin{bmatrix} F \\ G \\ H \end{bmatrix} \in R^{(\alpha+1+\beta)\times(1+\alpha+\beta)}.$$

Now consider the following lexico inequality system.

$$\text{(7-2)} \qquad \begin{aligned} &\sum_{\xi \in \Xi_0} L(\xi)\, W(\xi) + MX + NY = Q \\ &W(\xi) \succsim 0 \quad (\xi \in \Xi_0),\ X \succsim 0,\ Y \succsim 0, \end{aligned}$$

where

$$W(\xi) \in R^{1\times(\alpha+1+\beta)} \qquad (\xi \in \Xi_0),$$

$$X=\begin{bmatrix} X_{0\cdot} \\ \vdots \\ X_{\beta\cdot} \end{bmatrix} \in R^{(1+\beta)\times(1+\alpha+\beta)} \quad \text{and} \quad Y=\begin{bmatrix} Y_{1\cdot} \\ \vdots \\ Y_{\beta\cdot} \end{bmatrix} \in R^{\beta\times(1+\alpha+\beta)}$$

are unknown variable matrices. Every solution of (7-2) is denoted by a triplet (W, X, Y). Let us introduce some new symbols. For each solution (W, X, Y) of (7-2) we define $\tau(W)=\{\xi \in \Xi_0 \cdot W(\xi) \succ 0\}$, $I(X)=\{i : X_{i\cdot} \succ 0\}$, $J(Y)=\{j : Y_{j\cdot} \succ 0\}$ and $\eta(W, X, Y)=\{L(\xi) : \xi \in \tau(W)\} \cup \{M_{\cdot i} : i \in I(X)\} \cup \{N_{\cdot j} : j \in J(Y)\}$. We are interested in solutions (W, X, Y) satisfying

$$\text{(7-3)} \qquad \tau(W)=\sigma \quad \text{for some} \quad \sigma \in \Delta \quad \text{and} \quad I(X) \cap J(Y)=\phi$$

or

$$\text{(7-4)} \qquad \tau(W)=\sigma \quad \text{for some} \quad \sigma \in \Xi,\ 0 \notin I(X) \quad \text{and} \quad I(X) \cap J(Y)=\phi.$$

Let Ω^1 denote the set of all solutions satisfying (7-3), and Ω^2, (7-4). Every $(W, X, Y) \in \Omega^1 \cup \Omega^2$ is said to be a complementary solution of (7-2). Now we are in position to construct the graph H; $N(H)$ and $E(H)$ are defined as follows:

$$N(H)=\{W, X, Y) : \text{complementary basic solutions of (7-2)}\},$$

$$E(H)=\{\{(W, X, Y), (\overline{W}, \overline{X}, \overline{Y}\} :$$

$$(W, X, Y), (\bar{W}, \bar{X}, \bar{Y}) \in N(H) \quad \text{and}$$

$$(1-\lambda)(W, X, Y)+\lambda(\bar{W}, \bar{X}, \bar{Y}) \in \Omega^1 \cup \Omega^2 - N(H) \quad \text{for all} \quad \lambda \in (0, 1)\}.$$

The graph H is completely determined by $h(\cdot): R^\beta \to R^\beta$, $A_{\cdot i} \in R^{\alpha \times 1}(i=\alpha+1, \cdots, \beta)$, $b \in R^{\beta\times 1}$, $d \in R^{\beta \times 1}$ and Ξ. The remainder of this section is devoted to showing that the graph H satisfies Condition (3-3-b).

(7-5) **Lemma:** If $(W, X, Y) \in N(H) \cap \Omega^1$ then exactly one of the three cases (a), (b) and (c) occurs, and if $(W, X, Y) \in N(H) \cap \Omega^2$ then exactly one of the three cases (a), (d) and (e) occurs, where

(a) $\tau(W) \in \Delta_\alpha$ and $I(X) \cup J(Y)=\{1, 2, \cdots, \beta\}$,
(b) $\tau(W) \in \Delta_\alpha$ and $I(X) \cup J(Y)=\{0, 1, 2, \cdots, \beta\}-\{i^*\}$ for some $i^* \neq 0$,
(c) $\tau(W) \in \Delta_{\alpha-1}$ and $I(X) \cup J(Y)=\{0, 1, 2, \cdots, \beta\}$,
(d) $\tau(W) \in \Xi_\alpha - \Delta$ and $I(X) \cup J(Y)=\{1, 2, \cdots, \beta\}$,
(e) $\tau(W) \in \Xi_{\alpha+1}$ and $I(X) \cup J(Y)=\{1, 2, \cdots, \beta\}-\{i^*\}$ for some $i^* \neq 0$.

Proof. If $(W, X, Y) \in N(H)$, then it follows from rank $Q=1+\alpha+\beta$ that $|\tau(W)|+|I(X)|+|J(Y)|=1+\alpha+\beta$. We first consider the case that $(W, X, Y) \in N(H) \cap \Omega^1$. Since $I(X) \cap J(Y)=\phi$, the relation $|I(X)|+|J(Y)| \leqq \beta+1$ must holds. Hence $|\tau(W)| \geqq \alpha$. On the other hand, by $\tau(W) \in \Delta$ and $\dim \Delta=\alpha$, we see $|\tau(W)| \leqq \alpha+1$. Thus $|I(X)|+|J(Y)|=\beta$ or $\beta+1$. If $|I(X)|+|J(Y)|=\beta$, either (a) or (b) occurs, and otherwise (c).

Suppose that $(W, X, Y) \in N(H) \cap \Omega^2$. It follows from $I(X) \cap J(Y)=\phi$ and $0 \notin I(X)$ that $|I(X)|+|J(Y)| \leqq \beta$, which implies that $|\tau(W)| \geqq \alpha+1$. While, by $\tau(W) \in \Xi$ and $\dim \Xi=\alpha+1$, we see $|\tau(W)| \leqq \alpha+2$ Hence, $|\tau(W)|=\alpha+1$ or $\alpha+2$ If $|\tau(W)|=\alpha+1$ then either (a) or (d) occurs, and otherwise (e). Q.E.D.

For each $(W, X, Y) \in N(H)$, define the set

$$\zeta(W, X, Y)=\begin{cases} \{M_{\cdot 0}\} \cup \{L(\xi): \{\tau(W), \xi\} \in \Xi_{\alpha+1}\} & \text{if (a),} \\ \{M_{\cdot i^*}\} \cup \{N_{\cdot i^*}\} & \text{if (b) or (e),} \\ \{L(\xi): \{\tau(W), \xi\} \in \Delta_\alpha\} & \text{if (c),} \\ \{L(\xi): \{\tau(W), \xi\} \in \Xi_{\alpha+1}\} & \text{if (d)} \end{cases}$$

The following lemma follows from Lemma (2-2).

(7-6) **Lemma:** For every $(W, X, Y) \in N(H)$, $\zeta(W, X, Y)$ has two elements.

(7-7) **Lemma:** $(W, X, Y) \in N(H)$ and $(\bar{W}, \bar{X}, \bar{Y}) \in N(H)$ are adjacent if and only if $\eta(\bar{W}, \bar{X}, \bar{Y})$ contains one element of $\zeta(W, X, Y)$ and $\alpha+\beta$ elements of $\eta(W, X, Y)$.

Proof. Suppose that $\eta(\bar{W}, \bar{X}, \bar{Y})$ contains $K \in \zeta(W, X, Y)$ and $\alpha+\beta$ elements of $\eta(W, X, Y)$. For each $\lambda \in (0, 1)$, let

(7-8) $$W^\lambda=(1-\lambda) W+\lambda \bar{W}, X^\lambda=(1-\lambda)X+\lambda \bar{X} \quad \text{and} \quad Y^\lambda=(1-\lambda) Y+\lambda \bar{Y}$$

Then we see

(7-9) $$\eta(W^\lambda, X^\lambda, Y^\lambda)=\eta(W, X, Y) \cup K \qquad \text{for every} \quad \lambda \in (0, 1).$$

This implies $(W^\lambda, X^\lambda, Y^\lambda)\in\Omega^1\cup\Omega^2-N(H)$ for every $\lambda\in(0,1)$. Thus (W, X, Y) and $(\bar{W}, \bar{X}, \bar{Y})$ are adjacent.

Conversely, suppose that $(W, X, Y)\in N(H)$ and $(\bar{W}, \bar{X}, \bar{Y})\in N(H)$ are adjacent. By defining W^λ, X^λ and Y^λ by (7-8), we obtain

$$\eta(W^\lambda, X^\lambda, Y^\lambda)=\eta(W, X, Y)\cup\eta(\bar{W}, \bar{X}, \bar{Y}) \qquad \text{for every} \quad \lambda\in(0,1).$$

Since $(W^\lambda, X^\lambda, Y^\lambda)\in\Omega^1\cup\Omega^2-N(H)$, we see

$$\eta(W, X, Y)\cup\eta(\bar{W}, \bar{X}, \bar{Y})=\eta(W, X, Y)\cup K \qquad \text{for some} \quad K\in\zeta(\bar{W}, \bar{X}, \bar{Y}).$$ Q.E.D.

(7-10) **Lemma:** For a given $(W, X, Y)\in N(H)$ and $K\in\zeta(W, X, Y)$, there is no $(\bar{W}, \bar{X}, \bar{Y})\in N(H)$ which is adjacent to (W, X, Y) and uses K if and only if a unique solution $[\lambda(\xi)(\xi\in\tau(W)), \mu_i(i\in I(X)), \nu_j(j\in J(Y))]$ of the linear equation

$$\sum_{\xi\in\tau(W)}\lambda(\xi)L(\xi)+\sum_{i\in I(X)}\mu_i M_{\cdot i}+\sum_{j\in J(Y)}\nu_j N_{\cdot j}=K \tag{7-11}$$

has no positive element.

Proof. By applying Lemma (1-4) to the lexico inequality system (7-2), we obtain the desired result.

(7-12) **Theorem:** For each $(W, X, Y)\in N(H)$, $\deg(W, X, Y)=0, 1$ or 2. More precisely, $\deg(W, X, Y)=1$ or 2 if (a) occurs, and $\deg(W, X, Y)=2$ if (c) or (d) occurs.

Proof. The first part of the theorem follows from Lemmas (7-6) and (7-7). Suppose that $L(\xi)\in\zeta(W, X, Y)$ for some $\xi\in\Xi_0$. For $K=L(\xi)$, a unique solution $[\lambda(\xi)(\xi\in\tau(W)), \mu_i(i\in I(X)), \nu_j(j\in J(Y))]$ of the linear equation (7-11) has at least one positive element. In fact, the $(\alpha+1)$-th row of (7-11) turns out to be $\sum_{\xi\in\tau(W)}\lambda(\xi)=1$; hence, at least one $\lambda(\xi)$ is positive. Therefore, by Lemma (7-10), there is a $(\bar{W}, \bar{X}, \bar{Y})\in N(H)$ which is adjacent to (W, X, Y) and uses $L(\xi)$. In the case (a), $\zeta(W, X, Y)$ has one $L(\xi)$ for some $\xi\in\Xi_0$. Hence, by the above argument, we see $\deg(W, X, Y)\geqq 1$. While, in the case (c) or (d), $\zeta(W, X, Y)$ consists of $L(\xi^1)$ and $L(\xi^2)$ for some distinct $\xi^1, \xi^2\in\Xi_0$, so that $\deg(W, X, Y)=2$. Q.E.D.

§8. A computational method based on LEMKE's method

By Theorem (7-12), we have shown that the graph H satisfies Condition (3-3-b). In order to apply Iteration procedure (3-2) to H, we must find a node $(W, X, Y)\in N(H)$ such that $\deg(W, X, Y)=0$ or 1. By the construction of $W^0(\xi)$ $(\xi\in\Xi_0)$, we see

$$-\sum_{\xi\in\Xi_0}\phi_1(\xi)W^0(\xi)=F \quad\text{and}\quad \sum_{\xi\in\Xi_0}W^0(\xi)=G \tag{8-1}$$

Since $d\in R^\beta_+$ satisfies (7-1), there is an $X^0_{0\cdot}\succeq 0$ such that

$$H+\sum_{\xi\in\Xi_0}h(\phi_1(\xi))W^0(\xi)+dX^0_{0\cdot}\succeq 0$$

and that

$$X_{0\cdot} \gtrsim X_{0\cdot}^0 \quad \text{if } H + \sum_{\xi\in\Xi_0} h(\psi_1(\xi))W^0(\xi) + dX_{0\cdot} \gtrsim 0 \quad \text{and} \quad X_{0\cdot} \gtrsim 0.$$

By defining

$$X_{i\cdot}^0 = 0 \quad \text{for} \quad i=1,2,\cdots,\beta, \quad \text{and} \quad Y^0 = H + \sum_{\xi\in\Xi_0} h(\psi_1(\xi))W^0(\xi) + dX_{0\cdot}^0,$$

then $(W^0, X^0, Y^0)\in N(H)$.

(8-2) **Lemma:** $(W^0, X^0, Y^0)\in N(H)$ satisfies either (7-5-a) or (7-5-b). If it satisfies the former than $\deg((W^0, X^0, Y^0)=1$, and if the latter, $\deg(W^0, X^0, Y^0)=0$ or 1.

Proof. The first part of the lemma is trivial from $\tau(W^0)=\sigma^0\in\Delta_\alpha$. Suppose that (W^0, X^0, Y^0) satisfies (7-5-a). Then $I(X^0)=\phi, J(Y^0)=\{1,2,\cdots,\beta\}$ and $\zeta(W^0, X^0, Y^0)=\{M_{\cdot 0}\}\cup\{L(\xi):\{\tau(W^0),\xi\}\in\Xi_{\alpha-1}\}$. For $K=M_{\cdot 0}$ (7-11) becomes

$$\sum_{\xi\in\tau(W^0)} \lambda(\xi)L(\xi) + \sum_{j=1}^{\beta} \nu_j N_{\cdot j} = M_{\cdot 0},$$

which implies $\lambda(\xi)=0$ $(\xi\in\tau(W^0))$ and $\nu_j=-d_j$ $(j=1,2,\cdots,\beta)$. By Lemma (7-10), there is no $(\bar{W},\bar{X},\bar{Y})\in N(H)$ which is adjacent to (W^0, X^0, Y^0) and uses $M_{\cdot 0}$. Hence $\deg(W^0, X^0, Y^0)\leqq 1$. On the other hand, by Theorem (7-12), we have $\deg(W^0, X^0, Y^0)\geqq 1$. Therefore, $\deg(W^0, X^0, Y^0)=1$.

Now we consider the case that (W^0, X^0, Y^0) satisfies (7-5-b). Then we see that for some $i^*\neq 0$,

$$I(X^0)=\{0\}, J(Y^0)=\{1,2,\cdots,\beta\}-\{i^*\},$$

$$\zeta(W^0, X^0, Y^0)=\{M_{\cdot i^*}, N_{\cdot i^*}\}.$$

By letting $K=N_{\cdot i^*}$, (7-11) becomes

$$\sum_{\xi\in\tau(W^0)} \lambda(\xi)L(\xi) + \sum_{j\neq i^*} \nu_i N_{\cdot j} + \mu_0 M_{\cdot 0} = N_{\cdot i^*}.$$

$[\lambda(\xi)=0$ $(\xi\in\tau(W^0))$, $\mu_0=-1/d_{i^*}$, $\nu_j=-d_j/d_{i^*}$ $(j\neq i^*)]$ is the solution of the above system. By Lemma (7-10), there is no $(\bar{W},\bar{X},\bar{Y})\in N(H)$ which is adjacent to (W^0, X^0, Y^0) and uses $N_{\cdot i^*}$. Hence $\deg(W^0, X^0, Y^0)\leqq 1$. Q.E.D.

Consequently Iteration procedure (3-2) can be uniquely applied to the graph H. That is, the following algorithm can be executed.

(8-3) **Algorithm:**

(i) Let $p=1$ and $(W^1, X^1, Y^1)=(W^0, X^0, Y^0)$.

(ii) If (W^p, X^p, Y^p) satisfies (7-5-a) then find $\bar{\xi}\in\Xi_0$ such that $\{\tau(W^p),\bar{\xi}\}\in\Xi_{\alpha+1}$, compute $L(\bar{\xi})$ and let $K=L(\bar{\xi})$. Otherwise, let $K=M_{\cdot i^*}$.

(iii) If there is a basic solution $(\bar{W},\bar{X},\bar{Y})$ of (7-2) which uses K and a subset of $\eta(W^p, X^p, Y^p)$, let $(W^{p+1}, X^{p+1}, Y^{p+1})=(\bar{W},\bar{X},\bar{Y})$ and increase p by 1. Otherwise stop.

(iv) Let $K=\zeta(W^p, X^p, Y^p)-\eta(W^{p-1}, X^{p-1}, Y^{p-1})$. When $K=L(\bar{\xi})$ for some $\bar{\xi}\in\Xi_0$, find such a $\bar{\xi}\in\Xi_0$ and compute $L(\bar{\xi})$. Return to (iii).

This is another computational method which we propose in this thesis. As the result of the execution of the algorithm, we obtain a finite or infinite sequence $\{(W^p, X^p, Y^p)\}$ of distinct complementary basic solutions of (7-2). Each (W^p, X^p, Y^p) satisfies

(8-4)
$$\begin{bmatrix} X^p_{1\cdot} \\ \vdots \\ X^p_{\alpha\cdot} \end{bmatrix} = \sum_{\xi\in\tau(W^p)} \psi_1(\xi) W^p(\xi) + F,$$
$$\sum_{\xi\in\tau(W^p)} W^p(\xi) = G,$$
$$Y^p = \sum_{\xi\in\tau(W^p)} W^p(\xi) h(\psi_1(\xi)) + \sum_{i=\alpha+1}^{\beta} A_{\cdot i} X^p_{i\cdot} + dX^p_{0\cdot} + H,$$
$$X^p_{i\cdot} = 0 \quad \text{or} \quad Y^p_{i\cdot} = 0 \qquad (i=1, 2, \cdots, \beta).$$

By defining, for $p=1, 2, \cdots$

(8-5)
$$x^p_0 = X^p_{01}, \qquad x^p = (X^p_{11}, X^p_{21}, \cdots, X^p_{\beta 1}),$$
$$y^p = (Y^p_{11}, Y^p_{21}, \cdots, Y^p_{\beta 1}) \quad \text{and} \quad w^p(\xi) = W^p(\xi)_{\cdot 1} \quad (\xi\in\Xi_0)$$

we further obtain, for $p=1, 2, \cdots$

(8-6)
$$x^p = \sum_{\xi\in\tau(W^p)} w^p(\xi)(\psi_1(\xi), x^p_{\alpha+1}, \cdots, x^p_\beta) \geqq 0,$$
$$y^p = \sum_{\xi\in\tau(W^p)} w^p(\xi) f((\psi_1(\xi), x^p_{\alpha+1}, \cdots, x^p_\beta)) + dx^p_0 \geqq 0,$$
$$\sum_{\xi\in\tau(W^p)} w^p(\xi) = 1, \qquad w^p(\xi) \geqq 0 \qquad (\xi\in\tau(W^p)),$$
$$x^p_i = 0 \quad \text{or} \quad y^p_i = 0 \qquad (i=1, 2, \cdots, \beta),$$
$$\operatorname{diam}\{(\psi_1(\xi), x^p_{\alpha+1}, \cdots, x^p_\beta) : \xi\in\tau(W^p)\} < \varepsilon^*.$$

The union $\cup\{(1-t)(x^p_0, x^p) + t(x^{p+1}_0, x^{p+1}) : 0 \leqq t \leqq 1\}$ is said to be a path.

§9 Unbounded paths

When f is linear, by taking $\alpha=0$, we see that Algorithm (8-3) coincides with LEMKE's method for the linear CP. In such a case Algorithm (8-3) terminates in a finite number of steps, and the path is always bounded. When f is not linear, however, the algorithm does not necessarily terminate. In this section we deal with the case that the algorithm continues infinitely. Let $\{(W^p, X^p, Y^p)\}$ be an infinite sequence, generated by the algorithm, of distinct complementary basic solutions of (7-2).

(9-1) **Lemma:** If $\tau(W^p)\in\Delta$ for $p=1, 2, \cdots,$ then $X^p_{0\cdot} \succ 0$ for $p=1, 2, \cdots$ and the sequence $\{(x^p_1, x^p_2, \cdots, x^p_\alpha)\}$ is unbounded.

Proof. If $X^1_{0\cdot}=0$ then $\tau(W^2)\notin\Delta$, which is a contradiction. Assume that $X^p_{0\cdot}=0$ for some $p>1$. Then (W^p, X^p, Y^p) satisfies one of (a), (d) and (e) of Lemma (7-5). It follows from $\tau(W^p)\in\Delta$ that (d) and (e) can not occur. While, by Lemma (7-7), (a) implies either $\tau(W^{p-1})\notin\Delta$ or $\tau(W^{p+1})\notin\Delta$. Hence we also obtain a contradiction. Thus $X^p_{0\cdot}>0$ for $p=1, 2, \cdots$.

Assume now that $\{(x^p_1, x^p_2, \cdots, x^p_\alpha)\}$ is bounded. Then there is a bounded set $U\subset R^\alpha$ such that $\mathrm{co}\{\tau(W^p)\}\subset U\times\{0\}$ for $p=1, 2, \cdots$. By Condition (2-1-c), we can find a subsequence $\{\tau(W^{p_r}): r=1, 2, \cdots\}$ and a simplex $\sigma^*\in\Xi$ such that $\sigma^*=\tau(W^{p_r})$ for $r=1, 2, \cdots$. $(W^{p_r}, X^{p_r}, Y^{p_r})$ $(r=1, 2, \cdots)$ are distinct basic solutions of $\sum_{\xi\in\sigma^*} L(\xi)W(\xi)+MX+NY=Q$. But, since the number of variable vectors of the above system is finite, the system has at most a finite number of distinct basic solutions. This is a contradiction. Q.E.D.

(9-2) **Lemma:** If the sequence $\{(x^p_1, x^p_2, \cdots, x^p_\alpha)\}$ is bounded, then so is the sequence $\{x^p\}$.

Proof. We shall assume that $\{(x^p_1, x^p_2, \cdots, x^p_\alpha)\}$ is bounded and that $\{x^p\}$ is unbounded, and exhibit a contradiction. We assume that $\{x^p_\beta\}$ is a unbounded sequence. By taking an appropriate subsequence $\{(W^{p_r}, X^{p_r}, Y^{p_r}): r=1, 2, \cdots\}$, we obtain that $\lim_{r\to\infty} x^{p_r}_\beta=+\infty$, $\{\beta\}\subset I(X^{p_r})=I^*(r=1, 2, \cdots)$, $J(Y^{p_r})=J^*(r=1, 2, \cdots)$ and $\lim_{r\to\infty}(x^{p_r}_1, x^{p_r}_2, \cdots x^{p_r}_\alpha)=(x^*_1, x^*_2, \cdots, x^*_\alpha)$ for some $I^*\subset\{0, 1, \cdots, \beta\}$, $J^*\subset\{1, 2, \cdots, \beta\}$ and $(x^*_1, x^*_2, \cdots, x^*_\alpha)\in R^\alpha$. Hence for $r=1, 2, \cdots$

$$\sum_{i\in I^*} M_{\cdot i}x^{p_r}_i+\sum_{j\in J^*} M_{\cdot j}y^{p_r}_j=Q_{\cdot 1}-\sum_{\xi\in\tau(W^{p_r})} L(\xi)w^{p_r}(\xi).$$

The right term of the above system converges to

$$\begin{bmatrix} 0 \\ \\ \\ 1 \\ b \end{bmatrix}+\begin{bmatrix} \begin{pmatrix} x^*_1 \\ \vdots \\ x^*_\alpha \end{pmatrix} \\ -1 \\ h(x^*_1, \cdots, x^*_\alpha) \end{bmatrix}\in R^{(\alpha+1+\beta)\times 1}$$

Since $\{M_{\cdot i}(i\in I^*), N_{\cdot j}(j\in J^*)\}$ forms a linearly independent subset of $R^{(\alpha+1+\beta)\times 1}$, the sequence $\{x^{p_r}_i\ (i\in I^*), y^{p_r}_j\ (j\in J^*)\}$ must converge. This is a contradiction. Q.E.D.

Let S^* be the set of all cluster points of the sequence $\{x^p\}$. By the above lemma, if the sequence $\{(x^p_1, x^p_2, \cdots, x^p_\alpha)\}$ is bounded then S^* is a nonempty compact set.

(9-3) **Theorem:** If S^* is nonempty, S^* is a closed subset of R^β_+ and every $x^*\in S^*$ is a solution of the CP.

Proof. Clearly S^* is a closed subset of R^β_+. Let $x^*\in S^*$; for simplicity, we assume that the sequence $\{x^p\}$ itself converges to x^*. Then $\{(\phi_1(\tau(W^p))\}$ is contained in some bounded set $U\in R^\beta$. By Condition (2-1-c), $\lim_{p\to\infty}\min\phi_2(\tau(W^p))=+\infty$. Hence, by Condition (2-3-b), $\lim_{p\to\infty}\mathrm{diam}\,\phi_1(\tau(W^p))=0$. We also see that $X^p_{0\cdot}=0$ for every $p\geqq p^*$ and some p^*. Now we take the limit of (8-6) as $p\to+\infty$. Then we get

$$x^* \geqq 0,\ y^* = f(x^*) \geqq 0 \quad \text{and} \quad \langle x^*, y^* \rangle = 0 \qquad \text{for some } y^* \in R^{\beta}_{+};$$

x^* is a solution of the CP. Q.E.D.

§10. Unbounded rays

This section is concerned with the case that Algorithm (8-3) terminates in a finite number of steps. Let $\{(W^p, X^p, Y^p) : p=1, 2, \cdots, r\}$ be the sequence generated by the algorithm. Then $\deg(W^p, X^p, Y^p)=0$ or 1 (see Lemma (3-4)). By Theorem (7-12), (W^r, X^r, Y^r) satisfies one of the three cases (a), (b) and (e) of Lemma (7-5).

(10-1) **Lemma:** Suppose that (W^r, X^r, Y^r) satisfies (7-5-a). Then $r>1$ and there is a nonzero $(\varDelta x_0, \varDelta x, \varDelta y) \in R^{1+2\beta}$ such that

(i) $\qquad \varDelta x_i = 0 \qquad (i=1, 2, \cdots, \alpha),$

(ii) $\qquad \varDelta y = \sum_{i=\alpha+1}^{\beta} A_{\cdot i} \varDelta x_i + d \varDelta x_0$

(iii) $\qquad \varDelta x_i \cdot Y^r_i = 0 \qquad (i=1, 2, \cdots, \beta),$

(iv) $\qquad \varDelta y_i \cdot X^r_i = 0 \qquad (i=1, 2, \cdots, \beta),$

(v) $\qquad \varDelta x_i \varDelta y_i = 0 \qquad (i=1, 2, \cdots, \beta),$

and $\varDelta x_0 > 0$. Furthermore, if, in addition, $d>0$ then we can take $\varDelta x \neq 0$.

Proof. Let $\zeta(W^r, X^r, Y^r) = \{M_{\cdot 0}, L(\bar{\xi})\}$. By Theorem (7-12) and its proof, there is a basic solution $(W, X, Y) \in N(H)$ of (7-2) which is adjacent to (W^r, X^r, Y^r) and uses $L(\bar{\xi})$. Since $\deg(W^r, X^r, Y^r)=1$, there is no basic solution $(W, X, Y) \in N(H)$ of (7-2) which is adjacent to (W^r, X^r, Y^r) and uses $M_{\cdot 0}$. By Lemma (7-10), the linear equation

$$\sum_{\xi \in \tau(W^r)} \lambda(\xi) L(\xi) + \sum_{i \in I(X^r)} \mu_i M_{\cdot i} + \sum_{j \in J(Y^r)} \nu_j N_{\cdot j} = M_{\cdot 0}$$

has a unique nonpositive solution $[\lambda(\xi)\ (\xi \in \tau(W^r)), \mu_i\ (i \in I(X^r)), \nu_j\ (j \in J(Y^r))]$. It follows from the $(\alpha+1)$-th row of the above system that $\lambda(\xi)=0\ (\xi \in \tau(W^r))$, $\mu_i = 0 (i \in I(X^r) \cap \{1, 2, \cdots, \alpha\})$. Let

$$\begin{aligned}
&\varDelta x_0 = 1, \\
&\varDelta x_i = 0 && (i \notin I(X^r), i \neq 0), \\
&\varDelta x_i = -\mu_i && (i \in I(X^r), \\
&\varDelta y_j = 0 && (j \notin J(Y^r)), \\
&\varDelta y_j = -\nu_j && (j \in J(Y^r)).
\end{aligned}$$

Then we get the relations (i)-(v) and $\varDelta x_0 > 0$.

In order to prove the last part of the lemma, we assume that $d>0$ and $\varDelta x = 0$.

Then, by (ii), $\Delta y>0$. Hence, by (iv), $X^r_{j\cdot}=0 (j=1,2,\cdots,\beta)$, which implies $(W^r, X^r, Y^r)=(W^0, Y^0, Y^0)$. Consequently, by Lemma (3-4), we see $\deg(W^r, X^r, Y^r)=0$. This contradicts $\deg(W^r, X^r, Y^r)=1$. Q.E.D.

By the similar argument as the above proof we obtain the following two lemmas.

(10-2) **Lemma:** If (W^r, X^r, Y^r) satisfies (7-5-b), there is a nonzero $(\Delta x_0, \Delta x, \Delta y)\in R^{1+2\beta}$ satisfying (i)-(v) of Lemma (10-1). If, in addition, $r=1$ or $d>0$, then a nonzero Δx can be chosen.

(10-3) **Lemma:** If (W^r, X^r, Y^r) satisfies (7-5-e), then $r>1$ and there is a $(\Delta x_0, \Delta x, \Delta y)\in R^{1+2\beta}$ which satisfies $\Delta x_0=0$, $\Delta x\neq 0$ and (i)-(v) of Lemma (10-1).

If $(\Delta x_0, \Delta x, \Delta y)\in R^{1+2\beta}$ satisfies (i)-(v) then for all $s\in R_+$

(vi) $$x^r+s\Delta x=\sum_{\xi\in\tau(W^r)} w^r(\xi)\{(\psi_1(\xi), x^r_{\alpha+1}, \cdots, x^r_\beta)+s\Delta x\},$$

(vii) $$\sum_{\xi\in\tau(W^r)} w^r(\xi)=1, \quad w^r(\xi)\geqq 0 \quad (\xi\in\tau(W^r)),$$

(viii) $$y^r+s\Delta y=\sum_{\xi\in\tau(W^r)} w^r(\xi)f((\psi_1(\xi), x^r_{\alpha+1}, \cdots, x^r_\beta)+s\Delta x)+d(x^r_0+s\Delta x_0),$$

(ix) $$\langle x^r+s\Delta x, y^r+s\Delta y\rangle=0,$$

(x) $$\operatorname{diam}\{(\psi_1(\xi), x^r_{\alpha+1}, \cdots, x^r_\beta)+s\Delta x : \xi\in\tau(W^r)\}<\varepsilon^*.$$

Consequently, the above lemmas can be summarized in the following theorem.

(10-4) **Theorem:** There is a nonzero $(\Delta x_0, \Delta x, \Delta y)\in R^{1+2\beta}$ which satisfies (i)-(x). If $r=1$ or $d>0$, a nonzero Δx can be chosen.

Especially, Theorem (10-4) implies that if we take $d>0$ and $\alpha=\beta$ then Algorithm (8-3) continues infinitely.

§11. The convergence of approximate solutions

In this section, we impose Condition (5-1) on the function f, and show that the set S^* of all cluster points of the sequence $\{x^p\}$ is nonempty and compact. In other words, Algorithm (8-3) computes approximate solutions of the CP which satisfy the same condition that is assumed when Algorithm (6-1) is applied to the CP.

(11-1) **Lemma:** There is an $\varepsilon^*>0$ such that if $C\subset R^\beta$, $C\cap B\neq\phi$ and $\operatorname{diam} C<\varepsilon^*$ then $\langle x-x', d\rangle<0$ and $\langle x-x', f(x'')\rangle<0$ for some $x\in R^\beta_+$ and any $x', x''\in C$.

Proof. Let $\bar{x}\in B$. By Condition (5-1), there is an $\bar{x}\in R^\beta_+$ for which $\langle x-\bar{x}, d\rangle<0$ and $\langle x-\bar{x}, f(\bar{x})\rangle<0$. Since f is continuous, there is an $\varepsilon>0$ such that $\langle x-x', d\rangle<0$ and $\langle x-x', f(x'')\rangle<0$ for any $x', x''\in B_\varepsilon(\bar{x})$. By applying Borel-Lebesgue theorem in the same manner as in the proof of Lemma (5-3), we get the desired result.

Q.E.D.

(11-2) **Theorem:** If the simplicial subdivision Ξ of $R^{\alpha}\times(0,\infty)$ satisfies Condition (2-3) for the $\varepsilon^*>0$ whose existence is ensured by the above lemma, then
(a) the sequence $\{(W^p, X^p, Y^p)\}$ is infinite,
(b) $x^p\in D_{\rho^+}$ for $p=1,2,\cdots$, where $\rho^+=\max\{\langle x, d\rangle : x\in B\}$.

Proof. (a) We assume that Algorithm (8-3) generates only a finite sequence $\{(W^p, X^p, Y^p): p=1,2,\cdots,r\}$, and exhibit a contradiction. By Theorem (10-4), there is a $(\Delta x_0, \Delta x, \Delta y)\in R^{1+2\beta}$ satisfying (vi)-(x) and $\Delta x\neq 0$. Let $T=\left[\bigcup_{p=1}^{r-1}\{\mathrm{co}\, x^p, x^{p+1}\}\right]\cup\{x^r+t\Delta x : t\geqq 0\}$. Then T is a unbounded closed connected subset of R^{β}_+ which contains the origin. Hence T intersects B. First we consider the case that $\bar{x}\in\left[\bigcup_{p=1}^{r-1}\{\mathrm{co}\, x^p, x^{p+1}\}\right]\cap B$ for some $\bar{x}\in R^{\beta}_+$. Then we can find a positive integer p^* and a nonnegative number $\lambda\in[0,1]$ such that $\bar{x}=(1-\lambda)x^{p^*}+\lambda x^{p^*+1}$. By letting

$$\bar{w}(\xi)=(1-\lambda)w^{p^*}(\xi)+\lambda w^{p^*+1}(\xi)\quad(\xi\in\Xi_0),\ \bar{W}(\xi)=(1-\lambda)W^{p^*}(\xi)+W^{p^*+1}(\xi)\quad(\xi\in\Xi_0)$$

$$\bar{y}=(1-\lambda)y^{p^*}+\lambda y^{p^*+1}\quad\text{and}\quad \bar{x}_0=(1-\lambda)x^p_0+\lambda x^{p^*+1},$$

then we see

$$\begin{aligned}
&\bar{x}=\sum_{\xi\in\tau(\bar{W})}\bar{w}(\xi)\cdot(\psi_1(\xi),\bar{x}_{\alpha+1},\cdots,\bar{x}_\beta)\geqq 0,\\
&\bar{y}=\sum_{\xi\in\tau(\bar{W})}\bar{w}(\xi)f((\psi_1(\xi),\bar{x}_{\alpha+1},\cdots,\bar{x}_\beta))+d\bar{x}_0\geqq 0,\\
&\sum_{\xi\in\tau(\bar{W})}\bar{w}(\xi)=1,\qquad \bar{w}(\xi)\geqq 0\qquad(\xi\in\tau(\bar{W})),\\
&\langle\bar{x},\bar{y}\rangle=0\quad\text{and}\quad \mathrm{diam}\{(\psi_1(\xi),\bar{x}_{\nu+1},\cdots,\bar{x}_\beta):\xi\in\tau(\bar{W})\}<\varepsilon^*.
\end{aligned}\tag{11-3}$$

By Lemma (11-1), there is an $x\in R^{\beta}$ such that for any $\xi',\xi''\in\tau(\bar{W})$

$$\begin{aligned}
&\langle x-(\psi_1(\xi'),\bar{x}_{\alpha+1},\cdots,\bar{x}_\beta), d\rangle<0,\\
&\langle x-(\psi_1(\xi'),\bar{x}_{\alpha+1},\cdots,\bar{x}_\beta), f((\psi_1(\xi''),\bar{x}_{\alpha+1},\cdots,\bar{x}_\beta))\rangle<0,
\end{aligned}\tag{11-4}$$

It follows from (11-3) and (11-4) that

$$\begin{aligned}
0&\leqq\langle x,\bar{y}\rangle\\
&=\langle x-\bar{x},\bar{y}\rangle\\
&=\left\langle\sum_{\xi'\in\tau(\bar{W})}\bar{w}(\xi')\{x-(\psi_1(\xi'),\bar{x}_{\alpha+1},\cdots,\bar{x}_\beta)\},\right.\\
&\qquad\left.\sum_{\xi''\in\tau(\bar{W})}w(\xi'')\{f((\psi_1(\xi''),\bar{x}_{\alpha+1},\cdots,\bar{x}_\beta))+d\bar{x}_0\}\right\rangle\\
&=\sum_{\xi'\in\tau(\bar{W})}\bar{w}(\xi')\Big\{\langle x-(\psi_1(\xi'),\bar{x}_{\alpha+1},\cdots,\bar{x}_\beta), d\bar{x}_0\rangle\\
&\quad+\Big\langle x-(\psi_1(\xi'),\bar{x}_{\alpha+1},\cdots,\bar{x}_\beta), \sum_{\xi''\in\tau(\bar{W})}\bar{w}(\xi'')f((\psi_1(\xi''),\bar{x}_{\alpha+1},\cdots,\bar{x}_\beta))\Big\rangle\Big\}<0.
\end{aligned}$$

This is a contradiction. Now suppose that $\bar{x}\in\{x^r+t\Delta x : t\geqq 0\}\cap B$. Let $\bar{x}_0=x_0^r+t^*\Delta x_0$, $\bar{y}=y^r+t^*\Delta y$, $\bar{W}(\xi)=W^r(\xi)$ $(\xi\in\Xi_0)$ and $\bar{w}(\xi)=w^r(\xi)$ $(\xi\in\Xi_0)$, where t^* is a nonnegative number such that $\bar{x}=x^r+t^*\Delta x$. Then we have (11-3), from which a contradiction follows in the same manner as above.

(b) Assume that $x^q\notin D_{\rho+}$. Then the close connected set $\bigcap_{p=1}^{q-1}[\mathrm{co}\,\{x^p, x^{p+1}\}]$ and the set B have a common point $\bar{x}$. Hence we have a contradiction by the same argument as above, too. Q.E.D.

The theorem implies that the set S^* of all cluster points of $\{x^p\}$ is nonempty and compact. Therefore, by Theorem (9-3), S^* is a subset of solutions of the CP.

The above discussion has not supply any information regarding the convergence rate of the approximate solutions. We now give a brief theoretical consideration of it; the accuracy of approximate solutions x^p is evaluated in terms of diam $\psi_1(\tau((W^p))$. Suppose that the sequence $\{x^p\}$ converges to an exact solution $\hat{x}$ of the CP. Then $\lim_{p\to\infty}\mathrm{diam}\,\psi_1(\tau(W^p))=0$ and $\lim_{p\to\infty}\min\psi_2(\tau(W^p))=+\infty$. It follows from $\lim_{p\to\infty}\min\psi_2(\tau(W^p))=+\infty$ that there is a positive integer r for which $\psi_2(\tau(W^p))>0$ if $p\geqq r$. Let $p>r$ be fixed. Then $x_0^p=0$; hence we have

$$y^p=\sum_{\xi\in\tau(W^p)} w^p(\xi)f((\psi_1(\xi), x_{\alpha+1}^p, \cdots, x_\beta^p))$$

$$=\sum_{\xi\in\tau(W^p)} w^p(\xi)h(\psi_1(\xi))+\sum_{i=\alpha+1}^{\beta} A_{\cdot i}x_i^p+b\geqq 0.$$

We also see, for each $\xi\in\tau(W^p)$ and $k=1, 2, \cdots, \beta$,

$$f_k(x^p)=h_k(x_1^p, \cdots, x_\alpha^p)+\sum_{i=\alpha+1}^{\beta} A_{ki}x_i^p+b_k$$

$$=f_k((\psi_1(\xi), x_{\alpha+1}^p, \cdots, x_\beta^p))+h(x_1^p, \cdots, x_\alpha^p)-h_k(\psi_1(\xi)).$$

Thus we obtain

$$\begin{aligned} f_k(x^p)&=y_k^p+\sum_{\xi\in\tau(W^p)} w^p(\xi)\{h_k(x_1^p, \cdots, x_\alpha^p)-h_k(\psi_1(\xi))\} \\ &\geqq \sum_{\xi\in\tau(W^p)} w^p(\xi)\{h_k(x_1^p, \cdots, x_\alpha^p)-h_k(\psi_1(\xi))\} \end{aligned} \tag{11-5}$$

and

$$f_k(x^p)=\sum_{\xi\in\tau(W^p)} w^p(\xi)\{h_k(x_1^p, \cdots, x_\alpha^p)-h_k(\psi_1(\xi))\} \quad \text{if} \quad x_k^p>0. \tag{11-6}$$

The term $\sum_{\xi\in\tau(W^p)} w^p(\xi)\{h_k(x_1^p, \cdots, x_\alpha^p)-h_k(\psi_1(\xi))\}$ is approximately zero if the diameter of $\psi_1(\tau(W^p))$ is sufficiently small. If f is twice continuously differentiable, the term satisfies the following relation.

$$\sum_{\xi\in\tau(W^p)} w^p(\xi)\{h_k(x_1^p, \cdots, x_\alpha^p)-h_k(\psi_1(\xi))\}$$

$$\leqq -\mu_k\{\mathrm{diam}\,\psi_1(\tau(W^p))\}^2,$$

where $\mu_k=\max\{\Delta z\nabla^2 h_k(x_1, \cdots, x_\alpha)\Delta z^T : (x_1, \cdots, x_\alpha)\in D_{\rho 1},\ \Delta z=(\Delta z_1, \cdots, \Delta z_\alpha),\ ||z||=1\}$. Therefore, we see, for $k=1, 2, \cdots,$

$$
(11\text{-}7)\qquad
\begin{aligned}
&f_k(x^p)\geqq -\mu_k\{\operatorname{diam}\phi_1(\tau(W^p))\}^2 && && \text{(from (11-5))},\\
&|f_k(x^p)|\leqq \mu_k\{\operatorname{diam}\phi_1(\tau(W^p))\}^2 && \text{if}\quad x_k^p>0 && \text{(from (11-6))}.
\end{aligned}
$$

Chap. 4. Applications

This chapter which consists of §12, §13, §14 and §15, is devoted to applications of Algorithm (8-3). In §12 Algorithm (8-3) is used to prove EAVES's basic theorem of the CP (Theorem (4-4)). In §13, it is shown that the BROUWER fixed point problem can be converted into the nonlinear CP which satisfies Condition (5-1). In §14 we deal with a certain type of the nonlinear CP for which Algorithm (8-3) can be applied to compute approximate solutions. Problems of finding equilibria of n-person noncooperative games, which include a mathematical programming problem as a special case, can be reduced to the above nonlinear CP. In §15 we consider balanced cooperative games without sidepayments; the method for finding a point of the core of the game is given.

§12. A semi-constructive proof of EAVES's basic theorem

In what follows we prove EAVES's basic theorem of the CP (Theorem (4-4)) by using Algorithm (8-3). Let $\{\varepsilon^k\}$ be an infinite sequence of positive numbers such that $\lim_{k\to\infty}\varepsilon^k=0$. Corresponding to each ε^k, let Ξ^k be a simplicial subdivision $R^\beta\times[0,\infty)$ such that $\operatorname{diam}\phi_1(\sigma)<\varepsilon^k$ for all $\sigma\in\Xi^k$. By using the function $h(\cdot)=f(\cdot)$, the simplicial subdivision Ξ^k and $d>0$, we construct the graph H^k $(k=1, 2, \cdots)$, where we take $\alpha=\beta$. Apply Algorithm (8-3) to each H^k. Then we obtain an infinite sequence $\{({}^kW^p, {}^kX^p, {}^kY^p) : p=1, 2, \cdots\}$ of $N(H^k)$ (see the final remark of §10). Define, for $p=1, 2, \cdots$ and $k=1, 2, \cdots,$

$$
\begin{aligned}
&{}^kx_0^p={}^kX_{01}^p,\ {}^kx^p=({}^kX_{11}^p, \cdots, {}^kX_{\beta 1}^p),\ {}^ky^p=({}^kY_{11}^p, \cdots, {}^kY_{\beta 1}^p),\\
&{}^kw^p(\xi)={}^kW^p(\xi)_{\cdot 1}\quad(\xi\in\Xi_0^k)\quad\text{and}\\
&T^k=\begin{cases}\{{}^kx^p : p=1, 2, \cdots\} & \text{if}\quad (A): \tau({}^kW^p)\in\Delta^k\quad(p=1, 2, \cdots),\\ \{{}^kx^p : p=1, 2, \cdots, p^*(k)\} & \text{if}\quad (B): \tau({}^kW^p)\in\Delta^k\quad(p=1, 2, \cdots, p^*(k))\\ & \qquad\text{and}\quad \tau({}^kW^{p^*(k)+1})\notin\Delta^k.\end{cases}
\end{aligned}
$$

Then, for each $k=1, 2, \cdots,$

$$
(12\text{-}1)\qquad
\begin{aligned}
&{}^kx^0=0,\\
&{}^kx_0^{p^*(k)}=0\qquad \text{if}\quad (B),\\
&T^k \text{ is unbounded if } (A),
\end{aligned}
$$

$$||^k x^{p+1} - {}^k x^p|| < \varepsilon^k \qquad \text{for} \quad p = 1, 2, \cdots.$$

Furthermore, for each $k=1,2,\cdots$, we define the function $g^k(\cdot): R^\beta \to R^\beta$ as follows:

$$g^k(x) = \sum_{\xi\in\sigma} w(\xi)\cdot f(\psi_1(\xi))$$

$$\text{if} \quad x = \sum_{\xi\in\sigma} w(\xi)\psi_1(\xi), \ \sum_{\xi\in\sigma} w(\xi) = 1, w(\xi) \geqq 0 \ (\xi\in\sigma) \quad \text{and} \quad \sigma\in\Delta^k,$$

where $\Delta^k = \{\sigma\in\Xi^k : \sigma\subset R^\beta\times\{0\}\}$. Then we see, for $k=1,2,\cdots$ and $p=1,2,\cdots$,

$$(12\text{-}2) \qquad {}^k x_0^p \geqq 0, \ {}^k x^p \geqq 0,$$

$$(12\text{-}3) \qquad {}^k y^p = g^k({}^k x^p) + d\cdot {}^k x_0^p \geqq 0,$$

$$(12\text{-}4) \qquad \langle {}^k x^p, {}^k y^p \rangle = 0.$$

Let T^∞ be the set of all z such that $z^q \to z$ as $q\to\infty$ where $z^q\in T^{k(q)}$ and $k(q)\to\infty$ as $q\to\infty$. Now we shall show that T^∞ or its connected component containing the origin satisfies (a) and (b) of Theorem (4-4). Let $x\in T^\infty$. If $x=0$ then trivially $x\in\Gamma(x;0)$. Suppose that $x\neq 0$. Then there is an infinite sequence $\{z^q\in R^\beta\}$ such that $z^q\to x$ as $q\to\infty$ where $0\neq z^q = {}^{k(q)}x^{p(q)}\in T^{k(q)}$ for some $k(q)$ and $p(q)$, and $k(q)\to\infty$ as $q\to\infty$. It follows from (12-3) and (12-4) that

$${}^{k(q)}x_0^{p(q)} = \frac{\langle {}^{k(q)}x^{p(q)}, g^{k(q)}({}^{k(q)}x^{p(q)})\rangle}{\langle {}^{k(q)}x^{p(q)}, d\rangle} \geqq 0 \qquad \text{for} \quad q = 1, 2, \cdots.$$

Hence we obtain, for some $x_0\in R$,

$$x_0 = \lim_{q\to\infty} {}^{k(q)}x_0^{p(q)} = -\frac{\langle x, f(x)\rangle}{\langle x, d\rangle} \geqq 0.$$

By substituting ${}^{k(q)}x_0^{p(q)}$, ${}^{k(q)}x^{p(q)}$ and ${}^{k(q)}y^{p(q)}$ into (12-2), (12-3) and (12-4), and taking the limit of them, we see that x_0, x and $y=f(x)+dx_0$ satisfies (4-3) for $\rho=\langle d, x\rangle$. Thus $x\in\Gamma(x;\rho)$; we have shown that T^∞ satisfies (a) of Theorem (4-4).

In order to see the connectedness of T^∞, we introduce some notions. We say that a finite sequence $\{z^q : q=1,2,\cdots,n\}$ is an ε chain if $||z^q - z^{q+1}|| < \varepsilon$ for $q=1,2,\cdots, n-1$. We say that $C\subset R^\beta$ contains an ε path from a to b if there is an ε chain $\{z^q : q=1,2,\cdots,n\}$ in C such that $||a-z^1||<\varepsilon$ and $||b-z^n||<\varepsilon$. We say that C is ε-connected if it contains an ε path between any two points in C. The following theorem is due to EAVES and SAIGAL (1972).

(12-5) **Theorem:** If C^k is ε^k-connected, $\inf\{||z|| : z\in C^k\}\leqq\varepsilon^k$, and $\varepsilon^k\to 0$ as $k\to\infty$, then the set C^∞ of all z such that $z^q\to z$ as $q\to\infty$ where $z^q\in C^{k(q)}$ and $k(q)\to\infty$ as $q\to\infty$ is closed, $0\in C^\infty$ and either C^∞ is connected or each of its connected component is unbounded.

Since the sequence $\{T^k : k=1,2,\cdots\}$ satisfies the assumptions of the above theorem, T^∞ is closed, $0\in T^\infty$ and either T^∞ is connected or each of its connected component is unbounded. Let $\hat S$ be the connected component of T^∞ containing the origin. Suppose that $\hat S$ is bounded. Then T^∞ itself is connected. Let $\rho^+ = \max\{\langle x, d\rangle : x\in T^\infty\}$. Let $\rho\in[0,\rho^+]$. Since T^∞ is a compact connected subset of R^β such that $\min\{\langle x, d\rangle : x\in T^{\infty}\}=0$ and $\max\{\langle x, d\rangle : x\in T^{\infty}\}=\rho^{+}$, there is an $x\in T^{\infty}$ such that $\langle x, d\rangle=\rho$; $x\in\Gamma(x;\rho)$. On the other hand, it follows from the boundedness of T^{∞} that $T^{k}\subset D_{\rho^*}$ for all $k\geqq k^*$, some k^* and some $\rho^*\geqq 0$. Hence, for each $k\geqq k^*$,

$$ {}^{k}x^{p^*(k)}\in D_{\rho^*},\quad g^{k}({}^{k}x^{p^*(k)})\geqq 0 \quad\text{and}\quad \langle {}^{k}x^{p^*(k)}, g^{k}({}^{k}x^{p^*(k)})\rangle=0. $$

Let $\bar{x}$ be a cluster points of the sequence $\{{}^{k}x^{p^*(k)} : k=k^*, k^*+1, \cdots\}$. Then $\bar{x}\in\Gamma(\bar{x};\rho)$ for all $\rho\geqq\rho^{+}$. Thus $\hat{S}=T^{\infty}$ satisfies (b) of Theorem (4-4).

Finally suppose that $\hat{S}$ is unbounded. In this case, for each $\rho\geqq 0$, there is an $x\in\hat{S}$ such that $\langle x, d\rangle=\rho$; $x\in\Gamma(x;\rho)$ Hence $\hat{S}$ satisfies (b) of Theorem (4-4).

§13. The Brouwer fixed point problem

Let $D=\left\{x\in R^{\beta}_{+} : \sum_{i=1}^{\beta} x_i\leqq 1\right\}$, and g be a continuous function from D into itself. Then the following Brouwer fixed point theorem holds.

(13-1) **Theorem:** There exists a fixed point of g, a point $x\in D$ such that $g(x)=x$.

As stated in Introduction, there are some computational methods (Scarf (1967-b)), Kuhn (1968) and Eaves (1970 and 1972-b) which caluculate approximate fixed point of g. Here we demonstrate that the problem of finding a fixed point of g can be reduced to the CP. Define

(13-2) $$ f(x)=x-g\left(\frac{(|x_1|, \cdots, |x_\beta|)}{\max\left\{1, \sum_{i=1}^{\beta}|x_i|\right\}}\right) \quad\text{for each}\quad x\in R^{\beta}. $$

Then x is a fixed point of g if and only if it is a solution of the CP with the function f just defined. Let $B=\left\{x\in R^{\beta}_{+} : \sum_{i=1}^{\beta} x_i=2\beta\right\}$. Then it is easily shown that for each $\bar{x}\in B$ $\langle\bar{x}, f(\bar{x})\rangle>0$. Hence $f(\cdot)$ satisfies Condition (5-1). This implies that Algorithm (6-1) as well as Algorithm (8-3) computes approximate fixed points of g.

§14. Mathematical programming problems and n-person noncooperative games

Let $N=\{1, 2, \cdots, n\}$ denote the set of players, and for each $i\in N$, $Z(i)$ the set of strategies available to the player i For each $i\in N$, let $\theta_i(z^1, \cdots, z^n)$ denote the pay-off of the player i when the players j ($j=1, 2, \cdots, n$) take $z^j\in Z(j)$ ($j=1, 2, \cdots, n$) as their strategies, respectively. We assume the following condition:

(14-1) **Condition:** For each i, the following (i)-(iv) hold:

(i) $Z(i)$ is a compact convex set of the form $\{z^i\in R^{\alpha(i)\times 1}_{+} : B^i z^i+c^i\geqq 0 \text{ and } z^i\geqq 0\}$ where $B^i\in R^{m(i)\times\alpha(i)}$ and $c^i\in R^{m(i)\times 1}$.

(ii) $\theta_i(z^1, \cdots, z^n)$ is continuous on R^{α}, where $\alpha=\sum_{i=1}^{n}\alpha(i)$.

(iii) $\theta_i(z^1, \cdots, z^n)$ is concave with respect to $z^i \in R^{\alpha(i)}$.

(iv) $\theta_i(z^1, \cdots, z^n)$ is continuously differentiable with respect to $z^i \in R^{\alpha(i)}$.

(14-2) **Problem:** Find a $(\bar{z}^1, \cdots, \bar{z}^n) \in Z(1) \times \cdots \times Z(n)$ such that for each i

(14-3) $$\max_{z^i \in Z(i)} \theta_i(\bar{z}^1, \cdots, \bar{z}^{i-1}, z^i, \bar{z}^{i+1}, \cdots, \bar{z}^n) = \theta(\bar{z}^1, \cdots, \bar{z}^{i-1}, \bar{z}^i, \bar{z}^{i+1}, \cdots, \bar{z}^n)$$

A solution to the above problem is said to be an equilibrium of the n-person noncooperative game. The existence of an equilibrium was proved by NASH (1951). It is obvious that Problem (14-2) includes a concave maximization problem and a saddle-point problem as special cases. The above formulation of the n-person game includes that of ROSENMÜLLER (1971), WILSON (1971) and HOWSON (1972). EAVES (1972-a) showed that equilibria can be computed by the fixed point algorithm developed in (1972-b). In this section, we show that Problem (14-2) can be reduced to a certain type of the CP to which we apply Algorithm (8-3).

By KUHN and TUCKER (1951), $\bar{z}^i \in Z(i)$ satisfies (14-3) if and only if

$$\bar{z}^i \geqq 0, \; \bar{u}^i \geqq 0, \; -\nabla_i \theta_i(\bar{z}^1, \cdots, \bar{z}^n) - (B^i)^T \bar{u}^i \geqq 0, \; B^i \bar{z}^i + c^i \geqq 0,$$

$$\langle \bar{z}^i, -\nabla_i \theta_i(\bar{z}^1, \cdots, \bar{z}^n) - (B^i)^T \bar{u}^i \rangle = 0 \qquad \text{and} \qquad \langle \bar{u}^i, B^i \bar{z}^i + c^i \rangle = 0$$

for some $\bar{u}^i \in R^{m(i) \times 1}$, where

$$\nabla^i \theta^i(\bar{z}^1, \cdots, \bar{z}^n) = \left. \begin{bmatrix} \dfrac{\partial \theta_i(\bar{z}^1, \cdots, \bar{z}^{i-1}, z^i, \bar{z}^{i+1}, \cdots, \bar{z}^n)}{\partial z_1^i} \\ \vdots \\ \dfrac{\partial \theta_i(\bar{z}^1, \cdots, \bar{z}^{i-1}, z^i, \bar{z}^{i+1}, \cdots, \bar{z}^n)}{\partial z_{\alpha(i)}^i} \end{bmatrix} \right|_{z^i = \bar{z}^i} \in R^{\alpha(i) \times 1}.$$

Let

$$\phi(z^1, \cdots, z^n) = \begin{bmatrix} -\nabla_1 \theta_1(z^1, \cdots, z^n) \\ \vdots \\ -\nabla_n \theta_n(z^1, \cdots, z^n) \end{bmatrix} \in R^{\alpha \times 1},$$

$$B = \begin{bmatrix} B^1 & & 0 \\ & \ddots & \\ 0 & & B^n \end{bmatrix} \in R^{m \times \alpha}, \qquad c = \begin{bmatrix} c^1 \\ \vdots \\ c^n \end{bmatrix} \in R^{m \times 1},$$

$$z = \begin{bmatrix} z^1 \\ \vdots \\ z^n \end{bmatrix} \in R^{\alpha \times 1} \qquad \text{and} \qquad u = \begin{bmatrix} u^1 \\ \vdots \\ u^n \end{bmatrix} \in R^{m \times 1},$$

where $m = \sum_{i=1}^{n} m(i)$. Then $\bar{z}$ is an equilibrium if and only if for some $\bar{u}$ $(\bar{z}, \bar{u})$ is a solution to the following complementarity problem:

(14-4) **Problem:** Find a $(\bar{z}, \bar{u}) \in R^{\alpha+m}$ such that

$$(\bar{z}, \bar{u}) \geqq 0, \; \phi(\bar{z}) - B^T \bar{u} \geqq 0, \; B\bar{z} + c \geqq 0$$

$$\langle \bar{z}, \phi(\bar{z}) - B^T \bar{u} \rangle = 0 \qquad \text{and} \qquad \langle \bar{u}, B\bar{z} + c \rangle = 0.$$

We assume without loss of generality that $c \geqq 0$; in fact, an appropriate linear transformation provides such a situation. Let

$$h(x_1, \cdots, x_\alpha) = \begin{bmatrix} \psi(x_1, \cdots, x_\alpha) \\ \sum_{i=1}^{\alpha} B_{\cdot i} x_i \end{bmatrix} \in R^{(\alpha+m)\times 1} \qquad \text{for each} \quad (x_1, \cdots, x_\alpha) \in R^\alpha,$$

$$A_{\cdot i} = \begin{bmatrix} (-B_{i\cdot})^T \\ 0 \end{bmatrix} \in R^{(\alpha+m)\times 1} \qquad (i = \alpha+1, \cdots, \beta),$$

$$b = \begin{bmatrix} 0 \\ c \end{bmatrix} \in R^{(\alpha+m)\times 1}, \qquad d = \begin{bmatrix} e \\ 0 \end{bmatrix} \in R^{(\alpha+m)\times 1}$$

and

$$\beta = \alpha + m$$

By using these $h(\cdot): R^\alpha \to R^\beta$, $A_{\cdot i}$ $(i=\alpha+1, \cdots, \beta)$, b and d, we construct the lexico inequality system (7-2) and apply Algorithm (8-3) to it. Then we obtain a sequence $\{(W^p, X^p, Y^p)\}$ of distinct complementary basic solution of (7-2).

(14-5) **Lemma:** $\{(W^p, X^p, Y^p)\}$ is an infinite sequence.

Proof. Assume that Algorithm (8-3) generates only a finite sequence $\{(W^p, X^p, Y^p) : p=1, 2, \cdots, r\}$. Then, by Theorem (10-4), there is a nonzero $(\Delta x_0, \Delta x, \Delta y) \in R^{1+2\beta}$ which satisfies (i)-(x) of §10, and $\Delta x \neq 0$ if $(W^r, X^r, Y^r) = (W^0, X^0, Y^0)$. Let

$$\tilde{Z} = \begin{bmatrix} X^r_{1\cdot} \\ \vdots \\ X^r_{\alpha\cdot} \end{bmatrix} \in R^{\alpha\times(\alpha+1+\beta)}, \qquad \tilde{U} = \begin{bmatrix} X^r_{\alpha+1\cdot} \\ \vdots \\ X^r_{\beta\cdot} \end{bmatrix} \in R^{\alpha\times(\alpha+1+\beta)},$$

$$\Delta u = \begin{bmatrix} \Delta x_{\alpha+1} \\ \vdots \\ \Delta x_\beta \end{bmatrix} \in R^{m\times 1} \qquad \text{and} \quad C = \begin{bmatrix} H_{\alpha+1\cdot} \\ \vdots \\ H_{\beta\cdot} \end{bmatrix} \in R^{m\times(\alpha+1+\beta)}.$$

Then $\tilde{Z} \succeq 0$, $\tilde{U} \succeq 0$, $\Delta u \geqq 0$, $B\tilde{Z}+C \succeq 0$, $\tilde{Z}^T(-B^T\Delta u + e\Delta x_0) = 0$ and $\Delta u^T(B\tilde{Z}+C) = 0$. Hence $(e^T\tilde{Z})\Delta x_0 + \Delta u^T C = 0$. It follows from $C_{j\cdot} \succ 0$ $(j=1, 2, \cdots, m)$ and $(e^T\tilde{Z})\Delta x_0 \geqq 0$ that $\Delta u = 0$ and $(e^T\tilde{Z})\Delta x_0 = 0$. Since $(\Delta x_0, \Delta x, \Delta y)$ is nonzero, we see $\Delta x_0 > 0$. Hence $\Delta y_j = \Delta x_0$ $(j=1, 2, \cdots, \alpha)$, $X^r_{j\cdot} = 0$ $(j=1, 2, \cdots, \alpha)$, $Y_{j\cdot} = C_{j-\alpha\cdot} \succ 0$ $(j=\alpha+1, \cdots, \beta)$ and $X^r_{j\cdot} = 0$ $(j=\alpha+1, \cdots, \beta)$. Consequently, we obtain $(W^r, X^r, Y^r) = (W^0, X^0, Y^0)$, which contradicts $\Delta x = 0$. Q.E.D.

Obviously, the infinite sequence $\{(x_1^p, \cdots, x_\alpha^p) : p=1, 2, \cdots\}$ is in the compact set $\{z : Bz + c \geqq 0, z \geqq 0\}$. By Lemma (9-2), the sequence $\{x^p\}$ is bounded. Therefore, the set of all cluster points of $\{x^p\}$ is a nonempty, compact and connected subset of solutions to the CP.

§15. *N*-person cooperative games

Let $N = \{1, 2, \cdots, n\}$ denote the set of players, and 2^N the collection of all nonempty subsets of N. The sets contained in the collection 2^N are called coalitions.

(15-1) **Definition:** An *NSP* game $\Gamma=\{V(S)\epsilon 2^N\}$ is a collection of closed subsets of R^n satisfying the following conditions:
(i) If $u\epsilon V(S)$, $v\epsilon R^n$ and $u_i\geqq v_i$ for each $i\epsilon S$, then $v\epsilon V(S)$.
(ii) $V(\{i\})=\{u\epsilon R^n: u_i\leqq 0\}$ for $i=1, 2, \cdots, n$.
(iii) $u\leqq a$ for some $a\epsilon R^n$ and any $u\epsilon V(N)$.

(15-2) **Definition:** The core of an *NSP* game is the set $C(\Gamma)=V(N)-\bigcup_{S\epsilon 2^N}\{\mathrm{int}_0\, V(S)\}$, and the ε-core of an *NSP* game the set $C_\varepsilon(\Gamma)=V(N)-\bigcup_{S\epsilon 2^N}\{\mathrm{int}_\varepsilon\, V(S)\}$, where $\mathrm{int}_\varepsilon\, V=\{u\epsilon R^n: B_{\varepsilon'}(u)\subset V$ for some $\varepsilon'>\varepsilon\}$ ($\varepsilon\geqq 0$, $V\subset R^n$).

Note that $C(\Gamma)\subset C_\varepsilon(\Gamma)\subset C_{\varepsilon'}(\Gamma)$ if $\varepsilon'\geqq\varepsilon\geqq 0$ and that $C(\Gamma)=\bigcap_{\varepsilon>0} C_\varepsilon(\Gamma)$. Since $C_\varepsilon(\Gamma)$ is compact for each $\varepsilon>0$, $C(\Gamma)\neq\phi$ if $C_\varepsilon(\Gamma)\neq\phi$ for each $\varepsilon>0$. For a given *NSP* game Γ, $C(\Gamma)$ is not necessarily nonempty. When the *NSP* game is balanced, however, SCARF (1967) proposed a combinatorial method for obtaining a point of $C_\varepsilon(\Gamma)$; in this case $C(\Gamma)\neq\phi$. The notion of the "balancedness" was extended by BILLERA (1970 and 1971). He showed that if the *NSP* game is π-balanced then $C(\Gamma)\neq\phi$ by using the above SCARF's method. In this section, we shall show that, for any given $\varepsilon>0$, Algorithm (8-3) can be used for caluculating a point of the ε-core of a π-balanced *NSP* game. One defect of SCARF's method is that the storage space required to compute points of $C_\varepsilon(\Gamma)$ increases infinitely as ε approaches zero. While, the storage space used by our method is independent of ε.

First we describe the notion of the "π-balanceness".

(15-3) **Definition:** Let $c^S=(c_1^S, \cdots, c_n^S)^T\epsilon R^{n\times 1}(S\epsilon 2^N)$ be vectors satisfying the relations

$$0\neq c^S\geqq 0 \qquad \text{for all } S\epsilon 2^N,$$

$$c_i^S=0 \qquad \text{for any } i\notin S \text{ and all } S\epsilon 2^N,$$

$$c^N>0$$

and

$$c_i^{\{i\}}=1 \qquad \text{for } i=1, 2, \cdots, n.$$

A collection $\Pi\epsilon 2^N$ is said to be π-balanced if there exist nonnegative numbers $\delta_S(S\epsilon\Pi)$ such that $\sum_{S\epsilon\Pi}\delta_S c^S=c^N$. An *NSP* game Γ is said to be π-balanced if $\bigcap_{S\epsilon\pi} V(S)\subset V(N)$ for each π-balanced collection Π. In what follows, only π-balanced games will be considered.

Let m be the number of coalitions contained in $2^N-\bigcup_{i=1}^{n}\{i\}$, $\alpha=n$ and $\beta=n+m$. Let $\gamma(\cdot)$ be an arbitrary one-to-one mapping from $\{\alpha+1, \cdots, \beta\}$ to $2^N-\bigcup_{i=1}^{n}\{i\}$. Define $h(\cdot)\cdot R^\alpha\to R^\beta$, $A_{\cdot j}\epsilon R^{(\alpha+m)\times 1}(j=\alpha+1, \cdots, \beta)$, $b\epsilon R^{(\alpha+m)\times 1}$ and $d\epsilon R^{(\alpha+m)\times 1}$ as follows:

$$h_i(u)=\begin{cases} 0 & \text{if } i\epsilon\{1, 2, \cdots, \alpha\} \\ -1 & \text{if } i\epsilon\{\alpha+1, \cdots, \beta\}, u\epsilon V(\gamma(i)) \text{ and } u\leqq a+\varepsilon^* e \\ +1 & \text{otherwise,} \end{cases}$$

$$A_{\cdot j}=\begin{bmatrix} -c^{\tau(j)} \\ 0 \end{bmatrix}\in R^{(\alpha+m)\times 1} \qquad (j=\alpha+1,\cdots,\beta),$$

$$b=\begin{bmatrix} c^N \\ 0 \end{bmatrix}\in R^{(\alpha+m)\times 1} \qquad \text{and} \qquad d=\begin{bmatrix} 0 \\ e \end{bmatrix}\in R^{(\alpha+m)\times 1}.$$

Now we construct the lexico inequality system (7-2) by using them, and apply Algorithm (8-3) to it. Then a sequence $\{(W^p, X^p, Y^p)\}$ of distinct complementary basic solutions of (7-2) is obtained. $[w^p(\xi)(\xi\in\tau(W^p)), x_0^p, x^p, y^p]$ defined by (8-5) satisfies

(15-4) $$\begin{bmatrix} x_1^p \\ \vdots \\ x_\alpha^p \end{bmatrix}=\sum_{\xi\in\tau(W^p)} w^p(\xi)\psi_1(\xi)\geqq 0,\ \sum_{\xi\in\tau(W^p)} w^p(\xi)=1, w^p(\xi)\geqq 0 \ \ (\xi\in\tau(W^p)),$$

(15-5) $$\begin{bmatrix} y_1^p \\ \vdots \\ y_\alpha^p \end{bmatrix}=\sum_{j=\alpha+1}^{\beta}(-c^{\tau(j)})x_j^p+c^N\geqq 0,$$

and

(15-6) $$y_j^p=\sum_{\xi\in\tau(W^p)} h_j(\psi_1(\xi))w^p(\xi)+x_0^p\geqq 0 \qquad (j=\alpha+1,\cdots,\beta).$$

(15-7) **Lemma:** $\{(W^p, X^p, Y^p)\}$ is an infinite sequence.

Proof. Assume that Algorithm (8-3) generates only a finite sequence $\{(W^p, X^p, Y^p): p=1,2,\cdots,r\}$. By applying Theorem (10-4) we have a nonzero $(\varDelta x_0, \varDelta x, \varDelta y)\in R_+^{1+2\beta}$ such that

$$\varDelta x_i=0 \qquad (i=1,2,\cdots,\alpha),$$

(15-8) $$\begin{bmatrix} \varDelta y_1 \\ \vdots \\ \varDelta y_\alpha \end{bmatrix}=\sum_{j=\alpha+1}^{\beta}(-c^{\tau(j)})\varDelta x_j\geqq 0,$$

$$\varDelta y_i=\varDelta x_0 \ (i=\alpha+1,\cdots,\beta) \qquad \text{and} \qquad \varDelta y_j X_{j\cdot}^r=0 \ (j=1,2,\cdots,\beta).$$

By the definition of c^S $(S\in 2^N)$ and the second relation of (15-8), we see

$$\varDelta x_i=0 \ (i=\alpha+1,\cdots,\beta) \qquad \text{and} \qquad \varDelta y_j=0 \ (j=1,2,\cdots,\alpha).$$

Hence $\varDelta y_i=\varDelta x_0>0$ $(i=\alpha+1,\cdots,\beta)$, which imply

(15-9) $$X_{i\cdot}^r=0 \qquad (i=\alpha+1,\cdots,\beta)$$

and $x_j^r=0$ $(j=\alpha+1,\cdots,\beta)$. Thus, by (15-5), we have

$$\begin{bmatrix} y_1^r \\ \vdots \\ y_\alpha^r \end{bmatrix}=c^N>0; \qquad \text{hence} \qquad X_{i\cdot}^r=0 \ (i=1,2,\cdots,\alpha).$$

It follows from (15–9) and the above relation that $(W^r, X^r, Y^r)=(W^0, X^0, Y^0)$. In this case we can take nonzero Δx. But we have observed $\Delta x=0$. This is a contradiction. We have thereby shown that $\{(W^p, X^p, Y^p)\}$ is infinite. Q.E.D.

(15–10) **Lemma:** For $p=1, 2, \cdots$

$$\psi_1(\tau(W^p))\subset\{u\in R^n: -\varepsilon^*e\leqq u\leqq a+2\varepsilon^*e\}.$$

Proof. Assume that $\psi_1(\tau(W^p))\not\subset\{u\in R^n: u\leqq a+2\varepsilon^*e\}$ for some p. Since $\operatorname{diam}\psi_1(\tau(W^p))<\varepsilon^*$,

(15–11) $$\operatorname{co}\psi_1(\tau(W^p))\cap\{u\in R^n: u\leqq a+\varepsilon^*e\}=\phi.$$

Hence, by the definition of h and (15–6), we have

$$y_i^p=1+x_0^p>0\ (i=\alpha+1, \cdots, \beta),\ x_i^p=0\ (i=\alpha+1, \cdots, \beta),$$

$$y_j^p=c_j^N>0\ (j=1, 2, \cdots, \alpha)\quad\text{and}\quad x_j^p=0\ (j=1, 2, \cdots, \alpha).$$

Hence, by (15–4),

$$0=\begin{bmatrix} x_1^p \\ \vdots \\ x_\alpha^p \end{bmatrix}\in\operatorname{co}\psi_1(\tau(W^p)).$$

But obviously

$$0\in\{u\in R^n: u\leqq a+\varepsilon^*e\}.$$

The last two relations contradicts (15–11). Thus we have shown

$$\psi_1(\tau(W^p))\subset\{u\in R^n: u\leqq a+2\varepsilon^*e\}.$$

While (15–4) together with $\operatorname{diam}\psi_1(\tau(W^p))<\varepsilon^*$ implies that

$$\psi_1(\tau(W^p))\subset\{u\in R^n: -\varepsilon^*e\leqq u\}.$$

Q.E.D.

By condition (2–1–c) and the above lemma, we see

$$\lim_{p\to\infty}\min\psi_2(\tau(W^p))=+\infty.$$

Hence we can find a positive integer p^* such that $x_0^p=0$ for all $p\geqq p^*$. From Condition (2–3–b) we also obtain

(15–12) $$\lim_{p\to\infty}\operatorname{diam}\psi_1(\tau(W^p))=0.$$

Define, for $p=1, 2, \cdots$,

$$u^p=(x_1^p, \cdots, x_\alpha^p)=\sum_{\xi\in\tau(W^p)}w^p(\xi)\psi_1(\xi)\geqq 0,$$

$$\bar{u}_i^p=\max\{\psi_1(\xi)_i: \xi\in\tau(W^p)\}\qquad(i=1, 2, \cdots, n),$$

(15–13) $$\underline{u}_i^p=\min\{\psi_1(\xi)_i: \xi\in\tau(W^p)\}\qquad(i=1, 2, \cdots, n),\qquad\text{and}$$

$$\varepsilon^p = \|\bar{u}^p - \underline{u}^p\|.$$

Then (15-12) implies that $\lim_{p\to\infty} \varepsilon^p = 0$.

(15-14) **Lemma:** $\underline{u}^p \in V(N)$ for each $p \geqq p^*$.

Proof. By the definition of the π-balanced game, it is sufficient to show that $\underline{u}^p \in V(S)$ of all $S \in \Pi$ and some π-balanced collection Π. Let $\Pi_1 = \{i : y_i^p > 0 \text{ and } 1 \leqq i \leqq \alpha\}$ and $\Pi_2 = \{\gamma(j) : x_j^p > 0 \text{ and } \alpha+1 \leqq j \leqq \beta\}$. From (15-5) we see that $\Pi_1 \cup \Pi_2$ forms a π-balanced collection. If $i \in \Pi_1$ then $x_i^p = 0$; hence $\underline{u}^p \in V(\{i\})$. Suppose that $\gamma(j) \in \Pi_2$. Then $y_j^p = \sum_{\xi \in \tau(W^p)} h_j(\phi_1(\xi)) w^p(\xi) = 0$. By the construction of $h_j(u)$, there is a $\bar{\xi} \in \tau(W^p)$ for which $\phi_1(\bar{\xi}) \in V(\gamma(j))$ and $\phi_1(\bar{\xi}) \leqq a + \varepsilon^* e$. But $\underline{u}^p \leqq \phi_1(\bar{\xi})$. Hence $\underline{u}^p \in V(\gamma(j))$ (see (15-1-i). Thus we have shown $\underline{u}^p \in V(S)$ for all $S \in \Pi_1 \cup \Pi_2$. Q.E.D.

(15-15) **Lemma:** If $p \geqq p^*$ then

$$\bar{u}^p \notin \text{int } V(S) \qquad \text{for all} \quad S \in 2^N.$$

Proof. Since $\bar{u}^p \geqq u^p \geqq 0$, $\bar{u}^p \notin \text{int } V(\{i\})$ for $i = 1, 2, \cdots, n$. While, corresponding to each $S \in 2^N - \bigcup_{i=1}^{n} \{i\}$, there is a $j \in \{\alpha+1, \cdots, \beta\}$ such that $S = \gamma(j)$. It follows from (15-6), $x_0^p = 0$ and the construction of $h(\cdot)$ that $h_j(\phi_1(\hat{\xi})) = 1$ for some $\hat{\xi} \in \tau(W^p)$. From Lemma (15-14) and (15-13) we see $\phi_1(\hat{\xi}) \leqq a + \varepsilon^* e$. Thus $\phi_1(\hat{\xi}) \notin V(\gamma(j))$, which implies $\bar{u}^p \notin V(\gamma(j))$. Q.E.D.

We have thereby shown the following theorem and its corollary.

(15-16) **Theorem:** $\underline{u}^p \in C_{\varepsilon p}(\Gamma)$ for all $p \geqq p^*$.

(15-16) **Corollary:** $C(\Gamma) \neq \phi$.

Chap. 5. Concluding discussion

In this chapter we compare five algorithms for solving the nonlinear CP, the algorithm given in Chap. 2, the algorithm given in Chap. 3, the algorithm developed by SEKINE, NISHINO and the author (1973), the algorithm developed by FISHER and GOULD (1973), and the algorithm developed by GALCIA (1973). This part consists of §16 and §17. In §16 we consider the effective range of the algorithms. In §17 we examine the computational efficiency of the algorithms.

§16. On the effective range of the algorithm

Suppose that each $\sigma \in \Xi$ has a sufficiently small diameter. Algorithm (8-3) generates a sequence of adjacent simplices such that a point $\bar{x}$ of the convex hull of each simplex satisfies

(16-1) $\quad \bar{x}_0 \geqq 0, \bar{x} \geqq 0, f(\bar{x}) + d\bar{x}_0 \geqq 0 \quad \text{and} \quad \langle \bar{x}, f(\bar{x}) + d\bar{x}_0 \rangle \doteqdot 0 \qquad \text{for some} \quad \bar{x}_0 \in R,$

where " $\varepsilon \doteqdot 0$ " indicates " ε is approximately zero ", While, Algorithm (6-1) generates a sequence of adjacent simplices such that a point $\tilde{x}$ of the convex hull of each simplices is an approximate fixed point of $I'(\cdot\,;\rho)$ which was defined by (4-2). $\tilde{x}$ also satisfies (16-1). In order for the function $I'(\cdot\,;\rho)$ to be defined consistently, however, we took $d\in R^\beta$ a positive vector. This fact limits the effective range of Algorithm (6-1). On the other hand, Algorithm (8-3) requires merely that $d\in R^\beta$ satisfies (7-1), but not the positivity of d. In fact, when we used Algorithm (8-3) in §14 and §15, we did not take positive d. Therefore, we conclude that the class of the complementarity problems for which Algorithm (8-3) is applicable contains that for which Algorithm (6-1) is applicable.

In the following discussion, the computational methods developed by FISHER and GOULD (1973) and by GARCIA (1973) are denoted by Algorithm $F-G$ and Algorithm G, respectively. Algorithm $F-G$ works roughly as follows. First, a labeling function $L(\cdot): R^\beta \to \{1, 2, \cdots, \beta+1\}$ is introduced.

$$(16\text{-}2)\qquad L(x)=\begin{cases}\beta+1 & \text{if } x>0 \text{ and } f(x)\geqq 0.\\ j & \text{if } x>0, f(x)\ngeqq 0, \text{ and } j \text{ such that } f_j(x)=\min_i f_i(x) \text{ (take}\\ & \text{the least such } j \text{ in the case of ties.)}\\ j & \text{if } x\not> 0, \text{ and } j \text{ such that } x_i>0 \text{ for } i=1,2,\cdots,j-1\\ & \text{and } x_j=0\end{cases}$$

Second, R^β_+ is triangulated by a simplicial subdivision such that there is a unique boundary $(\beta-1)$-simplex with labels 1 through β; the simplex is called $(\beta+1)$-almost completely labeled boundary set. Then, begining with the unique $(\beta+1)$-almost completely labeled boundary set, a complementary pivoting method generates a sequence of distinct adjacent β-simpleces such that each intersecting $(\beta-1)$-simplex of successive β-simplices possesses all labeles except $\beta+1$. The algorithm terminates when and only when it produces a fully labeled β-simplex. Each point of the convex hull of any fully labeled simplex is an approximate solution of the CP. FISHER and GOULD provided the following sufficient condition under which the algorithm terminates in a finite number of steps.

(16-3) **Condition:** There is a compact set B in $R^\beta-\{0\}$ such that $\bar{x}\in B$ implies $\langle\bar{x}, f(\bar{x})\rangle>0$ and such that B separates the origin from ∞.

The above condition is stronger than Condition (5-1). If we make a slight modification, however, the algorithm can be used to compute approximate solutions of the CP which merely satisfies Condition (5-1). Define $L(\cdot): R^\beta \to \{1, 2, \cdots, \beta+1\}$ by

$$L(x)=\begin{cases}\beta+1 & \text{if } x>0, f(x)\geqq 0\\ j & \text{if } x>0, f(x)\ngeqq 0, \text{ and } j \text{ such that } f_j(x)/d_j=\min_i f_i(x)/d_i,\\ & \text{(take the least such } j \text{ in case of ties)}\\ j & \text{if } x\not> 0, \text{ and } j \text{ such that } x_i>0 \text{ for } i=1,2,\cdots,j-1 \text{ and}\\ & x_j=0,\end{cases}$$

instead of by (16-2), where d is a positive vector of R^{β}. If the mesh of the triangulation is sufficiently small, a point $\bar{x}$ of the convex hull of each β-simplex generated by the algorithm satisfies (16-1). If the algorithm continues infinitely, we can find a point $\bar{x}\in B$ in the convex hull of a generated simplex. By Condition (5-1), there is an $x\in R^{\beta}_{+}$ such that $\langle x-\bar{x}, d\rangle<0$ and $\langle x-\bar{x}, f(\bar{x})\rangle<-\delta$, where δ is a positive number which only depends on the function f and the shape of the set B. It follows that

$$\begin{aligned} 0 &\leqq \langle x, f(\bar{x})+d\bar{x}_0\rangle \\ &\doteqdot \langle x-\bar{x}, f(\bar{x})+d\bar{x}_0\rangle \\ &= \langle x-\bar{x}, f(\bar{x})\rangle + \langle x-\bar{x}, d\rangle x_0 \\ &< -\delta. \end{aligned}$$

This is a contradiction. Hence the algorithm terminates in finite number of steps and generates an approximate solution of the CP.

Algorithm G generates a sequence of distinct adjacent simplices such that a point $\bar{x}$ of the convex hull of each simplex satisfies (16-1), too. Therefore, the effective range of his algorithm also covers the CP satisfying Condition (5-1). The following condition is assumed in GARCIA (1973).

(16-4) **Condition:** There is a compact nonempty set C in R^{β} such that for each $x\in R^{\beta}-C$, there is an $x\in C$ such that $\langle x-\bar{x}, f(\bar{x})\rangle\leqq 0$.

The above condition is similar in nature to Condition (5-1). Note that both conditions are stronger than that of Corollary (4-6).

§17. On the computational efficiency of the algorithms

First we give a small numerical example. We consider the CP with $f(\cdot): R^4\to R^4$ of the form

$$\begin{aligned} f_1(x) &= 3x_1^2+2x_1x_2+2x_2^2+x_3+3x_4-6, \\ f_2(x) &= 2x_1^2+x_1+x_2^2+3x_3+2x_4-2, \\ f_3(x) &= 3x_1^2+x_1x_2+2x_2^2+2x_3+3x_4-1, \\ f_4(x) &= x_1^2+3x_2^2+2x_3+3x_4-3. \end{aligned}$$

The unique solution of the problem is

$$\begin{aligned} \hat{x}_1 &= \sqrt{6}/2 = 1.2247449, \\ \hat{x}_2 &= 0, \\ \hat{x}_3 &= 0, \\ \hat{x}_4 &= 0.5. \end{aligned}$$

For each approximate solution x of the CP define

$$ACC(x)=\sum_{i=1}^{4} \delta_i(x),$$

where

$$\delta_i(x)=\begin{cases} |f_i(x)| & \text{if} \quad x_i>0, \\ -\min\{0, f_i(x)\} & \text{if} \quad x_i=0. \end{cases}$$

Then $ACC(x)$ represents an accuracy of the approximation. As stated in Introduction, Algorithm (8-3) is based on the algorithm developed by SEKINE, NISHINO and the author (1973). Computer programs in FORTRAN IV were writen by the author for Algorithm (6-1), Algorithm (8-3) and their algorithm. In each execution of the algorithms, the iteration was stopped when an approximate solution $\tilde{x}$ such that $ACC(\tilde{x})\leqq 0.001$ was computed. The programs were solved on a HITAC 8700 at Tokyo University Computer Center. The compile time of the programs as well as the time required for the computation of the test problem depends on not only the programs and the problem but also the situation under which the computer handles the programs. The following symbols are employed in Tables (17-1), (17-2), (17-3), (17-7) and (17-8).

Algorithm S: The algorithm developed by SEKINE, NISHINO and the author.
dim: the dimension of the simplicial subdivision used in the algorithm.
r: the number of iterations.
σ^0: initial simplex of the algorithm.
σ^r: the final simplex.
$d(\sigma)$: the diameter of $\psi_1(\sigma)$ (in Tables (17-1), (17-3), (17-7) and (17-8), or the diameter of σ itself (in Table (17-2)).
t_0: the compile time (second) of the program.
t_1: the time (second) required for compution $\tilde{x}$.
$\tilde{x}$: the approximate solution computed by the algorithm.

Table 17-1. Computational results of Algorithm 6-1 (dim $\Xi=5$)

$d(\sigma^0)$	r	t_0	t_1	t_0+t_1	$d(\sigma^r)$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
0.02828	1110	9.19	6.23	15.42	0.001925	1.22487	0	0	0.49972
0.04243	815	5.06	4.20	9.26	0.002563	1.22479	0	0	0.50004
0.05657	635	8.64	5.73	14.37	0.003647	1.22476	0	0	0.49988
0.07071	582	5.52	3.14	8.66	0.002839	1.22489	0	0	0.49961
0.08485	570	3.16	4.11	7.28	0.002969	1.22466	0	0	0.50014
0.9899	386	5.35	3.94	9.29	0.006382	1.22458	0	0	0.50013
0.11328	448	4.50	3.06	7.56	0.004146	1.22471	0	0	0.50017
0.12728	411	7.47	3.75	11.23	0.006926	1.22484	0	0	0.49992
0.14142	816	6.83	3.76	10.59	0.002221	1.22496	0	0	0.49998

Table 17-2. Computational results of Algorithm S (dim $\Xi=4$)

$d(\sigma^0)$	r	t_0	t_1	t_0+t_1	$d(\sigma^r)$	$\tilde{x}_1$	$\hat{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
0.04243	228	6.83	1.94	8.77	0.05196	1.22468	0	0	0.50001

Table 17-3. Computational results of Algorithm 8-3 (dim $\Xi=3$)

$d(\sigma^0)$	r	t_0	t_1	t_0+t_1	$d(\sigma^r)$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
0.07071	52	7.24	0.33	7.57	0.05590	1.22474	0	0	0.50000
0.14142	31	5.42	0.17	5.59	0.06009	1.22473	0	0	0.50000
0.28284	28	10.39	0.39	10.78	0.05207	1.22466	0	0	0.50000
0.42426	34	8.07	0.40	8.47	0.04714	1.22465	0	0	0.50000

We observe that Algorithm (dim $\Xi=3$) solves the test problem more efficiently than others. Now we point out three noteworthy features of Algorithm 8-3.

(17-4) If $f: R^{\beta}\to R^{\beta}$ has the form

$$h(x_1, \cdots, x_\alpha)+\sum_{i=\alpha+1}^{\beta} A_{\cdot i}x_i+b,$$

where $h: R^{\alpha}\to R^{\beta}$ is nonlinear, $A_{i\cdot}\in R^{\beta\times 1}$ $(i=\alpha+1, \cdots, \beta)$ and $b\in R^{\beta\times 1}$, the dimension of the simplicial subdivision used in the algorithm is merely $\alpha+1$.

(17-5) The algorithm generates an infinite sequence of approximate solutions of the CP which automatically converges to exact solution of the CP.

(17-6) In the algorithm, a solution of the CP is approximated by a point $\tilde{x}$, and if f is continuously twice differentiable the accuracy of the approximate solution $\tilde{x}$ is represented by

$$f_i(\tilde{x})>-\mu\{d(\sigma^r)\}^2 \qquad (i=1, 2, \cdots, \beta),$$

$$|f_i(x)|<\mu\{d(\sigma^r)\}^2 \qquad \text{if} \quad x_i>0,$$

$$ACC(x)<\beta\mu\{d(\sigma^r)\}^2$$

(see the last part of §11).

These features of the algorithm directly affect the computational efficiency; they operates to reduce the total number of iterations required to compute an approximate solution with a given accuracy. In the following, we shall show the effectiveness of the features more precisely.

The test problem has a function f of the form just described in (17-4);

$$\alpha=2,$$

$$h_1(x_1, x_2)=3x_1^2+2x_1x_2+2x_2^2,$$

$$h_2(x_1, x_2)=2x_1^2+x_1+x_2^2,$$

$$h_3(x_1, x_2)=3x_1^2+x_1x_2+2x_2^2,$$

$$h_4(x_1, x_2) = x_1^2 + 3x_2^2,$$

$$A_{\cdot 3} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \quad A_{\cdot 4} = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} -6 \\ -2 \\ -1 \\ -3 \end{bmatrix}.$$

When Algorithm 8-3 was applied to the test problem before, we took $\alpha, h, A_{\cdot i}$ $(i=3, 4)$ and b as above and obtained Table (17-3). In the case that we took

$$\alpha = 3,$$

$$h_1(x_1, x_2, x_3) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3,$$

$$h_2(x_1, x_2, x_3) = 2x_1^2 + x_1 + x_2^2 + 3x_3$$

$$h_3(x_1, x_2, x_3) = 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3,$$

$$h_4(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + 2x_3,$$

$$A_{\cdot 4} = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} -6 \\ -2 \\ -1 \\ -3 \end{bmatrix},$$

Table (17-7) was obtained. In the case that we took

$$\alpha = 4,$$

$$h_1(x_1, x_2, x_3, x_4) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4,$$

$$h_2(x_1, x_2, x_3, x_4) = 2x_1^2 + x_1 + x_2^2 + 3x_3 + 2x_4,$$

$$h_3(x_1, x_2, x_3, x_4) = 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 3x_4,$$

$$h_4(x_1, x_2, x_3, x_4) = x_1^2 + 3x_2^2 + 2x_3 + 3x_4,$$

$$b = \begin{bmatrix} -6 \\ -2 \\ -1 \\ -3 \end{bmatrix},$$

Table 17-7. Computational results of Algorithm (8-3) (dim $\Xi = 4$)

$d(\rho^0)$	r	t_0	t_1	t_0+t_1	$d(\rho^r)$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
0.07071	76	6.97	0.45	7.41	0.07071	1.22474	0	0	0.50000
0.14142	44	6.47	0.81	7.28	0.07071	1.22466	0	0	0.50000
0.28284	49	9.47	0.63	10.10	0.05774	1.22466	0	0	0.50000
0.42426	44	8.12	0.31	8.43	0.05303	1.22466	0	0	0.50000

Table 17-8. Computational results of Algorithm (8-3) (dim $\Xi=5$)

$d(\sigma^0)$	r	t_0	t_1	t_0+t_1	$d(\sigma^r)$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
0.07071	143	8.14	1.07	9.21	0.03536	1.22474	0	0	0.50001
0.14142	85	5.26	0.91	6.17	0.05774	1.22466	0	0	0.50000
0.28284	72	9.26	0.88	10.14	0.07118	1.22465	0	0	0.50001
0.42426	85	8.07	0.94	9.01	0.07500	1.22464	0	0	0.50000

Table (17-8) was obtained. Graph (17-9) shows the effectiveness of the feature (17-4).

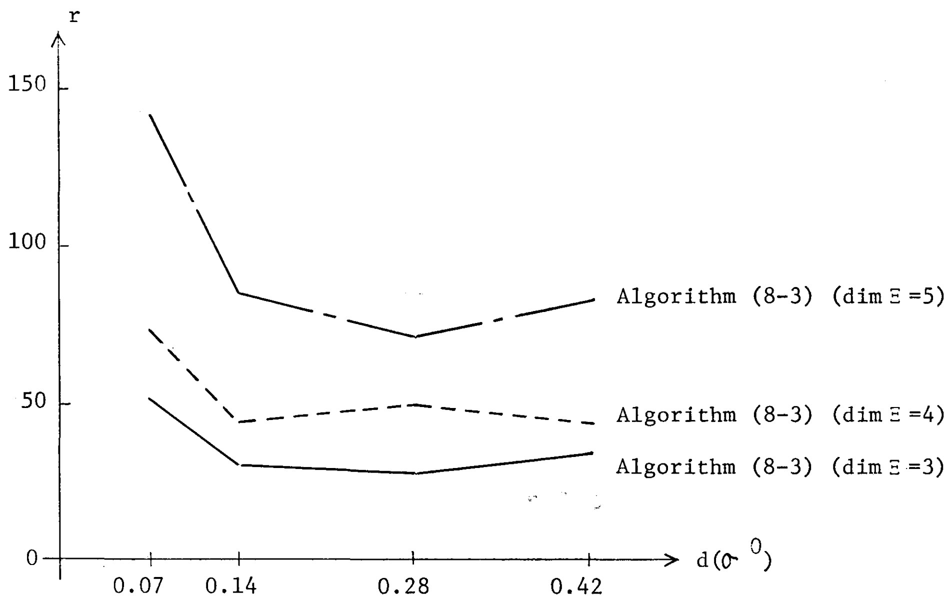
In spite of the relation that the dimension of the simplicial subdivision used in Table (17-2) is less than the one used in Table (17-8), the number of the iterations in the former is greater than all the numbers of the iterations in the latter. We shall consider the reason why this is so. Corresponding to each simplicial subdivision Ξ of R^β such that $\sup\{\text{diam}\,\sigma : \sigma\in\Xi\}=\varepsilon$, Algorithm S computes an approximate solution $\tilde{x}$ such that

$$f_i(\tilde{x}) > -\mu\varepsilon^2 \qquad (i=1, 2, \cdots, \beta),$$

$$|f_i(\tilde{x})| < \mu\varepsilon^2 \qquad \text{if } x_i>0,$$

$$ACC(\tilde{x}) < \beta\mu\varepsilon^2.$$

Therefore, if we want to obtain an approximate solution with high accuracy, we



Graph 17-9. The effectiveness of the feature (17-4)

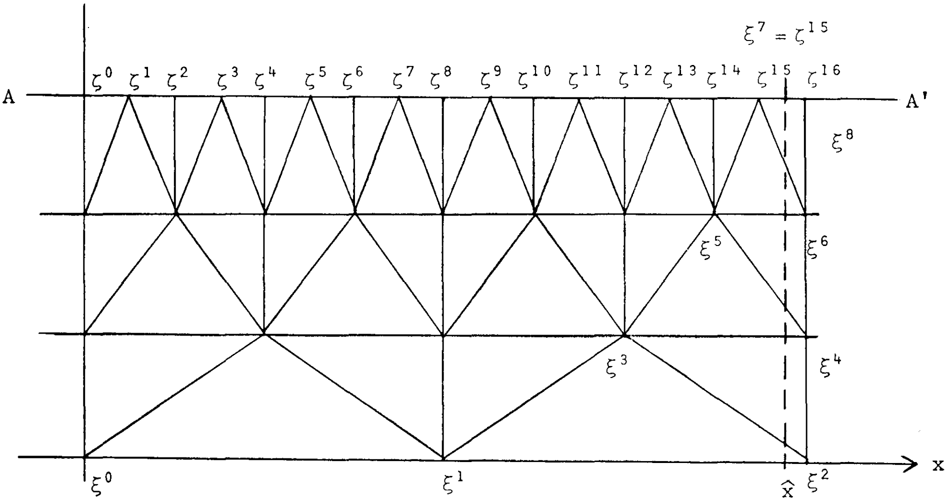


Figure 17-10. The effectiveness of the feature (17-5)

must take sufficiently small ε. But the smaller ε we take, the more iterations are required to obtain an approximate solution. It seems that if $\varepsilon/\eta=\varepsilon'$, where η is a positive integer, then the number of iterations corresponding to ε' is about η times as large as that corresponding to ε. In Algorithm 8-3, this difficulty are solved by the feature (17-5). In order to illustrate this fact, we consider the CP with $f: R^1 \to R^1$ and the simplicial subdivision in Figure (17-10).

Suppose that $\hat{x}$ is a unique solution of the CP. Then Algorithm (8-3) generates a sequence of simplices, $\{\zeta^0\zeta^1\}\{\zeta^1\zeta^2\}\{\zeta^2\zeta^3\}\{\zeta^3\zeta^4\}\{\zeta^4\zeta^5\}\{\zeta^5\zeta^6\}\{\zeta^6\zeta^7\}\{\zeta^7\zeta^8\}$ to compute an approximate solution contained in the convex hull of the simplex $\{\zeta^7\zeta^8\}$; the length of the sequence is 8. In the case that we compute an approximate solution with the same accuracy by using Algorithm S, we must take a simplicial subdivision Ξ such that diam $\sigma \leqq$ diam $\{\zeta^7\zeta^8\}$ for all $\sigma \in \Xi$. If we regard the line AA' as the x-axis, the sequence of simplices generated by the algorithm is $\{\zeta^0\zeta^1\}\{\zeta^1\zeta^2\}$, $\cdots, \{\zeta^{14}\zeta^{15}\}\{\zeta^{15}\zeta^{16}\}$; the length of the sequence is 16. The above argument explains the difference between the numbers of iterations in Table (17-8) and in Table (17-2).

Algorithm (6-1) has the feature (17-5). Comparing Table (17-1) with Table (17-8), however, we observe that each number of the iterations in Table (17-1) is about ten times as large as each number of the iterations in Table (17-8). This inefficiency of Algorithm (6-1) is caused by the fact that it has not the feature (17-6). In Algorithm (6-1), a solution of the CP is approximated by the set $\mathrm{co}\,\phi_1(\sigma^r) \cap R^\beta_+$; we have no information which ensures that a point $x \in \mathrm{co}\,\phi_1(\sigma^r) \cap R^\beta_+$ better approximates a solution of the CP than other points in $\mathrm{co}\,\phi_1(\sigma^r) \cap R^\beta_+$. Each point of the set can be regarded as an approximate solution of the CP, whose accuracy is expressed by the relations

$$f_i(x) > -\nu d(\sigma^r) \qquad (i=1,2,\cdots,\beta),$$

$$|f_i(x)| < \nu d(\sigma^r) \qquad \text{if } x_i > d(\sigma^r),$$

$$ACC(x) < \beta\nu d(\sigma^r)$$

(see the last part of §6). Therefore, with respect to the accuracy of the approximate solution in terms of the diameter of $\psi_1(\sigma^r)$, Algorithm (6-1) is inferior to Algorithm (8-3). In fact, the mean $d(^r\sigma)$ in Table (17-1) is 0.003735 and the one in Table (17-8) is 0.05982; the latter is about sixteen times as latge as the former.

Algorithm F—G and Algorithm G have none of the features (17-4), (17-5) and (17-6). Without reference to the form of the function f, Algorithm F—G and Algorithm G uses simplicial subdivisions with dimensions β and $\beta+1$, respectively. In their algorithms a solution of the CP is approximated by a simplex σ; such approximation is called simplicial approximation. In Algorithm F—G each x contained in the convex hull of σ satisfies

$$f_i(x) > -\nu \operatorname{diam} \sigma \qquad (i=1, 2, \cdots, \beta),$$

$$|f_i(x)| < \nu \operatorname{diam} \sigma \qquad \text{if } x_i > \operatorname{diam} \sigma,$$

$$ACC(x) < \beta\nu \operatorname{diam} \sigma.$$

While, in Algorithm G each x contained in the interior of the convex hull of σ satisfies

$$f_i(x) > -\nu \operatorname{diam} \sigma \qquad (i=1, 2, \cdots, \beta),$$

$$|f_i(x)| < \nu \operatorname{diam} \sigma \qquad \text{if } x_i > 0,$$

$$ACC(x) < \beta\nu \operatorname{diam} \sigma,$$

In their algorithms, however, there are some devices which increase the computational efficiency. As stated in the previous section, Algorithm F—G generates a sequence of distinct adjacent simplices such that any point x of each simplex satisfies (16-1). FISHER and GOULD showed in their paper that in certain instances a subsequence of distinct adjacent simplices can be anticipated, and that it is possible to accelate their algorithm by skipping over the anticipated sequence. In Algorithm G, only the nonnegative orthant of the β-dimensional Euclidean space is triangulated, and adjacency of boundary simplices as well as non-boundary simplices is introduced consistently. The introduction of the adjacency of boundary simplices operate to reduce the total number of iterations. It seems that by combining their devices and Algorithm (8-3) we can develope more efficient algorithm.

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