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# A NOTE ON A FUNCTIONAL CENTRAL LIMIT THEOREM FOR UNIFORM MIXING SEQUENCE

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# A NOTE ON A FUNCTIONAL CENTRAL LIMIT THEOREM FOR UNIFORM MIXING SEQUENCE

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**ABSTRACT**

In this paper we shall deal with the functional central limit theorem (the version of Donsker's invariance principle) for uniform mixing sequence.

BILLINGSLEY (1968) proved a version of Donsker's theorem for the variables of a stationary sequence satisfying a uniform mixing condition and extended the results to functions of such sequence.

We shall show that Billingsley's results also hold for the variables of a double sequence for which the variables in any row are stationary and uniform mixing sequence.

## 1. Introduction and Summary

We shall consider a double sequence of random variables

$$\begin{array}{l} \xi_{1,1}, \xi_{1,2}, \cdots\cdots, \xi_{1,k}, \cdots\cdots \\ \xi_{2,1}, \xi_{2,2}, \cdots\cdots, \xi_{2,k}, \cdots\cdots \\ \vdots \quad \vdots \qquad\qquad \vdots \\ \xi_{n,1}, \xi_{n,2}, \cdots\cdots, \xi_{n,k}, \cdots\cdots \\ \vdots \quad \vdots \qquad\qquad \vdots \end{array} \tag{S}$$

where the variables in the same row are defined on the same probability space and are strictly stationary and uniformly mixing sequence. Let $k_n$, $n=1, \cdots$, be positive integers going to infinity and $X_n(\omega)$ be the random element of Skorohod space $D[0,1]$ whose value at $t$ is

$$X_n(t, \omega)=1/\sigma\sqrt{k_n}\cdot\{\xi_{n,1}(\omega)+\cdots+\xi_{n,[k_nt]}(\omega)\}, \tag{1.1}$$

where $\sigma$ is a suitable positive number and $0 \leq t \leq 1$. We shall investigate the problem of finding the conditions which the relation

$$X_n \xrightarrow{D} W \tag{1.2}$$

holds, where $W$ denotes a Wiener process and $X_n \xrightarrow{D} W$ means that $X_n$ converges in distribution to $W$.

When $X_n$ is defined by

$$X_n(t, \omega) = 1/\sigma\sqrt{n} \cdot \{\xi_{1,1}(\omega) + \cdots + \xi_{1,[nt]}(\omega)\} \tag{1.3}$$

in stead of (1.1) and $\sum_{k=1}^{\infty} \varphi_1^{1/2}(k) < \infty$ for which $\varphi_1(k)$ is the mixing coefficient of $\{\xi_{1,j};\ j=1, 2, \cdots\}$, BILLINGSLEY (1968) proved that the relation (1.2) holds, by using the theorem which characterizes Wiener process, given by ROSÉN (1967).

In this paper, we shall show by the same manner as the proof of Theorem 20.1 of BILLINGSLEY (1968) that if $\lim_{n,k\to\infty} \left(\sum_{j=1}^{k} \varphi_n^{1/2}(j)\right) < \infty$ for which $\varphi_n(j)$ is the mixing coefficient of $\{\xi_{n,j};\ j \geq 1\}$ and $\lim_{n,k\to\infty} \left(E\{\xi_{n,1}^2\} + 2\sum_{j=2}^{\infty} E\{\xi_{n,1}\xi_{n,j}\}\right) < \infty$, and if $\{\xi^2_{n,1};\ n \geq 1\}$ is uniformly integrable, then the relation (1.2) holds. Furthermore, we shall extend the above result to functions of strictly stationary, uniformly mixing sequence. The proof is similar to the one of Theorem 21.1 of BILLINGSLEY (1968).

Given a one-sided strictly stationary and uniformly mixing sequence, we can always construct a two-sided sequence with the same finite-dimentional distributions and the same mixing coefficient as before, from the arguments in BILLINGSLEY (1968), pp 168—169. In what follows, therefore, we shall always consider a two-sided double sequence $\{\xi_{n,j};\ n=1, 2, \cdots, j=0, \pm 1, \pm 2, \cdots\}$ in stead of (S).

## 2. Notation and Results

Let $\{\xi_n;\ n=0, \pm 1, \pm 2, \cdots\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{B}, P)$. For $a \leq b$, define $\mathcal{M}_a^b$ as the $\sigma$-field generated by the random variables $\xi_a, \cdots, \xi_b$; define $\mathcal{M}_{-\infty}^a$ as the $\sigma$-field generated by $\cdots, \xi_{a-1}, \xi_a$; and define $\mathcal{M}_a^\infty$ as the $\sigma$-field generated by $\xi_a, \xi_{a+1}, \cdots$.

Consider a nonnegative functin $\varphi$ of nonnegative integers. We shall say that the sequence $\{\xi_n\}$ is $\varphi$-mixing (uniformly mixing) if, for each $k$ $(-\infty < k < \infty)$ and for each $n$ $(n \geq 1)$, $E_1 \in \mathcal{M}_{-\infty}^k$ and $E_2 \in \mathcal{M}_{k+n}^\infty$ together imply

$$|P(E_1 \cap E_2) - P(E_1)P(E_2)| \leq \varphi(n)P(E_1). \tag{2.1}$$

This is a joint property of $\{\xi_n\}$ and $\varphi$. We consider only functions $\varphi$ satisfying

$$\lim_{n\to\infty} \varphi(n) = 0, \tag{2.2}$$

and usually we require that $\varphi(n)$ go to 0 at some specified minimum rate. If we say that $\{\xi_n\}$ is $\varphi$-mixing without specifying $\varphi$, we mean that (2.1) holds for some $\varphi$ satisfying (2.2). The random variables $\xi_n$ are said to be uniformly integrable if

$$\lim_{\alpha\to\infty} \sup_n \int_{\{|\xi_n|\geq\alpha\}} |\xi_n| dP=0.$$

Given a double sequence of random variables $\{\xi_{n,j};\ n=1,2,\cdots, j=0,\pm1,\pm2,\cdots\}$ defined on $(\Omega,\mathcal{B},P)$, let $S_{n,j}=\xi_{n,1}+\cdots+\xi_{n,j}$ be the partial sums in the $n$-th row and $X_n$ be the random element of Skorohod space $D[1,0]$ whose value at $t$ is

$$X_n(t,\omega)=1/\sigma\sqrt{k_n}\cdot S_{n,[k_nt]}(\omega), \tag{2.3}$$

where $k_n$ are positive integers and $0\leq t\leq 1$. Then, we have the following theorem which is a simple generalization of Theorem 20.1 of BILLINGSLEY (1968).

THEOREM 1. Suppose that the following conditions hold:

(i) for each $n$, the sequence $\{\xi_{n,j};\ j=0,\pm1,\pm2,\cdots\}$ is strictly stationary and $\varphi_n$-mixing with $E\{\xi_{n,0}\}=0$,

(ii) there exists a $\lim_{n,k\to\infty} T_{n,k}(=T)$, where $T_{n,k}=\sum_{j=0}^{k}\varphi_n^{1/2}(j)$,

(iii) the random variables $\xi_{n,0}^2$, $n=1,2,\cdots$, are uniformly integrable,

(iv) there exists a $\lim_{n,k\to\infty}\sigma_{n,k}^2(=\sigma^2)$, where $\sigma_{n,k}^2=E\{\xi_{n,0}^2\}+2\sum_{j=1}^{k}E\{\xi_{n,0}\xi_{n,j}\}$.

If $\sigma^2>0$ and $X_n$ is defined by (2.3) for which $k_n$ are positive integers going to infinity, then

$$X_n \xrightarrow{D} W.$$

Next, we shall generalize Theorem 1 by analyzing sequence $\{\eta_{n,j};\ n=1,2,\cdots, j=0,\pm1,\pm2,\cdots\}$ for which each $\eta_{n,j}$ is a function of the entire process $\{\xi_n\}$, which we assume to be strictly stationary and $\varphi$-mixing.

For each $n$, let $f_n$ be a measurable mapping from the space of double infinite sequence $(\cdots,\alpha_{-1},\alpha_0,\alpha_1,\cdots)$ of real numbers into the real line:

$$f_n(\cdots,\alpha_{-1},\alpha_0,\alpha_1,\cdots)\in R^1.$$

Define random variables

$$\eta_{n,j}=f_n(\cdots,\xi_{j-1},\xi_j,\xi_{j+1},\cdots), \tag{2.4}$$

where $\xi_j$ occupies the 0-th place in the argument of $f_n$. Then, for each $n$, the sequence $\{\eta_{n,j};\ j=0,\pm1,\pm2,\cdots\}$ is strictly stationary, since $\{\xi_n\}$ is strictly stationary.

For each $h$, define random variables

$$\eta_{n,h,j}=E\{\eta_{n,j}\,||\,\mathcal{M}_{j-h}^{j+h}\}, \tag{2.5}$$

where $\mathcal{M}_{j-h}^{j+h}$ is the $\sigma$-field generated by $\xi_{j-h},\cdots,\xi_{j+h}$. Then, for each $h$, the sequence $\{\eta_{n,h,j};\ j=0,\pm1,\pm2,\cdots\}$ is strictly stationary and $\varphi^{(h)}$-mixing, where

$$\varphi^{(h)}(j)=\begin{cases}1, & \text{if } j\leq 2h\\ \varphi(j-2h), & \text{if } j>2h,\end{cases}$$

and hence we can apply Theorem 1 to the sequence $\{\eta_{n,h,j};\ n=1,2,\cdots,j=0,\pm1,\pm2,\cdots\}$. We shall obtain limit theorems for $\{\eta_{n,j};\ n=1,2,\cdots,j=0,\pm1,\pm2,\cdots\}$ under the assumption that $\{\eta_{n,j}\}$ can be closely approximated by $\{\eta_{n,h,j}\}$. Therefore, we shall assume for the functions $f_n$ and the process $\{\xi_n\}$ that $E\{\eta_{n,0}\}=0$ for each $n$, and that the $\eta_{n,0}^2$ are uniformly integrable. We shall assume further that if,

$$\nu_n(h)=E\{|\eta_{n,0}-\eta_{n,h,0}|^2\}$$
$$\psi_{n,k}=\sum_{h=1}^{k}\nu_n^{1/2}(h), \tag{2.6}$$

then there exists a $\lim_{n,k\to\infty}\psi_{n,k}(=\psi)$.

Write

$$S_{n,j}=\eta_{n,1}+\cdots+\eta_{n,j}$$

and define $X_n$ by, for $0\leq t\leq 1$,

$$X_n(t,\omega)=1/\sigma\sqrt{k_n}\cdot S_{n,[k_nt]}(\omega). \tag{2.7}$$

Furthermore, define $\delta_{n,h,j}=\eta_{n,j}-\eta_{n,h,j}$. Then, we have the following theorem which is simple generalization of Theorem 21.1 of BILLINGSLEY (1968).

THEOREM 2. Suppose that the following conditions hold:

(i) $\{\xi_n;\ n=0,\pm1,\cdots\}$ is strictly stationary and $\varphi$-mixing with $\sum_{n=0}^{\infty}\varphi^{1/2}(n)<\infty$,

(ii) the $\eta_{n,j}$ defined by (2.4) have mean 0 and the $\eta_{n,0}^2$ are uniformly integrable,

(iii) there exists a $\lim_{n,k\to\infty}\psi_{n,k}(=\psi)$, where $\psi_{n,k}=\sum_{h=1}^{k}\nu_n^{1/2}(h)$,

(iv) there exists a $\lim_{n,k\to\infty}\sigma_{n,k}^2(=\sigma^2)$, where $\sigma_{n,k}^2=E\{\eta_{n,0}^2\}+2\sum_{j=1}^{k}E\{\eta_{n,0}\eta_{n,j}\}$,

(v) there exists a $\lim_{n,k\to\infty}\sigma_{n,h,k}^2(=\sigma_h^2)$ uniformly with respect to $h$, where $\sigma_{n,h,k}^2=E\{\eta_{n,h,0}^2\}+2\sum_{j=1}^{k}E\{\eta_{n,h,0}\eta_{n,h,j}\}$,

(vi) there exists a $\lim_{n,k\to\infty}\tau^2_{n,h,k}(=\tau^2_h)$ uniformly with respect to $h$, where $\tau^2_{n,h,k}=E\{\delta^2_{n,h,0}\}+2\sum_{j=1}^{k}E\{\delta_{n,h,0}\delta_{n,h,j}\}$.

If $\sigma^2>0$ and $X_n$ is defined by (2.7) for which $k_n$ are positive integers going to infinity, then

$$X_n\xrightarrow{D}W. \tag{2.8}$$

## 3. Proof of Theorem 1

In order to prove Theorem 1, it is necessary to first state several definitions and lemmas. At first, we shall state the theorem characterizing Wiener process which is due to ROSEN (1967). Let $X_n$ be a random element of $D[0,1]$. We say $X_n$ has asymptotically independent increments if

$$0 \le s_1 \le t_1 < s_2 \le t_2 < \cdots < s_r \le t_r \le 1$$

implies, for all linear Borel sets $H_1, \cdots, H_r$, that the difference

$$P\{X_n(t_i) - X_n(s_i) \in H_i, i=1, \cdots, r\} - \prod_{i=1}^{r} P\{X_n(t_i) - X_n(s_i) \in H_i\}$$

converges 0 as $n \to \infty$. The modulus of continuity of $X_n$ is defined by

$$w(X_n, \delta) = \sup_{|t-s|<\delta} |X_n(t) - X_n(s)|, \qquad 0 < \delta \le 1.$$

Thereupon, Rosen proved the following result.

LEMMA 1 (ROSEN 1967). Suppose that $X_n$ has asymptotically independent increments, that $\{X_n^2(t); n \ge 1\}$ is uniformly integrable for each $t$, and that $E\{X_n(t)\} \to 0$ and $E\{X_n^2(t)\} \to t$ as $n \to \infty$. Suppose finally that, for each positive $\varepsilon$ and $\eta$, there exists a positive $\delta$ such that

$$P\{w(X_n, \delta) \ge \varepsilon\} \le \eta \tag{3.1}$$

for all sufficiently large $n$. Then $X_n \xrightarrow{D} W$.

Next, we shall state several properties for $\varphi$-mixing. In all that follows, $\{\xi_n;\ n=0, \pm 1, \pm 2, \cdots\}$ is assumed strictly stationary and $\varphi$-mixing unless the contrary is explicitly stated, and $\mathcal{M}_a^b$, $\mathcal{M}_{-\infty}^a$, $\mathcal{M}_a^\infty$ are $\sigma$-fields defined in § 2. For the proof, see IBRAGIMOV and LINNIK (1971), BILLINGSLEY (1968), for example.

LEMMA 2. If $\xi$ is measurable $\mathcal{M}_{-\infty}^k$ and $\eta$ is measurable $\mathcal{M}_{k+n}^\infty (n \ge 0)$, then

$$E\{|\xi|^r\} < \infty, \quad E\{|\eta|^s\} < \infty, \quad r, s > 1, \ 1/r + 1/s = 1,$$

implies

$$|E\{\xi\eta\} - E\{\xi\}E\{\eta\}| \le 2\varphi^{1/r}(n) E^{1/r}\{|\xi|^r\} E^{1/s}\{|\eta|^s\}.$$

LEMMA 3. If $\xi$ is measurable $\mathcal{M}_{-\infty}^k$ and $|\xi| \le C_1$, and if $\eta$ is measurable $\mathcal{M}_{k+n}^\infty$ $(n \ge 0)$ and $|\eta| \le C_2$, then

$$|E\{\xi\eta\} - E\{\xi\}E\{\eta\}| \le 2C_1C_2\varphi(n).$$

Let

$$S_n = \xi_1 + \cdots + \xi_n$$

and $S_0 = 0$. In order to prove the next lemma, it is sufficient to only assume that $\{\xi_n\}$ is weakly stationary.

LEMMA 4. Suppose that $\sum_{k=0}^{\infty} E|\{\xi_0 \xi_k\}| < \infty$. Then,

$$1/n \cdot E\{S_n^2\} \longrightarrow \sigma^2$$

where $\sigma^2=E\{\xi_0^2\}+2\sum_{k=1}^{\infty}E\{\xi_0\xi_k\}$.

Suppose that, for each $n$, $\{\xi_{n,j};\ j=0,\pm1,\pm2,\cdots\}$ is strictly stationary and $\varphi_n$-mixing. Let

$$S_{n,j}=\xi_{n,1}+\cdots+\xi_{n,j}$$

and $S_{n,0}=0$ for each $n$.

LEMMA 5. If the $\xi_{n,0}$, $n=1,2,\cdots$, are uniformly bounded by $C$ and $E\{\xi_{n,0}\}=0$ for each $n$, then

$$E\{S^4_{n,j}\}\leq 768C^4j^2\left[\sum_{i=0}^{j}\varphi_n^{1/2}(i)\right]^2.$$

The following lemma extends lemma 4 to double sequence. In order to prove this lemma, it is sufficient to assume that, for each $n$, $\{\xi_{n,j}\}$ is weakly stationary.

LEMMA 4′. If $\sup_n E\{\xi^2_{n,0}\}<\infty$, and if there exists a $\lim_{n,k\to\infty}\sigma^2_{n,k}(=\sigma^2)$, where $\sigma^2_{n,k}=E\{\xi^2_{n,0}\}+2\sum_{j=1}^{k}E\{\xi_{n,0}\xi_{n,j}\}$, then, for any positive integers $k_n$, $n=1,2,\cdots$, going to infinity as $n\to\infty$,

$$1/k_nE\{S^2_{n,k_n}\}\longrightarrow\sigma^2.$$

Now, we shall prove Theorem 1 by the aid of the above lemmas. First of all, we shall prove that

$$\lim_{\alpha\to\infty}\sup_{n,j\geqq n_0}E_\alpha\{1/j\cdot S^2_{n,j}\}=0,\tag{3.2}$$

where $n_0$ is a positive number for $\varepsilon>0$ determined by the convergency of condition (ii) and

$$E_\alpha\{X\}=\int_{\{X\geqq\alpha\}}XdP.$$

Define, for real number $x$ and positive number $u$,

$$f_u(x)=\begin{cases}x, & \text{if } |x|\leq u,\\ 0, & \text{if } |x|>u,\end{cases}\qquad g_u(x)=\begin{cases}0, & \text{if } |x|\leq u,\\ x, & \text{if } |x|>u,\end{cases}$$

and put

$$\bar f_u(x)=f_u(x)-E\{f_u\{\xi_{n,0})\},\ \bar g_u(x)=g_u(x)-E\{g_u(\xi_{n,0})\}.$$

Then $x=f_u(x)+g_u(x)=\bar f_u(x)+\bar g_u(x)$, so that, if $S_{n,j,u}=\sum_{i=1}^{j}\bar f_u(\xi_{n,i})$, $D_{n,j,u}=\sum_{i=1}^{j}\bar g_u(\xi_{n,i})$, then $S_{n,j}=S_{n,j,u}+D_{n,j,u}$ and hence

$$S^2_{n,j}\leq 2S^2_{n,j,u}+2D^2_{n,j,u}.\tag{3.3}$$

Since $\bar f_u(\xi_{n,0})$ is bounded by $2u$ and $\{\bar f_u(\xi_{n,j})\}$ is strictly stationary and $\varphi_n$-mixing, it follows by lemma 5 that

$$E_\alpha\{1/j\cdot S^2_{n,j,u}\}\leq 1/\alpha\cdot\phi_{n,j}\cdot(2u)^4, \tag{3.4}$$

where $\phi_{n,j}=768\left[\sum_{i=0}^{j}\varphi_n^{1/2}(\mathrm{i})\right]^2$. By lemma 2,

$$E\{\bar{g}_u(\xi_{n,0})\bar{g}_u(\xi_{n,k})\}\leq 2\varphi_n^{1/2}(k)E_{u^2}\{\xi^2{}_{n,0}\},$$

and it follows by lemma 4 that

$$E\{1/j\cdot D^2_{n,j,u}\}\leq 4\left[\sum_{k=0}^{j}\varphi_n^{1/2}(k)\right]\cdot E_{u^2}\{\xi^2{}_{n,0}\}. \tag{3.5}$$

From (3.3), the relation $E_\alpha\{U+V\}\leq 2E_{\alpha/2}\{U\}+2E\{V\}$, (3.4) and (3.5), we obtain

$$E_\alpha\{1/j\cdot S^2_{n,j}\}\leq 2^4\cdot\phi_{n,j}[u^4/\alpha+E_{u^2}\{\xi^2{}_{n,0}\}].$$

Since there exists a constant $\phi$ such that $\sup_{n,j\geqq n_0}\phi_{n,j}\leq\phi$ from condition (ii) and the $\xi^2{}_{n,0}$ are uniformly integrable, therefore, we obtain (3.2).

We shall show that $X_n$ defined by (2.3) is satisfied with all the conditions of lemma 1 by the aid of (3.2). We shall first prove that $X_n$ has asymptotically independent increments. Suppose that $s_i$ and $t_i$ are real numbers with $0\leq s_1\leq t_1<s_2\leq t_2<\cdots<s_r\leq t_r\leq 1$ and let $E_{n,i}$ be the event $\{X_n(t_i)-X_n(s_i)\in H_i\}$, where $H_i$, $i=1,2,\cdots,r$, are any linear Borel sets. Then $E_{n,i}$ lies in ${}_n\mathcal{M}^{[k_nt_i]}_{[k_ns_i]+1}$ and, if $\delta$ is the smallest difference $s_i-t_{i-1}$, then $[k_ns_i]+1-[k_nt_{i-1}]\geq[k_n\delta]$, so that, by the definition of $\varphi$-mixing, we have

$$|P\{X_n(t_i)-X_n(s_i)\in H_i,\ i=1,\cdots,r\}$$
$$-\prod_{i=1}^{r}P\{X_n(t_i)-X_n(s_i)\in H_i\}|\leq r\varphi_n([k_n\delta]).$$

Since $\delta$ is positive, therefore, $X_n$ does have asymptotically independent increments.

It is briefly proved by (3.2) that $\{X_n{}^2(t);\ n\geq 1\}$ is uniformly integrable for each $t$. Certainly, $E\{X_n(t)\}=0$, and lemma 4′ implies that $E\{X_n{}^2(t)\}\to t$. By stationarity, (3.1) will follow if we prove that, for each positive $\varepsilon$, there exist a $\lambda_0(\varepsilon)$, $\lambda_0(\varepsilon)>\sigma$, and an integer $m_0(\varepsilon)$ such that $n$, $m\geq m_0(\varepsilon)$ implies

$$P\left\{\max_{i\leq m}|S_{n,i}|\geq\lambda_0\sqrt{m}\right\}\leq\varepsilon/\lambda_0^2. \tag{3.6}$$

Let us set

$$S^*_{n,j}=\sum_{i=1}^{j}|\xi_{n,i}|.$$

Since $\{\xi^2{}_{n,0};\ n\geq 1\}$ is uniformly integrable, there exists an increasing sequence of integers $m_i$ independently with respect to $n$ such that, for each $n$ and each positive $\lambda$, $m\geq m_i$ implies

$$P\{|\xi_{n,0}|\geq\lambda\sqrt{m/i}\}\leq 1/\lambda^2mi^2.$$

If we define $p_m=i$ for $m_i\leq m<m_{i+1}$ (and $p_m=1$ for $m<m_1$), then $p_m$ goes to infinity but so slowly that

$$\lim_{m\to\infty}\ \sup_{n}\ mP\{S^*_{n,p_m}\geq\lambda\sqrt{m}\}=0 \tag{3.7}$$

for each positive $\lambda$. We may at the same time choose $p_m$ in a such way that $p_m \leq m$.

From (3.2), there exists a $\lambda_0(\varepsilon)$ for each positive $\varepsilon$ so that $\lambda_0>\sigma$ and so that

$$P\{|S_{n,j}|>\lambda_0\sqrt{j}\}<\varepsilon/\lambda_0^2$$

for $n,\ j\geq n_0(\varepsilon)$. If

$$E_{n,i}=\left\{\max_{j<i}|S_{n,j}|<3\lambda_0\sqrt{m}\leq|S_{n,i}|\right\}$$

then

$$P\left\{\max_{i\leqq m}|S_{n,i}|\geq 3\lambda_0\sqrt{m}\right\}$$

$$\leqslant P\left\{|S_{n,m}|\geq\lambda_0\sqrt{m}\right\}+\sum_{i=0}^{m-1}P\{E_{n,i}\cap[|S_{n,m}-S_{n,i}|\geq 2\lambda_0\sqrt{m}[\right\}.$$

With $p=p_m$, the sum here is at most

$$\sum_{i=1}^{m-p-1}P\{|S_{n,i}-S_{n,i+p}|\geq\lambda_0\sqrt{m}\}$$

$$+\sum_{i=1}^{m-p-1}P\{E_{n,i}\cap[|S_{n,m}-S_{n,i+p}|\geq\lambda_0\sqrt{m}]\right\}$$

$$+\sum_{i=m-p}^{m-1}P\{|S_{n,m}-S_{n,i}|\geq\lambda_0\sqrt{m}\}.$$

Each term in the first and third of these sums is at most $P\{S^*_{n,p}\geq\lambda_0\sqrt{m}\}$, and we can estimate the second sum by using the fact that $E_{n,i}\in{}_n\mathcal{M}^i_{-\infty}$:

$$P\left\{\max_{i\leqq m}|S_{n,i}|\geq 3\lambda_0\sqrt{m}\right\}$$

$$\leq P\{|S_{n,m}|\geq\lambda_0\sqrt{m}\}+(m-1)P\{S^*_{n,p}\geq\lambda_0\sqrt{m}\}$$

$$+\sum_{i=1}^{m-p-1}P(E_{n,i})[P\{|S_{n,m}-S_{n,i+p}|\geq\lambda_0\sqrt{m}\}+\varphi_n(p)].$$

And now (3.8) and the fact that the $E_{n,i}$ are disjoint yield that, for $n,\ m\geq n_0(\varepsilon)$,

$$P\left\{\max_{i\leqq m}|S_{n,i}|\geq 3\lambda_0\sqrt{m}\right\}$$

$$\leq 2\varepsilon/\lambda_0^2+mP\{S^*_{n,p}\geq\lambda_0\sqrt{m}\}+\varphi_n(p)$$

$$+\sum_{i=1}^{n_0(\varepsilon)}P(E_{n,m-i-p})P\{|S_{n,i}|\geq\lambda_0\sqrt{m}\}.$$

Since

$$\sum_{i=1}^{n_0(\varepsilon)} P(E_{n,m-i-p})P\{|S_{n,i}|\geq\lambda_0\sqrt{m}\}\leq n_0(\varepsilon)P\{|\xi_{n,0}|\geq\lambda_0\sqrt{m/n_0}\}$$

and $\{\xi^2_{n,0};\ n\geq 1\}$ is uniformly integrable, there exists a $m_1(\varepsilon)>0$ for each positive $\varepsilon$ such that $m\geq m_1(\varepsilon)$ implies

$$\sum_{i=1}^{n_0(\varepsilon)} P(E_{n,m-i-p})P\{|S_{n,i}|\geq\lambda_0\sqrt{m}\}\leqslant\varepsilon/\lambda_0^2$$

for each $n$, so that, from (3.7), condition (ii) and the fact that $p_m\to\infty$, it follows that the relation (3.6) holds. Therefore, this completes the proof of Theorem 1.

## 4. Proof of Theorem 2

In order to prove Theorem 2, it is necessary to first state several definitions and lemmas. At first, we shall show that a double sequence $\{X_{n,k}\}$ of random variables satisfies the analogous property for Theorem 5.4 of BILLINGSLEY (1968).

Let $S$ be a metric space and let $\zeta$ be a $\sigma$-field generated by the open sets in $S$. We shall say that, for a double sequence $\{P_{n,k}\}$ of probability measures on $(S,\zeta)$, $P_{n,k}$ converges weakly to $P$ and write $P_{n,k}\Rightarrow P$ if such probability measures $P_{n,k}$ and $P$ satisfy $\int_S fdP_{n,k}\to\int_S fdP$ for every bounded, continuous real function $f$ on $S$, i.e. if there exists a $n_0(\varepsilon)$, $n_0(\varepsilon)>0$, for each positive $\varepsilon$, such that $n,k\geq n_0(\varepsilon)$ implies

$$\left|\int_S fdP_{n,k}-\int_S fdP\right|<\varepsilon$$

for every bounded, continuous real function $f$ on $S$. Furthermore, we say a sequence $\{X_{n,k}\}$ of random elements on $S$ converges in distribution to the random element $X$, and we write $X_{n,k}\xrightarrow{D}X$, if the distributions $P_{n,k}$ of the $X_{n,k}$ converge weakly to the distribution $P$ of $X$, i.e. $P_{n,k}\Rightarrow P$. The random variables $X_{n,k}$ are said to be uniformly integrable if

$$\lim_{\alpha\to\infty}\sup_{n,k}\int_{\{|X_{n,k}|\geqq\alpha\}}|X_{n,k}|dP=0.$$

Then, we have the same property as Theorem 5.4 of BILLINGSLEY (1968).

LEMMA 6. Suppose $X_{n,k}\xrightarrow{D}X$. If the $X_{n,k}$ are uniformly integrable, then

$$\lim_{n,k\to\infty}E\{X_{n,k}\}=E\{X\};\tag{4.1}$$

if $X$ and the $X_{n,k}$ are nonnegative and integrable, then (4.1) implies that there exist a $n_0(\varepsilon)>0$ and $\alpha_0(\varepsilon)>0$ such that $\alpha\geq\alpha_0$ implies

$$\sup_{n,k\geqq n_0} \int_{\{|X_{n,k}|\geqq\alpha\}} |X_{n,k}|dP<\varepsilon.$$

The proof is completely analogous to that of Theorem 5.4 of BILLINGSLEY (1968) and therefore omitted.

The following lemma implies that, if $\eta_{n,h,j}$ is given by (2.9), then $\nu_n(h)$ is a non-increasing function of $h$ for each $n$.

LEMMA 7. Let $\mathcal{F}$ and $\mathcal{G}$ be $\sigma$-fields with $\mathcal{F}\subset\mathcal{G}$. If $E\{\xi^2\}<\infty$, then

$$E\{|\xi-E\{\xi||\mathcal{G}\}|^2\}\leqslant E\{|\xi-E\{\xi||\mathcal{F}\}|^2\}.$$

Now, we shall prove Theorem 2 by using the above lemmas and Theorem 1. We first prove

$$1/\sqrt{k_n}\cdot S_{n,k_n} \xrightarrow{D} N(0,\sigma^2). \tag{4.2}$$

Put

$$\delta_{n,h,j}=\eta_{n,j}-\eta_{n,h,j} \tag{4.3}$$

so that $\nu_n(h)=E\{\delta^2_{n,h,j}\}$ and $E\{\delta_{n,h,j}\}=E\{\eta_{n,h,j}\}=0$. We have

$$1/\sqrt{k_n}\cdot S_{n,k_n}=1/\sqrt{k_n}\cdot\sum_{j=1}^{k_n}\eta_{n,h,j}+1/\sqrt{k_n}\cdot\sum_{j=1}^{k_n}\delta_{n,h,j}, \tag{4.4}$$

and the idea in proving (4.2) is to show, by using Theorem 1, that the first sum on the right in (4.4) is approximately normal for large $n$ and then to show that the second sum is small for large $h$.

From conditions (iv) and (v), there exists a $n_0(\varepsilon)>0$ independently with respect to $h$ for each positive $\varepsilon$ such that $n$, $k\geq n_0(\varepsilon)$ implies

$$|\sigma^2_{n,k}-\sigma^2|<\varepsilon,\ \ |\sigma^2_{n,h,k}-\sigma^2_h|<\varepsilon$$

for each $h$, and from (iii), there exists a $h_0(\varepsilon)$, $h_0(\varepsilon)\geq n_0(\varepsilon)$, such that $n$, $h\geq h_0(\varepsilon)$ implies

$$\nu_n^{1/2}(h)<\varepsilon/\{n_0(\varepsilon)+1\}.$$

Furthermore, we have for each $j$ from the uniformly integrability $\{\eta^2_{n,0};\ n\geq1\}$

$$|E\{\eta_{n,0}\eta_{n,j}\}-E\{\eta_{n,h,0}\eta_{n,h,j}\}|\leq 2C\nu_n^{1/2}(h)$$

for some constant $C$. Therefore, we have that $h\geq h_0(\varepsilon)$ implies

$$|\sigma^2_h-\sigma^2|<2\varepsilon+4C,$$

i.e.

$$\lim_{h\to\infty}\sigma^2_h=\sigma^2. \tag{4.5}$$

Since $\{\eta_{n,h,j};\ n=1,2,\cdots,j=0,\pm1,\pm2,\cdots\}$ satisfies all the conditions of Theorem 1 for each $h$, we have

$$1/\sqrt{k_n}\cdot\sum_{j=1}^{k_n}\eta_{n,h,j} \xrightarrow{D} N(0,\sigma^2_h). \tag{4.6}$$

If $\sigma^2_h=0$, (4.6) is the same as $1/\sqrt{k_n}\cdot\sum_{j=1}^{k_n}\eta_{n,h,j}\xrightarrow{P}0$, which is satisfied by lemma 4′.

Now $N(0,\sigma^2_h)\xrightarrow{D}N(0,\sigma^2)$ by (4.5); because of (4.4) and (4.6), the relation (4.2) will follow by Theorem 4.2 of BILLINGSLEY (1968) if we show that

$$\lim_{h\to\infty}\limsup_{n\to\infty}P\left\{\left|1/\sqrt{k_n}\cdot\sum_{j=1}^{k_n}\delta_{n,h,j}\right|\geq\varepsilon\right\}=0 \tag{4.7}$$

for each positive $\varepsilon$.

From conditions (iii) and (iv), we have by the same way as proving (4.5),

$$\lim_{h\to\infty}\tau^2_h=0, \tag{4.8}$$

and $E\{\delta^2_{n,h,0}\}\leqslant E\{\eta^2_{n,0}\}$ by lemma 7. By lemma 4′ applied to $\{\delta_{n,h,\cdot,j}\}$,

$$\lim_{n\to\infty}1/k_n\cdot E\left\{\left|\sum_{j=1}^{kn}\delta_{n,h,j}\right|^2\right\}=\tau^2_h \tag{4.9}$$

for each $h$. Chebyshev's inequality, (4.8) and (4.9) now yield (4.7), which completes the proof of (4.2). If $\sigma^2=0$, (4.2) is the same as $1/\sqrt{k_n}\cdot S_{n,k_n}\xrightarrow{P}0$, but we have by lemma 4′ that

$$1/k_n\cdot E\{S^2_{n,k_n}\}\longrightarrow\sigma^2, \tag{4.10}$$

so that (4.2) is satisfied even if $\sigma^2=0$. From now on we assume that $\sigma^2>0$.

To prove (2.12), we establish the convergence of the finite-dimensional distributions and then tightness. Let $p_n$ be positive integers going to infinity at a rate to be specified later and define

$$U_{n,i}=E\{S_{n,i-2p_n}||\mathcal{M}_{-\infty}^{i-p_n}\} \tag{4.11}$$

$$V_{n,i}=E\{S_{n,k_n}-S_{n,i+2p_n}||\mathcal{M}_{i+p_n}^{\infty}\}. \tag{4.12}$$

In these definitions, we adopt the conventions that $S_{n,i-2p_n}=0$ if $i<2p_n$ and $S_{n,k_n}-S_{n,i+2p_n}=0$ if $k_n<i+2p_n$. We shall often write $p$ in place of $p_n$.

By Minkowski's inequality and lemma 7,

$$E^{1/2}\{|S_{n,k}-E\{S_{n,k}||\mathcal{M}_{-\infty}^{k+p}\}|^2\}\leq\sum_{j=p}^{k+p}\nu_n^{1/2}(j). \tag{4.13}$$

Then, it follows that

$$E\{|U_{n,i}-S_{n,i}|^2\}\leq 2E\{S^2_{n,2p}\}+2\left[\sum_{j=p}^{kn-i-p}\nu_n^{1/2}(j)\right]^2, \tag{4.14}$$

for all $i$. In the same way we obtain

$$E\{|V_{n,i}-(S_{n,k_n}-S_{n,i})|^2\}\leq 2E\{S^2_{n,2p}\}+2\left[\sum_{j=p}^{kn-i-p}\nu_n^{1/2}(j)\right]^2. \tag{4.15}$$

Since $U_{n,i}$ and $V_{n,i}$ are measurable $\mathcal{M}_{-\infty}^{i-p}$ and $\mathcal{M}_{i+p}^{\infty}$ respectively,

$$|P\{U_{n,i}\in H_1,\ V_{n,i}\in H_2\}-P\{U_{n,i}\in H_1\}\cdot P\{V_{n,i}\in H_2\}|\leq\varphi(2p) \tag{4.16}$$

for all linear Borel sets $H_1$ and $H_2$.

Consider now the finite-dimensional distributions. From (4.2), it follows that

$$X_n(t)-X_n(s) \xrightarrow{D} W_t - W_s. \tag{4.17}$$

We shall prove that

$$(X_n(t), X_n(1)-X_n(t)) \xrightarrow{D} (W_t, W_1 - W_t); \tag{4.18}$$

the argument is easily adapted to cases of dimension exceeding 2. Let $p_n$ go to infinity slowly enongh that $p_n/k_n \to 0$. By (4.14), Chebyshev's inequality, (4.10) and condition (iii), we have

$$1/\sigma\sqrt{k_n}\cdot U_{n,[k_n t]}-X_n(t) \xrightarrow{P} 0. \tag{4.19}$$

Similarly,

$$1/\sigma\sqrt{k_n}\cdot V_{n,[k_n t]}-(X_n(1)-X_n(t)) \xrightarrow{P} 0, \tag{4.20}$$

and (4.18) will follow by Theorem 4.1 of BILLINGSLEY (1968) if

$$(1/\sigma\sqrt{k_n}\cdot U_{n,[k_n t]}, 1/\sigma\sqrt{k_n}\cdot V_{n,[k_n t]}) \xrightarrow{D} (W_t, W_1 - W_t).$$

Because of (4.16), it is enough to show that there is convergence in distribution in each of the two coordinates here; but this follows from (4.17) by (4.19) and (4.20).

We turn now to the question of tightness. The tightness of $\{X_n(t); n\geq 1\}$ follows by Theorem 15.5 of BILLINGSLEY (1968) if we prove that there exist a $\lambda_0(\varepsilon)$, $\lambda_0(\varepsilon)>\sigma$, and an integer $m_0(\varepsilon)$ for each positive $\varepsilon$ such that $n, m\geq m_0(\varepsilon)$ implies

$$P\{\max_{i\leq m}|S_{n,i}|\geq \lambda_0\sqrt{m}\}\leq \varepsilon/\lambda_0^2. \tag{4.20}$$

Put $S^*_{n,j}=\sum_{i=1}^{j}|\eta_{n,i}|$. By the argument leading to (3.7), there is a sequence $p_m$ going to infinity so slowly that, if

$$\beta_{n,m}(\lambda)=P\{S^*_{n,2p_m}\geq \lambda\sqrt{m}\}, \tag{4.22}$$

then

$$\lim_{m\to\infty}\sup_n m\beta_{n,m}(\lambda)=0 \tag{4.23}$$

for each positive $\lambda$. Define

$$U_{n,m,i}=E\{S_{n,i-2p_m}||\mathcal{M}_{-\infty}^{i-p_m}\} \tag{4.11$'$}$$

$$V_{n,m,i}=E\{S_{n,m}-S_{n,i+2p_m}||\mathcal{M}_{i+p_m}^{\infty}\} \tag{4.12$'$}$$

then we have, by (4.13) and Chebyshev's inequality,

$$P\{|S_{n,i}-U_{n,m,i}|\geq\lambda\sqrt{m}\}\leqslant\beta_{n,m}(\lambda/2)+4/\lambda^2 m\cdot\left[\sum_{j=p_m}^{i-p_m}\nu_n^{1/2}(j)\right]^2, \tag{4.24}$$

and similarly

$$P\{|(S_{n,m}-S_{n,i})-V_{n,m,i}|\geq\lambda\sqrt{m}\}\leq\beta_{n,m}(\lambda/2)+4/\lambda^2m\cdot\left[\sum_{j=p_m}^{m-i-p_m}\nu_n^{1/2}(j)\right]^2. \tag{4.25}$$

Now, we shall prove that

$$1/\sqrt{k}\cdot S_{n,k}\xrightarrow{D}N(0,\sigma^2). \tag{4.26}$$

Since (4.6) holds for any $k_n$ going to infinity, it follows that

$$1/\sqrt{k}\cdot\sum_{j=1}^{k}\eta_{n,h,j}\xrightarrow{D}N(0,\sigma^2{}_h)$$

for each $h$ and $N(0,\sigma_h^2)\xrightarrow{D}N(0,\sigma^2)$ by (4.5), so that, (4.26) will follow by the version of Theorem 4.2 of BILLINGSLEY (1968) if we show that

$$\lim_{h\to\infty}\limsup_{n,k\to\infty}P\left\{\left|1/\sqrt{k}\cdot\sum_{j=1}^{k}\delta_{n,h,j}\right|\geq\varepsilon\right\}=0 \tag{4.27}$$

for each positive $\varepsilon$. Since (4.9) holds for any $k_n$ going to infinity, it follows that, for each $h$,

$$\lim_{n,k\to\infty}1/k\cdot E\left\{\left|\sum_{j=1}^{k}\delta_{n,h,j}\right|^2\right\}=\tau^2{}_h, \tag{4.28}$$

so that, Chebyshev's inequality and (4.8) yield (4.27) which completes the proof of (4.26). Furthermore, we have

$$\lim_{n,k\to\infty}1/k\cdot E\{S^2{}_{n,k}\}=\sigma^2,$$

so that, it follows via lemma 6 that there exist a $\lambda_0(\varepsilon)$, $\lambda_0(\varepsilon)>\sigma$, and an integer $n_0(\varepsilon)$ for each positive $\varepsilon$ such that $n,\ j\geq n_0(\varepsilon)$ implies

$$P\{|S_{n,j}|\geq\lambda_0\sqrt{j}\}\leq\varepsilon/\lambda_0^2.$$

By (4.24),

$$P\{\max_{i\leq m}|S_{n,i}|\geq6\lambda_0\sqrt{m}\}\leq P\{\max_{i\leq m}|U_{n,m,i}|\geq5\lambda_0\sqrt{m}\}$$

$$+m\beta_{n,m}(\lambda_0/2)+4/\lambda_0^2\cdot\left[\sum_{j=p_m}^{m}\nu_n^{1/2}(j)\right]^2. \tag{4.30}$$

Consider the sets

$$E_{n,m,i}=\{\max_{i\leq m}|U_{n,m,j}|<5\lambda_0\sqrt{m}\leq|U_{n,m,i}|\}.$$

We have

$$P\{\max_{i\leq m}|U_{n,m,i}|\geq5\lambda_0\sqrt{m}\}$$

$$\leq P\{|S_{n,m}|\geq\lambda_0\sqrt{m}\}+\sum_{i=1}^{m}P(E_{n,m,i}\cap\{|S_{n,m}-U_{n,m,i}|\geq4\lambda_0\sqrt{m}\}). \tag{4.31}$$

The $i$-th summand in (4.31) is at most

$$P\{|S_{n,m}-S_{n,i}-V_{n,m,i}|\geq\lambda_0\sqrt{m}\}$$

$$+P(E_{n,m,i}\cap\{|V_{n,m,i}|\geq 2\lambda_0\sqrt{m}\})$$

$$+P\{|S_{n,i}-U_{n,m,i}|\geq\lambda_0\sqrt{m}\}.$$

Since $E_{n,m,i}$ lies in $\mathcal{M}_{-\infty}^{i-P_m}$ and $V_{n,m,i}$ is measurable $\mathcal{M}_{i+P_m}^{\infty}$ we have

$$P(E_{n,m,i}\cap\{|V_{n,m,i}|\geq 2\lambda_0\sqrt{m}\})$$

$$\leq\beta_{n,m}(\lambda_0/2)+4/\lambda_0^2 m\cdot\left[\sum_{j=P_m}^{m}\nu_n^{1/2}(j)\right]^2+P(E_{n,m,i})\varphi(2p_m)$$

$$+P(E_{n,m,i})P\{|S_{n,m-i}|\geq\lambda_0\sqrt{m}\}$$

Using this estimate for the middle term in (4.32) and the estimates (4.24) and (4.25) for the other two, we see that the $i$-th summand in (4.23) is at most

$$12/\lambda_2^0 m\cdot\left[\sum_{j=p_m}^{m}\nu_n^{1/2}(j)\right]^2+3\beta_{n,m}(\lambda_0/2)+P(E_{n,m,i})\varphi(2p_m)$$

$$+P(E_{n,m,i})P\{|_{n,m-i}|\geq\lambda_0\sqrt{m}\}.$$

Therefore, by (4.29) and the disjointness of the $E_{n,m,i}$, it follows that $n,m\geq n_0(\varepsilon)$ implies

$$P\left\{\max_{i\leq m}|S_{n,i}|\geq 6\lambda_0\sqrt{m}\right\}$$

$$=2\varepsilon/\lambda_0^2+16/\lambda_0^2\cdot\left[\sum_{j=p_m}^{m}\nu_n^{1/2}(j)\right]^2+4m\beta_{n,m}(\lambda_0/2)+\varphi(2p_m)$$

$$+\sum_{i=m-n_0+1}^{m}P(E_{n,m,i})P\{|S_{n,m-i}|\geq\lambda_0\sqrt{m}\}.$$

By the same argument as the one of Theorem 1, it follows from the uniformly integrability of $\{\eta^2_{n,0};\ n\geq 1\}$ that there exists an integer $m_1(\varepsilon)$ such that $m\geq m_1(\varepsilon)$ implies

$$\sum_{i=m-n_0(\varepsilon)+1}^{m}P(E_{n,m,i})P\{|S_{n,m-i}|\geq\lambda_0\sqrt{m}\}=\varepsilon/\lambda_0^2$$

for each $n$. Then, from conditions (i), (iii) and (4.23), we conclude that there exists an integer $m_0(\varepsilon)$ such that $n,m\geq m_0(\varepsilon)$ implies

$$P\left\{\max_{i\leq m}|S_{n,i}|\geq 6\lambda_0\sqrt{m}\right\}=4\varepsilon/\lambda_0^2.$$

This completes the proof of Theorem 2.

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