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BY

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A NOTE ON A RENEWAL REWARD PROCESS

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ABSTRACT

In this note, a renewal reward process is dealt with, and the asymptotic behavior of the expected total reward when time tends to infinity is studied in the case of independent and non-identically distributed random variables.

1. Let $\{X_i, i=1, 2, \cdots\}$ and $\{Y_i, i=1, 2, \cdots\}$ be sequences of independent, non-negative random variables with $0<EX_i=\mu_i<\infty$ and $0<EY_i=\lambda_i<\infty$, respectively. We shall consider the renewal process with the time interval X_i between the $(i-1)$-st and the i-th renewals, and Y_i is supposed to be a reward at the time of the i-th renewal. Y_i may depend on X_i, but we assume that the pairs (X_i, Y_i), $i=1, 2, \cdots$ are independent of each other.

Set $S_0=0$ and $S_n=\sum_{i=1}^{n} X_i, n\geqq 1$, and define $N(t)=\sup\{n ; S_n\leqq t\}$. Then if we let $Y(t)=\sum_{n=1}^{N(t)} Y_n$, then $Y(t)$ denotes the total reward earned by the time t. We call the process $Y(t)$ a renewal reward process. The purpose of this note is to show a theorem on the asymptotic behavior of the expected total reward $EY(t)$, in the case of not necessarily identically distributed random variables.

Renewal reward processes have been taken by Smith (1955) who call them cumulative processes, and Ross (1970) has discussed the case in which Y_i may depend upon X_i, but only the case where each of $\{X_i\}$ and $\{Y_i\}$ is identically distributed has been studied.

2. We first state some assumptions. Let $F_i(x)$ and $G_i(x)$ be the marginal distribution functions of X_i and Y_i, respectively. Set

$$\lambda_i(t)=\int_0^t [1-G_i(x)]dx. \tag{2.1}$$

Suppose that the following assumptions are satisfied:

(a) $\lim_{A\to\infty}\int_A^\infty x dF_i(x)=0$ holds uniformly with respect to i.

(b) $\mu=\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} \mu_i$ exists.
(c) $\lambda(t)=\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} \lambda_i(t)$ exists uniformly for t.
(d) $\lambda=\lim_{t\to\infty} \lambda(t)$ exists.

In view of the assumptions (c) and (d), it follows that

$$\lambda=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\lambda_i \tag{2.2}$$

exists.

Now, we state a lemma for the renewal process with non-identically distributed random variables.

LEMMA. *Let* $\{X_i, i=1,2,\cdots\}$ *be a sequence of independent, nonnegative random variables with* $0<EX_i=\mu_i<\infty$. *Under the assumption* (a), *we have*

$$EN^{\alpha}(t)<\infty \tag{2.3}$$

for $\alpha=1,2,\cdots$, *and if the assumption* (b) *is added, then*

$$\lim_{t\to\infty}\frac{EN^{\alpha}(t)}{t}=\frac{1}{\mu^{\alpha}} \tag{2.4}$$

for $\alpha=1,2,\cdots$.

This lemma was first proved by Kawata (1956) for $\alpha=1$, and Hatori (1960) proved it for any positive integer α.

The theorem we are going to show in the non-identically distributed case is the following.

THEOREM. *Let* $\{X_i, i=1,2,\cdots\}$ *and* $\{Y_i, i=1,2,\cdots\}$ *be sequences of independent, nonnegative random variables with* $0<EX_i=\mu_i<\infty$ *and* $0<EY_i=\lambda_i<\infty$, *respectively. We suppose that the pairs* $(X_i, Y_i), i=1,2,\cdots$, *are independent of each other, while* Y_i *may depend on* X_i. *Then, under the assumptions* (a), (b), (c) *and* (d), *we have*

$$\lim_{t\to\infty}\frac{EY(t)}{t}=\frac{\lambda}{\mu}.$$

Proof. Write

$$EY(t)=E\sum_{n=1}^{N(t)+1} Y_n-EY_{N(t)+1},$$

in which letting

$$Z_n=\begin{cases}1, & \text{if } n\leqq N(t)+1,\\ 0, & \text{otherwise,}\end{cases}$$

we have

$$E\sum_{n=1}^{N(t)+1} Y_n=E\sum_{n=1}^{\infty} Y_nZ_n=\sum_{n=1}^{\infty} EY_nZ_n.$$

Here, Z_n is independent of Y_n, because Y_n is independent of $\{X_1,\cdots,X_{n-1}\}$ and Z_n depends only on $\{X_1,\cdots,X_{n-1}\}$ as we see in the following way:

$$\{Z_n=0\}=\{N(t)+1<n\}$$
$$=\bigcup_{k=1}^{n-1}\{N(t)+1=k\}$$
$$=\{X_1>t\}\cup\left[\bigcup_{k=1}^{n-1}\{(X_1+\cdots+X_{k-1}\leqq t)\cap(X_1+\cdots+X_k>t)\}\right].$$

Hence,

$$E\sum_{n=1}^{N(t)+1}Y_n=\sum_{n=1}^{\infty}EY_nEZ_n$$
$$=\sum_{n=1}^{\infty}\lambda_n\Pr\{N(t)+1\geqq n\}. \tag{2.5}$$

We rewrite (2.2) as

$$\frac{1}{n}\sum_{i=1}^{n}\lambda_i=\lambda+\varepsilon_n,$$

from which we have

$$\lambda_n=\lambda+n\varepsilon_n-(n-1)\varepsilon_{n-1},$$

where $\varepsilon_n\to 0$ as $n\to\infty$. Then we get from (2.5) that

$$E\sum_{n=1}^{N(t)+1}Y_n=\lambda(EN(t)+1)+\sum_{n=1}^{\infty}(n\varepsilon_n-(n-1)\varepsilon_{n-1})\Pr\{N(t)+1\geqq n\}.$$

Noticing that

$$\sum_{n=1}^{\infty}|n\varepsilon_n\Pr\{N(t)+1\geqq n\}|\leqq\sup_n|\varepsilon_n|\sum_{n=1}^{\infty}n\Pr\{N(t)+1\geqq n\}$$
$$\leqq\sup_n|\varepsilon_n|(EN^2(t)+2)<\infty$$

holds by (2.3) with $\alpha=2$, we have

$$E\sum_{n=1}^{N(t)+1}Y_n=\lambda(EN(t)+1)+\sum_{n=1}^{\infty}n\varepsilon_n\Pr\{N(t)+1=n\}, \tag{2.6}$$

so that

$$EY(t)\leqq\lambda(EN(t)+1)+\sum_{n=1}^{\infty}n\varepsilon_n\Pr\{N(t)+1=n\}.$$

Now, since $\varepsilon_n\to 0$ as $n\to\infty$, for any given positive number ε, we can choose an integer N such that $|\varepsilon_n|<\varepsilon$ for $n>N$. We then have

$$\left|\sum_{n=1}^{\infty}n\varepsilon_n\Pr\{N(t)+1=n\}\right|$$
$$<\sum_{n=1}^{N}n|\varepsilon_n|\Pr\{N(t)+1=n\}+\varepsilon\sum_{n=N+1}^{\infty}n\Pr\{N(t)+1=n\}$$
$$<N^2C+\varepsilon(EN(t)+1),$$

where $C=\max_{1\leqq n\leqq N}|\varepsilon_n|<\infty$. Since ε is arbitrary, we get, using (2.4) with $\alpha=1$, (2.7) $$\lim_{t\to\infty}\frac{1}{t}\left|\sum_{n=1}^{\infty} n\varepsilon_n \Pr\{N(t)+1=n\}\right|=0,$$

so that

(2.8) $$\limsup_{t\to\infty}\frac{EY(t)}{t}\leqq\frac{\lambda}{\mu}.$$

Next, take $\delta>0$ arbitrarily and let Y_n^* represent the variables truncated according to the rule

$$Y_n^*=\begin{cases} Y_n, & \text{if } Y_n\leqq\delta t,\\ \delta t, & \text{otherwise.}\end{cases}$$

For this truncated variables, it is clear that $EY_n^*=\lambda_n(\delta t)$, where $\lambda_n(\cdot)$ is defined by (2.1). Repeating the same argument as (2.6) was obtained, we have

$$E\sum_{n=1}^{N(t)+1} Y_n^*=\lambda(\delta t)(EN(t)+1)+\sum_{n=1}^{\infty} n\varepsilon_n(\delta t)\Pr\{N(t)+1\geqq n\},$$

where $\varepsilon_n(\cdot)$ is defined by the relation

$$\lambda(t)=\frac{1}{n}\sum_{i=1}^{n}\lambda_i(t)+\varepsilon_n(t).$$

The assumption (c) says that $\varepsilon_n(t)\to 0$ uniformly for t as $n\to\infty$. Therefore, we have

$$\begin{aligned} EY^*(t)&\equiv E\sum_{n=1}^{N(t)} Y_n^*\\ &\geqq\lambda(\delta t)(EN(t)+1)+\sum_{n=1}^{\infty} n\varepsilon_n(\delta t)\Pr\{N(t)+1\geqq n\}-\delta t,\end{aligned}$$

and hence

$$\liminf_{t\to\infty}\frac{EY^*(t)}{t}\geqq\liminf_{t\to\infty}\frac{\lambda(\delta t)(EN(t)+1)}{t}-\delta,$$

by the same reasoning as in (2.7). For the fixed δ, $\lim_{t\to\infty}\lambda(\delta t)=\lambda$; we thus have

$$\liminf_{t\to\infty}\frac{EY^*(t)}{t}\geqq\frac{\lambda}{\mu}-\delta.$$

Since δ is arbitrary, we have

$$\liminf_{t\to\infty}\frac{EY^*(t)}{t}\geqq\frac{\lambda}{\mu}.$$

On the other hand, it is clear that $EY^*(t)\leqq EY(t)$; consequently we have

(2.9) $$\liminf_{t\to\infty}\frac{EY(t)}{t}\geqq\frac{\lambda}{\mu}.$$

The required result follows from (2.8) and (2.9).

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REFERENCES

HATORI, H., (1960): A note on a renewal theorem, *Kodai Math. Sem. Rep.*, **12** 28–37.

KAWATA, T., (1956): A renewal theorem, *J. Math. Soc. Japan*, **8**, 118–126.

ROSS, S. M., (1970): *Applied Probability Models with Optimization Applications*, Holden-Day, 51–53.

SMITH, W. L., (1955): Regenerative stochastic processes, *Proc. Roy. Soc.*, *A232*, 6–31.