Title	On a property of eigenvalues concerning some vibration problems
Sub Title	
Author	鬼頭, 史城(Kito, Fumiki)
Publisher	慶応義塾大学工学部
Publication year	1973
Jtitle	Keio engineering reports Vol.26, No.7 (1973.) ,p.77- 83
JaLC DOI	
Abstract	The author has previously made studies about existence of eigen-values for various cases of vibration problems. Recently, the author has been asked, how he could infer that these eigen-values are positive. In this supplementary note, the author has given an account, which shows the positivity of these eigen-values. The discussion is confined to two cases: (a) vibration of rectangular elastic plate fitted with a stiffener rib, (b) vibration of an elastic bar which is fitted in a water region.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00260007-0077

KEIO ENGINEERING REPORTS
VOL. 26 No. 7 YOKOHAMA 1973

ON A PROPERTY OF EIGENVALUES CONCERNING SOME VIBRATION PROBLEMS

BY

FUMIKI KITO

FACULTY OF ENGINEERING
KEIO UNIVERSITY

KEIO ENGINEERING REPORTS
VOL. 26, NO. 7, pp. 77-83, 1973

ON A PROPERTY OF EIGEN-VALUES CONCERNING SOME VIBRATION PROBLEMS

FUMIKI KITO

Dept. of Mechanical Engineering. Keio University. Yokohama 223, Japan

(Received Dec, 3, 1973)

ABSTRACT

The author has previously made studies about existence of eigen-values for various cases of vibration problems. Recently, the author has been asked, how he could infer that these eigen-values are positive. In this supplementary note, the author has given an account, which shows the positivity of these eigen-values. The discussion is confined to two cases: (a) vibration of rectangular elastic plate fitted with a stiffener rib, (b) vibration of an elastic bar which is fitted in a water region.

1. Introduction

The author has previously made studies about vibration of elastic plates (1968, 1969, 1970 and 1972), which is fitted with stiffener rib-bar, or which is in contact with a water region. For each one of these cases, the author has discussed the existence of eigen-values (natural frequencies), reducing the problem to that of linear integral equations. Recently, the author has been asked by a reader, how he could infer that the eigen-values have real positive values. One way to answer this question would be to point out that, we have the relation

$$\lambda\bar{T}=\bar{V}$$

where $\lambda\bar{T}$ and $\bar{V}$ are timely mean values of kinetic- and potential-energies of the whole dynamical system under consideration, however complicated it may be. Since $\bar{T}$ and $\bar{V}$ have positive real values, so we could conclude that the value $\lambda=\bar{V}/\bar{T}$ is always positive. Nevertheless, it was thought that, it may be of some interest to deduce the inference directly from the differential equation of vibration for respective cases. In what follows, this deduction will be made, for each individual cases.

2. Vibration of a rectangular elastic plate, fitted with a stiffener rib

Consider an elastic rectangular plate, as shown in Fig. 1, which is fitted with a stiffener rib-bar. In order to treat the small transverse vibration of this plate, we use the following notations:

Qx, Qy=vertical shearing forces acting on cross-section of the plate, (per its unit length), Mx, My=bending moments acting on cross-section of the plate (per its unit length), D=flexural rigidity of the plate$=Eh^3/[12(1-\nu^2)]$, h=thickness of the plate, E, ν=Young's modulus and Poisson's ratio of plate material, a, b=length and width of the rectangular plate, ρ=density of plate material.

As to quantities relating to the stiffener rib, we use following notations; G_1= shear modulus of elasticity, K_1=modulus of torsion, E_1=Young's modulus, I_1= secondary moment of sectional area of stiffener bar, ρ_1=material density, A_1=cross-sectional area, J_1=longitudinal secondary moment.

The fundamental equation of free transverse vibration, for small displacement w, is given by,

$$D\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right]+\rho h\frac{\partial^2 w}{\partial t^2}=0 \tag{1}$$

For the case of sustained free vibration with angular frequency $\omega=2\pi f$, we put

$$w=W\sin Wt,$$

and obtain following equation for $W(x, y)$,

$$D\left[\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2 \partial y^2}+\frac{\partial^4 W}{\partial y^4}\right]-\rho h\lambda W=0 \tag{2}$$

where we put, for shortness, $\lambda=\omega^2$. Boundary conditions to be satisfied by the solution W is (taking the case of fixed four edges, as an instance),

(a) for $x=0$ and $x=a$ $[0\leq y\leq b]$ $W=0$, $\partial W/\partial y=0$

(b) for $y=0$ and $y=b$ $[0\leq x\leq a]$ $W=0$, $\partial W/\partial x=0$

Values of shearing forces and bending moments are given by

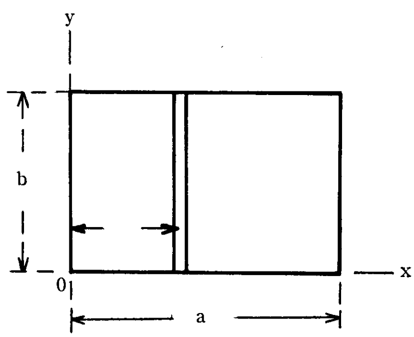

Fig. 1. Rectangular Elastic Plate fitted with a Stiffener Rib

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]$$

$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]$$

$$Q_x = -D\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]$$

$$Q_y = -D\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]$$

One way to express the fact that stiffener rib is attached to the elastic plate, at the location $x=\xi$, is to put

$$+|Q_x|_{-\varepsilon}^{+\varepsilon} = E_1 I_1 \frac{\partial^4 w}{\partial y^4} + \rho_1 A_1 \frac{\partial^2 w}{\partial t^2} \tag{3}^{(*)}$$

$$+|M_x + M_y|_{-\varepsilon}^{+\varepsilon} = G_1 K_1 \frac{\partial^3 w}{\partial x \partial y^2} - \rho_1 J_1 \frac{\partial^3 w}{\partial t^2 \partial x} \tag{4}^{(*)}$$

wherein, the right-hand sides must be taken values at $x=\xi$, and on the left-hand side we have to make ultimately $\varepsilon \to 0$.

Now, multiplying both sides of our equation (2) by W, and integrating over whole area of rectangular plate, we obtain an equation of the form,

$$I_1 + 2I_2 + I_3 = \lambda I_4 \tag{5}$$

where the put for shortness,

$$I_1 = \iint \frac{\partial^4 W}{\partial x^4} W dxdy, \qquad I_2 = \iint \frac{\partial^4 W}{\partial x^2 \partial y^2} W dxdy,$$

$$I_3 = \iint \frac{\partial^4 W}{\partial y^4} W dxdy, \qquad I_4 = \iint \rho h W^2 dxdy,$$

the integration being to be carried out for $y=0$ to b, and $x=0$ to $-\varepsilon, x=+\varepsilon$ to a. We are to take $\varepsilon \to 0$ ultimately. Also, we note that

$$W\frac{\partial^4 W}{\partial x^4} = \frac{\partial}{\partial x}\left[W\frac{\partial^3 W}{\partial x^3} - \frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2}\right] + \left(\frac{\partial^2 W}{\partial x^2}\right)^2$$

$$W\frac{\partial^4 W}{\partial x^2 \partial y^2} = \frac{\partial}{\partial y}\left[W\frac{\partial^3 W}{\partial x^2 \partial y} - \frac{\partial W}{\partial y}\frac{\partial^2 W}{\partial x^2}\right] + \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}$$

$$W\frac{\partial^4 W}{\partial y^4} = \frac{\partial}{\partial y}\left[W\frac{\partial^3 W}{\partial y^3} - \frac{\partial W}{\partial y}\frac{\partial^2 W}{\partial y^2}\right] + \left(\frac{\partial^2 W}{\partial y^2}\right)^2$$

and, by repeated application of partial integration, and taking into account boundary conditions along four edgelines, we obtain

$$I_1 + 2I_2 + I_3 = \iint \left[\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right]^2 dxdy$$

* Signs of left-hand side were mistaken, in previous paper.

$$+\int_0^b \left| W\frac{\partial^3 W}{\partial x^3} - \frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2} \right|_{+\varepsilon}^{-\varepsilon} dy \tag{6}$$

Furthermore, we have, by values of Qx and $Mx+My$ as given above,

$$W\frac{\partial^3 W}{\partial x^3}\Big|_{+\varepsilon}^{-\varepsilon} = -\frac{W}{D}|\bar{Q}x|_{+\varepsilon}^{-\varepsilon}$$

$$\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2}\Big|_{+\varepsilon}^{-\varepsilon} = \frac{-1}{(1+\nu)D}\frac{\partial W}{\partial x}|\bar{M}x+\bar{M}y|_{+\varepsilon}^{-\varepsilon}$$

In these equations $\bar{Q}x$, $\bar{M}x$ and $\bar{M}y$ are values of Qx, Mx and My wherein time factors $\sin \omega t$ are omitted. Thus we have, by relations (3) and (4),

$$J = \int_0^b \left| W\frac{\partial^3 W}{\partial x^3} - \frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2} \right|_{+\varepsilon}^{-\varepsilon} dy$$

$$= -\int_0^b \frac{W}{D}|\bar{Q}x|_{+\varepsilon}^{-\varepsilon} dy + \int_0^b \frac{1}{(1+\nu)D}\frac{\partial W}{\partial x}|\bar{M}x+\bar{M}y|_{+\varepsilon}^{-\varepsilon} dy$$

$$= \frac{1}{D}\int_0^b \left[E_1 I_1 \frac{\partial^4 W}{\partial y^4} W - \rho_1 A_1 \lambda W^2 \right] dy$$

$$-\frac{1}{(1+\nu)D}\int_0^b \frac{\partial W}{\partial x}\left[G_1 K_1 \frac{\partial^3 W}{\partial x \partial y^2} + \rho_1 J_1 \lambda \frac{\partial W}{\partial x} \right] dy$$

where the integrands on right-hand side are to take values for $x=\xi$. Reminding the relation

$$\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x \partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x \partial y} \right] - \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2$$

we have

$$J = \frac{1}{D}\int_0^b \left[E_1 I_1 \left(\frac{\partial^2 W}{\partial y^2} \right)^2 - \rho_1 A_1 \lambda W^2 \right] dy$$

$$+\frac{1}{(1+\nu)D}\int_0^b \left[G_1 K_1 \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 - \rho_1 J_1 \lambda \left(\frac{\partial W}{\partial x} \right)^2 \right] dy \tag{7}$$

Putting these values of equations (6) and (7) into the equation (5), we are led to an equation of the form

$$I_4 = \lambda I_5,$$

where I_4 and I_5 are values depending on W and its partial derivatives, which are always positive. Thus, we may conclude that λ is a positive real constant.

The above discussion was made about the case of a rectangular plate of uniform thickness having a homogeneous elastic property. The author has made some consideration about the case of free vibration of rectangular plate of non-uniform thickness, which has heterogeneous elastic property and which is in a stressed state. It may here be remarked that, again in this case, we can infer that eigen-values corresponding to free vibration have positive values, by an argument similar to the above discussion, that is multiplying both sides of equation [which is of more complicated form than the above equation (2)], and integrating over whole area of the plate.

3. Free vibration of an Elastic Bar placed in a Water Region

Let us consider a case of elastic bar, which is placed inside a water region, as sketched in Fig. 2(a). When this elastic bar is making free transverse vibration of small amplitude, its equation of motion will be given by,

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right]+\rho_m A\frac{\partial^2 w}{\partial t^2}+q=0 \tag{8}$$

In this equation, following notations are used: w=small transverse displacement of the elastic bar, EI=flexural rigidity of the bar, ρ_m, A=density and cross-sectional area of the bar, q=external force applied to the bar, which is due to action of surrounding water.

There will be set up, in surrounding water, a motion of water induced by vibration of the bar itself. Assuming that water is a non-viscous incompressible fluid, this motion of water may be given by velocity potential $\phi(x, y, z; t)$ which satisfy the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{9}$$

in water region. This velocity potential ϕ must also satisfy following boundary conditions.

(a) On the surface of rigid wall, we must have

$$\frac{\partial\phi}{\partial n}=0$$

where $\partial/\partial n$ denotes the derivative in direction normal to wall surface, which is understood to be drawn inwards into the fluid region.

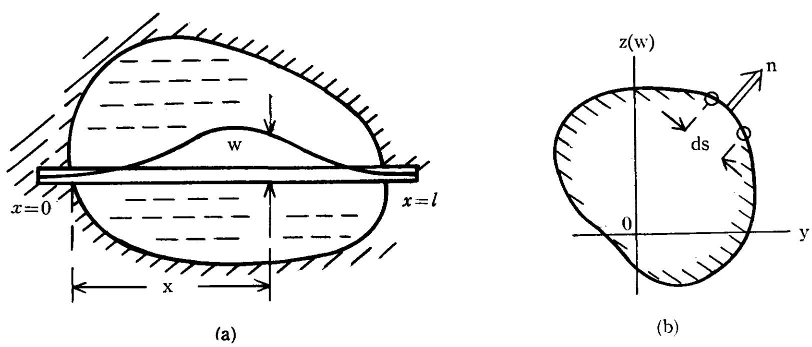

Fig. 2. Vibration of an Elastic Bar placed in a Water Region

(b) On the surface of vibrating bar we must have [see Fig. 2(b)]

$$\frac{\partial \phi}{\partial n}=\frac{\partial w}{\partial t} \cos (n, z)$$

where $\cos (n, z)$ is the cosine of angle subtended between the normal to surface of the vibrating bar and the z-axis. This normal is understood to be drawn outwards from body of bar, that is, inwards into fluid region. It is assumed that direction of transverse displacement of the bar is taking place in direction of z-axis.

For the case of free vibration with an angular frequency $\omega=2\pi f$, we may put

$$\phi=w\Phi \cos \omega t, \qquad w=W \sin \omega t, \qquad q=Q \sin \omega t \tag{11}$$

As to hydrodynamical pressure p of water, we may put approximately

$$p=\rho_w \frac{\partial \Phi}{\partial t}=-\rho_w \omega^2 \Phi \sin \omega t, \tag{12}$$

ρ_w being density of water. The equation of free vibration (8) becomes as follows,

$$\frac{d^2}{dx^2}\left[EI\frac{d^2 W}{dx^2}\right]-\rho_m \lambda A W+Q=0 \tag{13}$$

where we put $\lambda=\omega^2$. The function Φ must satisfy, together with Laplace equation

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}+\frac{\partial^2 \Phi}{\partial z^2}=0,$$

following boundary conditions.

(a) On the rigid wall surface,

$$\frac{\partial \Phi}{\partial n}=0$$

(b) On the surface of bar

$$\frac{\partial \Phi}{\partial n}=W \cos (n, z)$$

Moreover, we have

$$Q=-\lambda \rho_w R, \qquad R=\int \Phi \cos (n, z) ds$$

where ds is curvilinear element of closed curve of cross-section of the bar.

To fix ideas, we assume that ends of bar is kept in state of fixed ends, which is located at $x=0$ and $x=l$. Let us multiply by W the both sides of our equation (13), and integrate over the whole length of the bar. Then, we shall have (after making integration by parts, and taking into account above-mentioned end conditions at $x=0$ and $x=l$),

$$\int_0^l EI\left(\frac{d^2 W}{dx^2}\right)^2 dx-\rho_m \lambda \int_0^l A W^2 dx-\rho_w \lambda \int_0^l R W dx=0 \tag{14}$$

On the other hand, we have

$$J=\int_0^l R W dx=\int_0^l W dx\left[\int \Phi \cos (n, z) ds\right]=\iint W\Phi \cos (n, z) dS$$

where dS means elementary area of surface of the bar. But we have by above-mentioned condition (b)

$$J=\iint \Phi \frac{\partial \Phi}{\partial n} dS$$

and this surface integral is seen to be equal to following volume integral which extend to whole water region;

$$J=\iiint\left[\left(\frac{\partial \Phi}{\partial x}\right)^2+\left(\frac{\partial \Phi}{\partial y}\right)^2+\left(\frac{\partial \Phi}{\partial z}\right)^2\right]dxdydz$$

Summing up these results, we see that our equation (14) may be expressed in following form

$$I_1=\lambda[\rho_m I_2+\rho_w J] \tag{15}$$

where we have put

$$I_1=\int_0^l EI\left(\frac{d^2W}{dx^2}\right)^2 dx, \qquad I_2=\int_0^l AW^2 dx$$

Since I_1, I_2 and J are positive quantities, we may conclude from this relation (15) that eigen-value λ must always be positive.

As a final remark, it may be allowed to state that similar argument as given above may be made about more complicated vibration problems in elasticity and in hydro-elasticity.

REFERENCES

KITO, F.: Proc. Fac. Eng., Keio Univ., 21, 4 (1968); 22, 2 (1969).
KITO, F.: Jour. Coll. Eng., Nihon Univ., A-13 (1972).
KITO, F.: Principles of Hydro-elasticity, Yokendo Co. (Publishers), Tokyo (1970).