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ON EXPANSION OF FUNCTIONS IN FOURIER-TYPE SERIES ABOUT CN (x) AND SN (x)

By

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ON EXPANSION OF FUNCTIONS IN FOURIER-TYPE SERIES ABOUT $Cn(x)$ AND $Sn(x)$

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ABSTRACT

Given a differential equation $d^4w/dx^4+\lambda^4w=0$, under boundary conditions $w(\pm 1/2)=0, w'(\pm 1/2)=0$, it is known that there exist a set of normalized orthogonal functions $Cn(\mathrm{x}), Sn(x)$ which represent solution of eigen-value/function problem. It is also known that any function $f(x)$ which is (together with its derivatives f', f'', f''') are continuous in the region and satisfies the end-conditions $f(\pm 1/2)=0$, $f'(\pm 1/2)=0$, can be expanded into Fourier-type infinite series in $Cn(x)$ and $Sn(x)$. In the present note, the author discusses about the similar expansion, for the case in which the given function does not fully satisfy the required condition. It is also pointed out that similar line of thoughts can be applied to linear differential equation of any degree, with any number of independent variables.

1. Introduction

The set of orthogonal functions $Cn(\mathrm{x})$ and $Sn(x)$, which relate to free vibration of elastic bar, has been used since long years ago. In 1945, S. Chandrasekhar and W. H. Reid has given us more concise form to $Cn(x)$ and $Sn(x)$, which are normalized and orthogonal to each other. For any given function $f(x)$ which satisfy certain conditions (described below), it can be expanded into an infinite series of Fourier type, viz.,

$$f(x)=\sum_{n=1}^{\infty} a_nC_n(x)+\sum_{n=1}^{\infty} b_nS_n(x) \tag{1}$$

This fact can be proved by referring to theory of linear integral equations, by showing that the function $f(x)$ under consideration can be expressed in "source like (quellen-mässige) form as to be shown by Conrant, R. and Milbert. D. (1924)

or Kneser, A (1922). But the theorem does not supply any information about validity of expansion (1), for the case in which function $f(x)$ does not satisfy the abovementioned condition. In the present note the author gives some consideration about expansibility of some functions which do not satisfy the necessary condition.

2. Property of functions $Cn(x)$ and $Sn(x)$

2.1. Statement of our problem

We consider the differential equation

$$\frac{d^4 W}{dx^4}+\lambda^4 W=0 \tag{2}$$

the unknown function $W(x)$ being to be so chosen that it satisfies the equation (2), together with the following boundary condition,

$$W(x)=0,\ W'(x)=0; \qquad \text{at} \qquad x=\pm 1/2 \tag{3}$$

λ is the unknown constant, specific values of λ being eigen-values. Chandrasekhar and Reid has given us the following explicit form of solutions;

$$\begin{aligned} C_n(x, \lambda_n) &= \frac{\cosh(\lambda_n x)}{\cosh \lambda_n/2} - \frac{\cos(\lambda_n x)}{\cos \lambda_n/2} \\ S_n(x, \lambda_n) &= \frac{\sinh(\lambda_n x)}{\sin \lambda_n/2} - \frac{\sin \lambda_n x}{\sin \lambda_n/2} \end{aligned} \tag{4}$$

λ_n and μ_n are eigen-values which are roots of following equations,

$$\begin{aligned} &\tanh \lambda_n/2 = -\tan \lambda_n/2 \\ &\tanh \mu_n/2 = \tan \mu_n/2 \\ &(n=1, 2, 3, \cdots) \end{aligned} \tag{5}$$

For a large value of integer n, we have approximately,

$$\lambda n \doteqdot \left(2n-\frac{1}{2}\right)\pi, \qquad \mu_n \doteqdot \left(2n+\frac{1}{2}\right)\pi \tag{6}$$

The fact that set of functions (4) form a set of normalized orthogonal functions, can be verified by actual calculation of integrals. Its completeness was pointed out by A. Kneser.

It was stated, by means of theory of linear integral equation, by A. Kneser that any function $f(x)$ which satisfies the boundary conditions (3) and which is such that f, f', f'' and f''' are continuous, finite and integrable in $-1/2 \leq x \leq 1/2$, (being expressible in source-like form), can be expanded in form of (1), which converge in our region of x. But, this theory does not supply us any information about validity of expansion of the form of (1), for the case of function $f(x)$ which does not satisfy the above mentioned conditions.

In the present note, the author gives some consideration about the expansion of function $f(x)$ into form of (1), in the case in which the above-mentioned condition is not satisfied. The author has, in previous papers (1962, and 1970) used expansion of Fourier-type of (1), rather freely. It is hoped that the present note may serve to supplement them.

2.2. Some property of functions $\mathbf{C}n(x)$ and $\mathbf{S}n(x)$

Let us consider the expression

$$C(x, \lambda) = -\frac{\cos \lambda x}{\cos \lambda/2} + \gamma(x)$$
$$\gamma(x) = \frac{\cosh (\lambda x)}{\cosh \lambda/2} \tag{7}$$

where λ is a positive constant. For any value of x lying in the region $-1/2+\eta< x<1/2-\eta$ (η being a positive constant which we can take as small value as we please), we have,

$$|\gamma(x)| \leq \frac{\cosh (\lambda/2-\lambda_n)}{\cosh \lambda/2}$$
$$\leq e^{-\lambda\eta} \frac{e^{\lambda}+e^{2\lambda\eta}}{e^{\lambda}+1}$$

But the function

$$y(\lambda) = \frac{e^{\lambda}+e^{2\lambda\eta}}{e^{\lambda}+1} \tag{8}$$

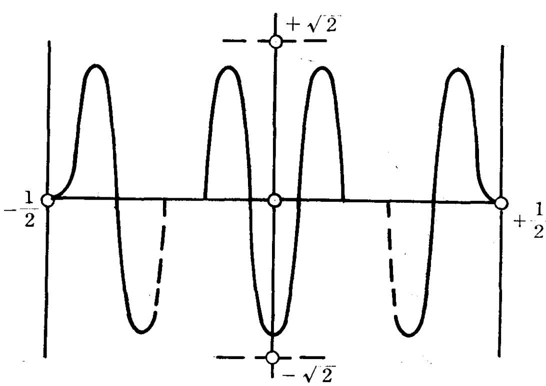

Fig. 1. Rough sketch of function $C_n(x)$

is such that $y(0)=1, y(+\infty)=1$, and $y(\lambda)$ has a simple maximum value y_m at a value of $\lambda=\lambda_m(0<\lambda_m<\infty)$. This maximum value of y, y_m, being a function of η, becomes as near as possible to unity, when we make $\eta\to 0$. For an instance, when $\eta<0.10$ we have $1<y(\lambda)<1.10$. Thus, we infer that for a value of x which lies in the domain $-1/2+\eta<x<1/2-\eta$, we have

$$|\gamma(x)|<Ae^{-\lambda\eta}$$

in which A is positive constant slightly larger than unity. For example, for $\eta<0.10$ we may put $A=1.10$.

On the other hand, when the value of λ takes the form $\lambda_n=(2n-1/2)\pi$ and becomes very large, we have

$$\left|\frac{\cos \lambda x}{\cos \lambda/2}\right|<\frac{1}{1/\sqrt{2}}=\sqrt{2}$$

These inferences show us that, for a fixed value of η, our series (1) behaves, for a large value of N and $N<n$, as near as possible with the trigonometric Fourier series.

3. Supplementary function $\varphi(x)$

Let us consider a function $f(x)$ which is, together with its derivatives f', f'' and f''' finite and continuous in our interval $-1/2\leq x\leq 1/2$. This function $f(x)$ is supposed not to satisfy end condition (3). This being so, this function itself cannot be expressed in source-like (quellenmässige) form, and therefore we have no right to insist about validity of expansion in form of (1), even though we could estimate its coefficients a_n and b_n. For that case, we form a new function $F(x)$ in following way:

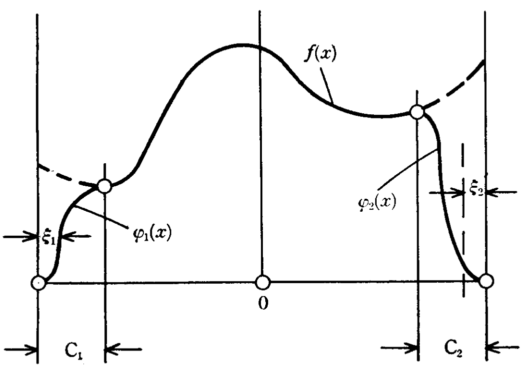

Fig. 2. Illustrating the use of function $\varphi(x)$

$$F(x)=\begin{cases}\varphi_1(\xi_1) & -1/2<x<-1/2+\eta_1\\ f(x) & -1/2+\eta_1<x<1/2-\eta_2\\ \varphi_2(\xi_2) & 1/2-\eta_2<x<1/2\end{cases}$$

in such manner that $F(x)$, together with its derivatives F', F'' and F''' are finite and continuous in the whole interval of $-1/2<x<1/2$, and also such that at $x=\pm 1/2$, $F(x)=0, F'(x)=0$.

One way to attain this purpose is to use a polynomial

$$\varphi(\xi)=a_2\xi^2+a_3\xi^3+a_4\xi^4+a_5\xi^5 \tag{9}$$

In order to make values of φ at $=c$ coincide with desired values $\varphi(c), \cdots$, we can take

$$a_2=10\varphi(c)-6\varphi'(c)+\frac{3}{2}\varphi''(c)-\frac{1}{6}\varphi'''(c)$$

$$-a_3=20\varphi(c)-14\varphi'(c)+4\varphi''(c)-\frac{1}{2}\varphi'''(c)$$

$$a_4=15\varphi(c)-11\varphi'(c)+\frac{7}{2}\varphi''(c)-\frac{1}{2}\varphi'''(c)$$

$$-a_5=4\varphi(c)-3\varphi'(c)+4\varphi''(c)-\frac{1}{6}\varphi'''(c)$$

[In actual application, we must of course, make change of variables.]

Modified function $F(x)$ thus defined satisfies all required conditions, and therefore it can be expressed in sourcelike form. Hence it follows that this function $F(x)$ can be expanded in Fourier-type series in form of (1). The numerical constant η used in above argument may take any positive value, as small as we please.

4. Derivatives of functions $Cn(x, \lambda_n)$ and $Sn(x, \mu_n)$

It can be verified, by actual differentiation, that all the derivatives of $Cn(x, \lambda_n)$ and $Sn(x, \mu_n)$ can be expressed by following eight functions (of course, to be multiplied by suitable powers of λ_n or μ_n).

$$F_1(x, \lambda_k)=\frac{\cosh \lambda_k x}{\cosh \lambda_k/2}\pm\frac{\cos \lambda_k x}{\cos \lambda_k/2} \tag{10}$$

$$F_2(x, \mu_k)=\frac{\cosh \mu_k x}{\sinh \mu_k/2}\pm\frac{\cos \mu_k x}{\sin \mu_k/2} \tag{11}$$

$$G_1(x, \mu_k)=\frac{\sinh \mu_k x}{\sinh \mu_k/2}\pm\frac{\sin \mu_k x}{\sin \mu_k/2} \tag{12}$$

$$G_2(x, \lambda_k)=\frac{\sinh \lambda_k x}{\cosh \lambda_k/2}\pm\frac{\sin \lambda_k 2}{\cos \lambda_k/2} \tag{13}$$

Let us expand these functions of x into infinite series (1) of Fourier type.

For the case of even functions (10), (11), we shall have

$$f_c=\sum_{n=0}^{\infty} a_n C_n(x, \lambda_n) \tag{14}$$

while for the case of odd functions (12), (13), we shall have

$$f_s=\sum_{n=0}^{\infty} a_n S_n(x, \mu_n) \tag{15}$$

Actual values of $a_n(n=1, 2, 3, \cdots)$ can be obtained by formula of integrals. Thus we obtain following results. In list shown below, the sign of approximate equality $\doteqdot$ means that we show case of large integer n.

For $F_1(x, \lambda_k)$,

$$a_n^k=\delta_{nk}+\begin{bmatrix}1\\0\end{bmatrix}\frac{8\lambda_n^2}{\lambda_k^4-\lambda_n^4}\left[\lambda_k \tanh\frac{\lambda_k}{2}-\lambda_n \tanh\frac{\lambda_n}{2}\right]$$

$$\doteqdot\delta_{nk}+\begin{bmatrix}1\\0\end{bmatrix}\frac{8(\lambda_k-\lambda_n)\lambda_n^2}{\lambda_k^4-\lambda_m^4}$$

For $F_2(x, \mu_k)$,

$$a_n^k=\left[\coth\frac{\mu_k}{2}\tanh\frac{\lambda_n}{2}\cdot\frac{8\lambda_n^3}{\lambda_m^4-\mu_k^4}\ ;\ -\frac{8\mu_k\lambda_n^2}{\lambda_n^4-\mu_k^4}\right]$$

$$\doteqdot\left[\frac{8\lambda_n^3}{\lambda_n^4-\lambda_k^4}\ ;\ -\frac{8\mu_k\lambda_n^2}{\lambda_n^4-\mu_k^4}\right]$$

For $G_1(x, \mu_k)$,

$$a_n^k=\delta_{nk}+\begin{bmatrix}1\\0\end{bmatrix}\left[\frac{8\mu_n^3}{\mu_n^4-\mu_k^4}\coth\frac{\mu_n}{2}-\frac{8\mu_k\mu_n^2}{\mu_n^4-\mu_k^4}\coth\frac{\mu_k}{2}\right]$$

$$\doteqdot\delta_{nk}+\begin{bmatrix}1\\0\end{bmatrix}\left[\frac{8\mu_n^2(\mu_n-\mu_k)}{\mu_n^4-\mu_k^4}\right]$$

For $G_2(x, \lambda_k)$,

$$a_n^k=\left[\frac{-8\lambda_k\mu_n^2}{\mu_n^4-\lambda_k^4}\ ;\ \coth\frac{\mu_n}{2}\tanh\frac{\lambda_k}{2}\cdot\frac{8\mu_n^3}{\mu_n^4-\lambda_k^4}\right]$$

$$\doteqdot\left[-\frac{8\lambda_k\mu_n^2}{\mu_n^4-\lambda_k^4}\ ;\ \frac{8\mu_n^3}{\mu_n^4-\lambda_k^4}\right]$$

Among the eight coefficients a_n^k, as shown above, 2 of them becomes of order of $1/n^2$ as $n\to\infty$. On the other hand, 4 of them becomes of order of $1/n$ as $n\to\infty$. It would be interesting if we could make a direct proof of convergence of an infinite series

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)}\cos(4n-1) \tag{16}$$

We remark that, we have

$$-\log\left|2\sin\frac{x}{2}\right|=\cos x+\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x+\cdots$$

[as pointed out in text book by Whittaker and Watson (1935)], which converge for all real values of x. At the present stage, the author cannot give direct proof about convergence of (16). We may remark that boundary values of functions F_i and G_i are as shown below:

$$F_1\left(\frac{1}{2}\right)=[2,0]\,;\quad F_1'\left(\frac{1}{2}\right)=\lambda_k\tanh\lambda_k/2$$

$$F_2\left(\frac{1}{2}\right)=[2,0]\coth\mu_k/2\,;\quad F_2'\left(\frac{1}{2}\right)=[0,2]\mu_k$$

$$G_1\left(\frac{1}{2}\right)=[2,0]\,;\quad G_1'\left(\frac{1}{2}\right)=[2,0]\mu_k\coth\mu_k/2$$

$$G_2\left(\frac{1}{2}\right)=[0,2]\tanh\lambda_k/2,\quad G_2'\left(\frac{1}{2}\right)=[2,0]\lambda_k$$

5. Some examples of Expansion of Fourier type about $Cn(x, \lambda_n)$ and $Sn(x, \lambda_n)$

Here we shall give some examples of given functions in infinite series of Fourier type, in form of (1).

5.1. The even function

$$y_1=1+\cos 2\pi x\qquad[-1/2\leq x\leq 1/2]$$

is one which satisfies all the required conditions. When expanded into form of (14), its coefficients are as follows:

$$a_n=4\left[\frac{(2\pi)^4}{\lambda_n^4-(2\pi)^4}\right]\tanh\lambda_n/2$$

5.2 The following even function

$$y_2=1-2|x|\qquad[-1/2\leq x\leq 1/2]$$

does not satisfy the required condition. Moreover, its derivative y_2' is discontinuous at $x=0$. Nevertheless we obtain

$$a_n=\frac{4}{\lambda_n^2}\left[2-\frac{1}{\cosh\lambda_n/2}-\frac{1}{\cos\lambda_n/2}\right]$$

and, for $n\to\infty$ we have

$$a_n\to\frac{4}{\lambda_n^2}\left[1-(-1)^n\sqrt{2}\right]\to\frac{A}{n^2}$$

5.3. Next, let us consider a function which is everywhere equal to unity, that is

$$y_3=1 \qquad [-1/2\leq x\leq 1/2]$$

This function does not satisfy the required condition. But, if we expand it formally in form of (14), we obtain,

$$a_n=\frac{4}{\lambda_n}\tanh \lambda_n/2,$$

and thus we see that as $n\to\infty$ we have

$$a_n=A/n$$

5.4. As an example of odd function, which does not satisfy the required condition, let us take

$$y_4=x \qquad [-1/2\leq x\leq 1/2]$$

We obtain for a_n in expansion of form of (15),

$$a_n=\frac{2}{\mu_n}\coth \mu_n/2-\frac{4}{\mu_n^2}$$

and we have, for $n\to\infty$,

$$a_n\to\frac{2}{(2n-1/2)\pi}$$

5.5 Since treatment of distribution (generalized function) by L. Schwartz, it became customary to use delta-function $\delta(x)$, even in the field of technology. Taking a point c at $-1/2<c<+1/2$, and expanding the delta function $\delta(x-c)$ merely formally, we obtain

$$\begin{aligned}\delta(x-c)=&\sum_{n=1}^{\infty} C_n(x,\lambda_n)C_n(c,\lambda_n)\\ &+\sum_{n=1}^{\infty} S_n(x,\mu_n)S_n(c,\mu_n)\end{aligned} \tag{17}$$

Reminding the expression (7), we observe that the above expansion (17) is one which correspond to expansion of $\delta(x-c)$ into usual Fourier series. On the other hand, we may derive the expansion (17) as the limiting case of expansion of following ordinary function $h_\nu(x)$ where η_ν

$$1/k=\int_{-1}^{+1}\exp\left[\frac{-1}{1-t^2}\right]dt$$

$$h_\nu(x)=\frac{k}{\eta_\nu}\exp\left[\frac{-1}{1-(x/\eta_\nu)^2}\right] \qquad |x|<\eta_\nu$$

$$=0 \qquad |x|>\eta_\nu$$

is a positive constant which we make, ultimately, $\eta_\nu\to 0$. This function $h_\nu(x)$ is, together with its derivatives of any degree, finite and continuous function of x, Moreover, it satisfies the boundary condition (3), so long as the constant is taken sufficiently small. Thus, we see that the function $h_\nu(x)$ is [when η_ν is taken sufficiently small] expressed in source-like form. The expansion (17) may be regarded to be limitting case of expansion of this function $h_\nu(x)$.

5.6. Here we shall deduce some absolutely convergent series, from expansion of derivatives of $Cn(x, \lambda_n)$ and $Sn(x, \lambda_n)$. From the above estimation it was seen that for the functions

$$F_1(x, \lambda_k)=\frac{\cosh \lambda_k x}{\cosh \lambda_k/2}+\frac{\cos \lambda_k x}{\cos \lambda_k/2}$$

we had, for $n\to\infty$,

$$a_n \doteqdot \frac{8\lambda_n^2(\lambda_n-\lambda_k)}{\lambda_n^4-\lambda_k^4}\to\frac{8}{(2n-1/2)\pi}$$

Thus, forming a new function

$$y_5=\frac{\cosh \lambda_k x}{\cosh \lambda_k/2}+\frac{\cos \lambda_k x}{\cos \lambda_k/2}-2,$$

we see that its expansion in infinite series of form (14) has the property that its coefficients a_n are such that, for $n\to\infty$ we have $a_n\to A/n^2$. That is, the infinite series (14) will be absolutely convergent.

6. Concluding Remarks

6.1. It was known, long since, that a function $f(x)$, which is, together with its derivatives f', f'' and f''', are continuous over the region $-1/2\leq x\leq 1/2$, and is such that $f(x)=0$ and $f'(x)=0$ at $x=\pm 1/2$, can be expanded into an infinite series of Fourier type in form of (1). In the present note, the author has shown that, even if the required conditions are not fully satisfied, we can obtain expansion of form of (1), by using modifying function $\varphi(x)$.

6.2. It was also shown that there are several functions which, when expanded into form of (1), are absolutely convergent, even if the required condition for $f(x)$ is not fully satisfied. It is also shown that, when the expansion for $f(x)$ is seen to be not absolutely convergent, we can obtain absolutely convergent series, by means of device such as shown in 5.6.

6.3. What will represent the expansion (1), for a function to which the required condition is not fully satisfied? We note that according to theorem of Fischer-Riesz, it is known that (ref. to N. Wiener's text-book (1933)) when we put

$$I_{m,n}=\int_a^b |f_m(x)-f_s(x)|^2 dx$$

and

$$\lim I_{m,s}\to 0$$

as $m \to \infty, s \to \infty$ independently, then there exist a function $f(x)$ such that

$$\lim_{m\to\infty}\int_b^a |f(x)-f_m(x)|^2dx=0$$

The property is said, in terms of Functional Analysis that the series of functions $f_m(x)$ ($m=1, 2, 3, \cdots \to \infty$) "converge" in the mean to the function $f(x)$. It is also said that the function $f(x)$ represent the limit of

$$\lim_{m\to\infty} f_m(x)$$

for values of x in the domain of $a \leq x \leq b$, "almost everywhere". In our case, we have only to put

$$y_s = \sum_{n=1}^{s} a_n C_n(x, \lambda_n)$$

$$y_m = \sum_{n=1}^{m} a_n C_n(x, \lambda_n)$$

to apply the Fischer-Riesz theorem to our present problem.

6.4. The above discussion was made with regard to the differential equation (2). But the principle of above treatment may be applied to eigen-value (eigen-function) problems about linear differential equation of any order with any number of independent variables.

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