Title	On the inversion of certain convolution operators related to feedback systems
Sub Title	
Author	川島, 弘尚(Kawashima, Hironao)
Publisher	慶応義塾大学工学部
Publication year	1973
Jtitle	Keio engineering reports Vol.26, No.4 (1973.) ,p.37- 53
JaLC DOI	
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Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00260004-0037

KEIO ENGINEERING REPORTS
VOL. 26 No. 4 YOKOHAMA 1973

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BY

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KEIO ENGINEERING REPORTS
VOL. 26, NO. 4, pp. 37-53, 1973

ON THE INVERSION OF CERTAIN CONVOLUTION OPERATORS RELATED TO FEEDBACK SYSTEMS

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(*Received Sep. 18, 1973*)

ABSTRACT

In this paper we shall investigate whether certain convolution operators related to feedback systems are invertible in some space of functions or distributions. The properties such as integrability and differentiability of the inverse are discussed.

1. Introduction

Consider a linear feedback system shown in Fig. 1, where g is a convolution operator defined by an integrable function over $(0, \infty)$. The equations governing the system are

$$\sigma(t)=l(t)-y(t), \qquad t\geqslant 0,$$

$$y(t)=g*\sigma(t), \qquad t\geqslant 0,$$

where $g*\sigma(t)=\int_0^t g(t-\tau)\sigma(\tau)\,d\tau$, $t\geqslant 0$. If $(\delta+g)$ is invertible in some class of distritions, then the equations

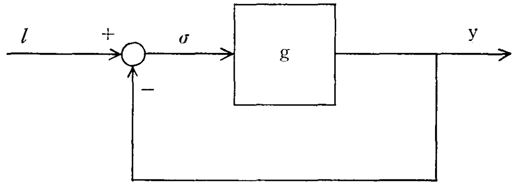

Fig. 1. A Linear Feedback System.

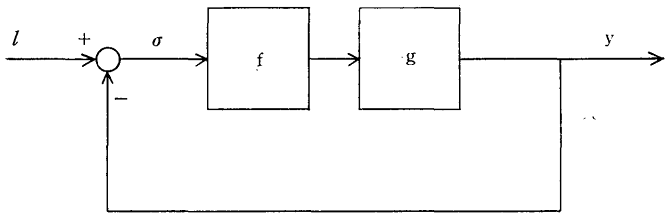

Fig. 2. A Nonlinear Feedback System.

$$\sigma(t)=(\delta+g)^{-1}*l(t), \qquad t\geqslant 0,$$

$$y(t)=g*(\delta+g)^{-1}*l(t), \qquad t\geqslant 0,$$

are meaningful. The system shown in Fig. 2 is called a nonlinear feedback system where f is a memoryless nonlinear function and g is the same as in the linear case. The system equation is given by

$$\sigma(t)=l(t)-g*f(\sigma)(t), \qquad t\geqslant 0.$$

If f satisfies a certain slop condition and if $(\delta+cg)^{-1}$ exists in some class of functions or distributions, then we have

$$\sigma(t)=(\delta+cg)^{-1}*l(t)-(\delta+cg)^{-1}*g*\hat{f}(\sigma)(t), \qquad t\geqslant 0,$$

where $\hat{f}$ is defined by $\hat{f}(\sigma)=f(\sigma)-c\sigma$. [See Kawashima, 1973-a]

These examples show that the investigation of conditions for the existence of the operators $(\delta+cg)^{-1}$ and $(\delta+cg)^{-1}*g$ are basic problems in the analysis of feedback systems. Note that if the operator $(\delta+cg)^{-1}*g$ exists, then we have $g*(\delta+cg)^{-1}=(\delta+cg)^{-1}*g$. This shows that we only need to study the properties of $(\delta+cg)^{-1}*g$, since $g*(\delta+g)^{-1}$ is a special case.

On the other hand, suppose that g is the impulse response of a linear constant coefficient differential equation, that is $g(t)=\sum_{i=1}^{r}\sum_{j=1}^{m_i}a_{ij}t^{j-1}e^{-b_it}$, $t\geqslant 0$, $\mathrm{Re}\, b_i>0$ and $\sum_{i=1}^{r}m_i=n$. In this case the existence of $(\delta+cg)^{-1}$ and $(\delta+cg)^{-1}*g$ is obvious. This can be easily shown, since the Laplace transform of g becomes a rational function of s and the method of partial fraction expansion can be used. Moreover, $(\delta+cg)^{-1}*g$ can be written as $(\delta+cg)^{-1}*g=\sum_{i=1}^{r'}\sum_{j=1}^{m'_i}a'_{ij}t^{j-1}e^{-b'_it}$, $t\geqslant 0$ and $\sum_{i=1}^{r'}m'_i=n$. With an additional restriction on g, we may assume that the real parts of b'_i are non-negative. Then the functions of the form $\sum_{i=1}^{r'}\sum_{j=1}^{m'_i}a'_{ij}t^{j-1}e^{-b'_it}$ have the following properties;

i) infinitely differentiable,

ii) belong to $L^p_{(0,\infty)}$ where $1\leqslant p<\infty$,

iii) their supports are not bounded in $(0, \infty)$.

A question now arises: What can we say about $(\delta+cg)^{-1}$ and $(\delta+cg)^{-1}*g$ if we can not assume that g is the impulse response of a differential equation? In this paper we shall deal with this problem in a more general setting in which g involves, in addition, the delta functional, derivatives of the delta functional and Heaviside unit step function. We first show an existence condition for the operators $(\delta+cg)^{-1}$ and $(\delta+cg)^{-1}*g$ in a certain class of distributions. Next, we shall examine the properties of these operators such as differentiability, integrability and unboundedness over $(0, \infty)$ of the support. The problems cited above are not only interesting in itself but also the solutions to the problems become a powerful tool in the analysis of nonlinear systems. For example see KAWASHIMA (1973-a, b) and also HOLTZMAN (1971), WILLEMS (1971), ZAMES (1964, 1966-a, b) and SANDBERG (1964).

In § 2 we shall give notations and definitions used in this paper. § 3 is devoted to show the existence of $(\delta+cg)^{-1}$ which gives as a special case the result of PALEY and WIENER (1934). In § 4 the integrability conditons are studied and in § 5 differentiability and unboundedness of the support are handled.

2. General Preliminaries

This preliminary section gives a brief sketch of Fourier and Laplace transforms for distributions and its related topics. [ZEMANIAN, 1965]

The spaces $L^p_{(R)}$, $1\leqslant p<\infty$, consist of all Lebesgue measurable functions $k(\cdot)$ which vanish for negative arguments with the property that

$$||k||_p\equiv\left\{\int_0^\infty |k(t)|^p\,dt\right\}^{1/p}<\infty.$$

The spaces $L^p_{(-\infty,\infty)}$ on the real axis $(-\infty, \infty)$ are similary defined. A Lebesgue measurable function is said to be of class $L^p{}_{\mathrm{loc}\,(R)}$, $1\leqslant p\leqslant\infty$, if it vanishes on negative arguments and satisfies

$$\int_0^T |k(t)|^p\,dt<\infty,$$

for any finite T.

The space of testing functions of bounded supports is denoted by $\mathfrak{D}$ and its dual space is denoted by $\mathfrak{D}'$. Furthermore, we let $\mathfrak{D}'_{(R)}$ be the space of distributions whose supports are contained in the non-negative real axis. The space of test ing functions of rapid descent is written by $\mathfrak{S}$ and its dual space is written by $\mathfrak{S}'$.

The Fourier transform of $k\in\mathfrak{S}'$ is denoted by

$$\langle\mathfrak{F}k, \phi\rangle=\langle k, \mathfrak{F}\phi\rangle, \qquad \phi\in\mathfrak{S}.$$

If $k(t)\in L^1_{(-\infty,\infty)}$, then the ordinary Fourier transform $K(i\lambda)$ of $k(t)$ exists and $\langle\mathfrak{F}k, \phi\rangle=\langle K(i\lambda), \phi\rangle$ for any $\phi\in\mathfrak{S}$. If $k(t)\in L^2_{(-\infty,\infty)}$, then the limit in the mean Fourier transform $K(i\lambda)$ of $k(t)$ exists and $\langle\mathfrak{F}k, \phi\rangle=\langle K(i\lambda), \phi\rangle$ for any $\phi\in\mathfrak{S}$. In both cases we may identify $\mathfrak{F}k$ with $K(i\lambda)$. Moreover, the Fourier transforms of the delta functional and its derivatives are given by

$$\mathfrak{F}\delta^{(n-1)}=(i\lambda)^{n-1}, \qquad n\geqslant 1.$$

If k belongs to $\mathfrak{D}'_{(R)}$ and if $e^{-\sigma t}k(t)\in\mathfrak{S}'$ for $\sigma>\sigma'$, then the Laplace transform of $k(t)$ is given by

$$K(s)=\mathfrak{L}k=\mathfrak{F}\{e^{-\sigma t}k(t)\},\quad \sigma>\sigma',\quad s=\sigma+i\lambda.$$

$K(s)$ is an analytic function in its region of convergence $\mathrm{Re}\, s>\sigma'$. The Laplace transforms of the delta functional and its derivatives are given by

$$\mathfrak{L}\delta^{(n-1)}=s^{n-1},\quad n\geqslant 1,\quad -\infty<\mathrm{Re}\, s<\infty.$$

Let $k*l$ denote the convolution of distributions k and l whenever the convolution is defined. Let u be a distribution with the representation

$$u(t)=\sum_{j=0}^{n}\delta^{(j)}*k_j(t),\quad k_j(t)\in L^p_{(-\infty,\infty)},\quad 0\leqslant n<\infty$$

and let v be a distribution with the representation

$$v(t)=\sum_{j=0}^{m}\delta^{(j)}*l_j(t),\quad l_j(t)\in L^{p'}_{(-\infty,\infty)},\quad 0\leqslant m<\infty,$$

where $(1/p)+(1/p')-1>0$. Then, it can be shown that the covolution $u*v$ is well defined and belongs to $\mathfrak{S}'$. Moreover, the Fourier transform of $u*v$ is given by

$$\mathfrak{F}[u*v]=\mathfrak{F}[u]\mathfrak{F}[v]=\sum_{j=0}^{n}(i\lambda)^j K_j(i\lambda)\sum_{j=0}^{m}(i\lambda)^j L_j(i\lambda), \tag{1}$$

where $K_j(i\lambda)$ and $L_j(i\lambda)$ are the Fourier transform of functions $k_j(t)$ and $l_j(t)$ respectively. (SCHWARTZ, 1966, p. 270)

3. Existence of $(\delta+cg)^{-1}$ in $\mathfrak{D}'_{(R)}$

3-1. A Theorem Related to the Result of Paley and Wiener

Let $\mathfrak{T}'_n$ be a class of distributions such that

$$g(t)=\sum_{k=1}^{n}a_k\delta^{(k-1)}(t)+a_0 1_+(t)+g_1(t),\qquad (1\leqslant n<\infty)$$

where

i) $g_1(t)$ is a real-valued function in $L^1_{(R)}$,

ii) a_k is a real constant and $a_n\neq 0$,

iii) $\delta^{(k)}(t)$ is the k-th derivative of the delta functional,

iv) $1_+(t)$ is the Heaviside unit step function.

PALEY and WIENER (1934) first showed the existence of $(\delta+cg)^{-1}$ in $\mathfrak{D}'_{(R)}$ when $g(t)=g_1(t)$. We shall show their result in a slightly generalized form.

Theorem 1. *Let $g(t)$ be in $\bigcap_{p=1}^{m} L^p_{(R)}$, where $1\leqslant m<\infty$. If the Laplace transform $G(s)$ of g satisfies $|1+cG(s)|\neq 0$ in $\mathrm{Re}\, s\geqslant 0$, then there exists a function $\hat{g}(t)$ in*

$\bigcap_{p=1}^{m} L^p_{(R)}$ *such that* $\hat{G}(i\lambda)=\mathfrak{F}\hat{g}=\dfrac{G(i\lambda)}{1+cG(i\lambda)}$.

Proof. We may assume $c=1$ without any loss of generality. Define the function $\phi_A(\lambda)$ by

$$\phi_A(\lambda)=\begin{cases} 1, & |\lambda|<A; \\ 2-\dfrac{|\lambda|}{A}, & A\leqslant|\lambda|\leqslant 2A; \\ 0, & |\lambda|>2A; \end{cases}$$

and put

$$\hat{G}(i\lambda)=\phi_A(\lambda)\hat{G}(i\lambda)+(1-\phi_A(\lambda))\hat{G}(i\lambda)$$
$$\equiv\hat{G}_1(i\lambda)+\hat{G}_2(i\lambda).$$

We first show that there exists a function $\hat{g}_1(t)$ in $\bigcap_{p=1}^{m} L^p_{(-\infty,\infty)}$ such that $\hat{G}_1(i\lambda)=\mathfrak{F}\hat{g}_1$. Now, from the definitions of $\hat{G}_1(i\lambda)$ and $\phi_A(\lambda)$, we have

$$\hat{G}_1(i\lambda)=\begin{cases} \dfrac{\phi_A(\lambda)G(i\lambda)}{\phi_{2A}(\lambda)[1+G(i\lambda)]}, & |\lambda|<2A; \\ 0, & |\lambda|\geqslant 2A. \end{cases}$$

Since $\phi_{2A}(\lambda)$ belongs to $L^1_{(-\infty,\infty)}$ the inverse Fourier transform exists and $\mathfrak{F}^{-1}\phi_{2A}(\lambda)=\dfrac{1}{\pi}\left\{\dfrac{\cos 2At-\cos 4At}{2At^2}\right\}\equiv p_{2A}(t)\in L^1_{(-\infty,\infty)}$. From the assumptions, $\phi_{2A}(\lambda)(1+G(i\lambda))$ is never zero for $\lambda\in[-2A, 2A]$. Thus, there exists a function q in $L^1_{(-\infty,\infty)}$ such that its Fourier transform $Q(i\lambda)$ has the property

$$Q(i\lambda)=\frac{1}{\phi_{2A}(\lambda)[1+G(i\lambda)]}, \qquad \lambda\in[-2A, 2A],$$

[WIENER, 1958, p. 91]. Therefore, we have $\hat{G}_1(i\lambda)=\mathfrak{F}\{q*p_A*g\}$ and $q*p_A*g\in\bigcap_{p=1}^{m} L^p_{(-\infty,\infty)}$ by the Hausdorff-Young inequality.

On the other hand $\hat{G}_2(i\lambda)$ can be written by

$$\hat{G}_2(i\lambda)=\begin{cases} [1-\phi_A(\lambda)]\dfrac{G(i\lambda)[1-\phi_{A/2}(\lambda)]}{1+G(i\lambda)[1-\phi_{A/2}(\lambda)]}, & |\lambda|\geqslant A; \\ 0, & |\lambda|<A. \end{cases}$$

Letting A sufficiently large, we have $|G(i\lambda)[1-\phi_{A/2}(\lambda)]|<1$, for all $\lambda\in(-\infty, \infty)$, from the Riemann-Lebesgue lemma. Hence, we get

$$\hat{G}_2(i\lambda)=[1-\phi_A(\lambda)]\hat{\Phi}(\lambda),$$

where $\hat{\Phi}(\lambda)=\sum_{n=1}^{\infty}(-1)^n[G(i\lambda)(1-\phi_{A/2}(\lambda))]^n$. From the denitions of $\phi_{A/2}$ and $G(i\lambda)$, we have

$$\hat{\Phi}(\lambda)=\sum_{n=1}^{\infty}(-1)^n[\mathfrak{F}(-\psi_{A/2})]^n=\sum_{n=1}^{\infty}\mathfrak{F}\{\psi^{n*}_{A/2}\},$$

where $\psi_{A/2}^{n*}=\psi_{A/2}*\psi_{A/2}\cdots*\psi_{A/2}$ (a convolution of n distributions) and $\mathfrak{F}\{-\psi_{A/2}\}=\mathfrak{F}\{g-g*p_{A/2}\}=G(i\lambda)[1-\hat{\phi}_{A/2}(\lambda)]$. Obviously, $\psi_{A/2}$ belongs to $\bigcap_{p=1}^{m} L^p_{(-\infty,\infty)}$. Therefore, we have $\|\psi_{A/2}^{n*}\|_1\leqslant\|\psi_{A/2}\|_1^n$ and $\|\psi_{A/2}^{n*}\|_p\leqslant\|\psi_{A/2}^{(n-1)*}\|_p\cdot\|\psi_{A/2}\|_1$, for $1<p\leqslant m$. This implies that $\psi_{A/2}^{n*}$ belongs to $\bigcap_{p=1}^{m} L^p_{(-\infty,\infty)}$ for any $n>0$. Moreover, after some calculations we have

$$\|\psi_{A/2}\|_p\leqslant 2\|g-F_A\|_p+\|g-F_{A/2}\|_p, \qquad 1\leqslant p\leqslant m,$$

where $F_A(t)=\dfrac{1}{2\pi A}\displaystyle\int_{-\infty}^{\infty} g(t+\tau)\left\{\frac{\sin A\tau/2}{\tau/2}\right\}^2 dt$. Since the Fejér integral $F_A(t)$ converges to g in L^p norm $(1\leqslant p\leqslant m)$, we have $\|\psi_{A/2}\|_p<1$ for sufficiently large A. Thus the series $\sum_{n=1}^{\infty}\psi_{A/2}^{n*}$ converges in L^p norm $(1\leqslant p\leqslant m)$. Since $L^p_{(-\infty,\infty)}$ is complete, there exists a function ϕ_p in $L^p_{(-\infty,\infty)}$, $1\leqslant p\leqslant m$, such that $\lim_{N\to\infty}\sum_{n=1}^{N}\psi_{A/2}^{n*}=\phi_p$ in $L^p_{(-\infty,\infty)}$. Next we shall show that there exists a common function ϕ in $\bigcap_{p=1}^{m} L^p_{(-\infty,\infty)}$ such that $\lim_{N\to\infty}\sum_{n=1}^{N}\psi_{A/2}^{n*}=\phi$ in L^p norm $(1\leqslant p\leqslant m)$. This can be easily shown, since for any compact set K in R^1, we have $\lim_{N\to\infty}\int_K\left|\sum_{n=1}^{N}\psi_{A/2}^{n*}-\phi_1\right|dt=0$ and $\lim_{N\to\infty}\int_K\left|\sum_{n=1}^{N}\psi_{A/2}^{n*}-\phi_p\right|\times dt\leqslant\lim_{N\to\infty}K^{1/q}\left\|\sum_{n=1}^{N}\psi_{A/2}^{n*}-\phi_p\right\|_p=0$, where $1/p+1/q=1$ with $1\leqslant p\leqslant m$. Therefore $\phi\equiv\phi_1=\phi_p$, a.e. in $(-\infty,\infty)$, where $1\leqslant p\leqslant m$, and we have $\mathfrak{F}\phi=\mathfrak{F}\sum_{n=1}^{\infty}\psi_{A/2}^{n*}=\sum_{n=1}^{\infty}\mathfrak{F}\psi_{A/2}^{n*}=\hat{\Phi}(\lambda)$ by interchanging the summation and the integral. Consequently, we have $\mathfrak{F}\hat{g}_2=[1-\hat{\phi}_A(\lambda)]\hat{\Phi}(\lambda)$ and $\hat{g}_2\in\bigcap_{p=1}^{m} L^p_{(-\infty,\infty)}$.

Now, we shall show that $g(t)=0$ for $t<0$. Since $g(t)\equiv g_1(t)+g_2(t)\in L^1_{(-\infty,\infty)}$, we have

$$\int_{-\infty}^{0}\hat{g}(t)e^{-i\lambda t}dt+\int_0^{\infty}\hat{g}(t)e^{-i\lambda t}dt=\frac{G(i\lambda)}{1+G(i\lambda)}. \tag{2}$$

On the other hand, $\dfrac{G(s)}{1+G(s)}$ is analytic in $\operatorname{Re}s>0$, continuous and bounded in $\operatorname{Re}s\geqslant 0$ by the assumption. Define $L_1(s)$ and $L_2(s)$ by

$$L_1(s)=\frac{G(s)}{1+G(s)}-\int_0^{\infty}\hat{g}(t)e^{-st}dt, \qquad \operatorname{Re}s\geqslant 0;$$

$$L_2(s)=\int_{-\infty}^{0}\hat{g}(t)e^{-st}dt, \qquad \operatorname{Re}s\leqslant 0.$$

Obviously, $L_1(s)$ is bounded and continuous in $\operatorname{Re}s\geqslant 0$ and analytic in $\operatorname{Re}s>0$; $L_2(s)$ is bounded and continuous in $\operatorname{Re}s\leqslant 0$ and analytic in $\operatorname{Re}s<0$. Moreover, from (2), we have $L_1(s)=L_2(s)$ on $\operatorname{Re}s=0$. Hence $L_1(s)$ and $L_2(s)$ are analytic continuations of each other and they reduce to a constant. Since $\lim_{|s|\to\infty}|L_2(s)|=0$ by the Riemann-Lebesgue lemma, this constant must be 0. This implies

$$\frac{G(s)}{1+G(s)}=\int_0^{\infty}\hat{g}(t)e^{-st}dt, \qquad \operatorname{Re}s\geqslant 0,$$

and

$$\int_{-\infty}^{0} \hat{g}(t)e^{-st}\,dt=0, \qquad \text{Re}\, s<0.$$

Thus, we have $\hat{g}(t)=0$, a.e. t, $t\in(-\infty, 0)$ and the theorem is established.

REMARK 1. Note that since $\dfrac{1}{1+cG(i\lambda)}=1-\dfrac{G(i\lambda)}{1+cG(i\lambda)}$ for any $\lambda\in(-\infty, \infty)$, Theorem 1 also states the existence of $(\delta+cg)^{-1}$ in $\mathfrak{D}'_{(R)}$. Actually, if $g\in L^1_{(R)}$ then $(\delta+cg)^{-1}$ can be written by $(\delta+cg)^{-1}=\delta+\hat{g}$ where $\hat{g}=-\mathfrak{F}^{-1}\left\{\dfrac{cG(i\lambda)}{1+cG(i\lambda)}\right\}\in L^1_{(R)}$.

3-2. Extension of the Result to a General Case

Next we shall show the conditions for existence of $(\delta+cg)^{-1}$ in $\mathfrak{D}'_{(R)}$ when g belongs to $\mathfrak{T}'_n$ and $n\geqslant 1$.

Theorem 2. *Let g be in $\mathfrak{T}'_n$. Suppose that g satisfies one of the following two conditions for some nonzero constant c;*

(3) $|\mathfrak{L}\{\delta+cg\}|\neq 0$, *in* $\text{Re}\, s\geqslant 0$ *when* $a_0=0$,

(4) $|\mathfrak{L}\{\delta^{(1)}+c\delta^{(1)}*g\}|\neq 0$, *in* $\text{Re}\, s\geqslant 0$ *when* $a_0\neq 0$.

Then the following results hold.

(A) *If $n=1$, $a_0\neq 0$, then there exists a function h in $L^2_{(R)}$ such that*

$$(\delta+cg)*(\delta^{(1)}*h)=\delta.$$

(B) *If $n=1$, $a_0=0$, then there exist a function h in $L^1_{(R)}$ and a nonzero number d such that*

$$(\delta+cg)*(d\delta+h)=\delta.$$

(C) *If $n\geqslant 2$, then there exists a function h in $L^2_{(R)}$ such that*

$$(\delta+cg)*h=\delta.$$

Proof of (A). From the assumption ii), $|1+cG(s)|$ is nonzero in $\text{Re}\, s>0$. Define $K_0(s)$ by

$$K_0(s)=\frac{1}{1+ca_1+ca_0/s+cG_1(s)}, \qquad \text{Re}\, s>0.$$

On the other hand we can define $K_1(s)$ and $H(s)$ by

$$K_1(s)=\frac{s}{a'_0+a'_1 s+csG_1(s)}\equiv sH(s), \quad \text{Re}\, s\geqslant 0,$$

where $a'_0=ca_0$, $a'_1=1+ca_1$. Moreover, we may assume $a'_1\neq 0$ without any loss of generality. The assumption ii) of this lemma implies that $H(s)$ and $K_1(s)$ are continuous in $\text{Re}\, s\geqslant 0$ and regular in $\text{Re}\, s>0$.

We shall show next that $H(s)\equiv H(\sigma+i\lambda)$ belongs to the Hardy class H^2 for $\sigma>0$. Since $g_1(t)$ belongs to $L^1_{(R)}$, we see that for any $\varepsilon>0$, there exists a constant M such that $|G(s)|<\varepsilon$ for any s with $|s|>M$. Setting ε safficiently small, we can choose M which satisfies the following two conditions for any s with $|s|>M$;

$$|sa_1'+a_0'|\geqslant 2\varepsilon|c||s|$$

and

$$|s|\geqslant 2\left|\frac{a_0'}{a_1'}\right|.$$

We, thus, have

$$|H(s)|=\left|\frac{1}{sa_1'+a_0'+cs\,G_1(s)}\right|\leqslant\frac{1}{|sa_1'+a_0'|-\varepsilon|c||s|}$$

$$\leqslant\frac{2}{|sa_1'+a_0'|}\leqslant\frac{4}{|a_1'|}\frac{1}{|s|}$$

$$\leqslant N\frac{1}{|s|}\leqslant\frac{N}{M}. \tag{5}$$

Let Γ be the bounded closed region defined by $\Gamma=\{s;\ |s|\leqslant M,\ \mathrm{Re}\,s\geqslant 0\}$, where M is the constant defined previously. Then, from the continuity of $H(s)$, we have $|H(s)|\leqslant N_0$ in Γ. Therefore, in $\mathrm{Re}\,s\geqslant 0$, we get

$$|H(s)|\leqslant N_1, \tag{6}$$

where $N_1=\max\{N_0,\ N/M\}$. Now, from (5) and (6) we see

$$\int_{-\infty}^{\infty}|H(\sigma+i\lambda)|^2\,d\lambda=\int_{|\lambda|\leqslant M}|H(\sigma+i\lambda)|^2\,d\lambda+\int_{|\lambda|>M}|H(\sigma+i\lambda)|^2\,d\lambda$$

$$\leqslant 2N_1^2M+\int_{|\lambda|>M}\frac{N^2}{\sigma^2+\lambda^2}\,d\lambda\leqslant 2N_1^2M+\int_{|\lambda|>M}\frac{N^2}{\lambda^2}\,d\lambda<\infty,$$

for any $\sigma\geqslant 0$. This implies that $H(s)$ belongs to H^2 in $\mathrm{Re}\,s>0$. From the known theorem (YOSHIDA 1965, p. 163) there exists a function $\widetilde{H}(i\lambda)$ in $L^2_{(-\infty,\infty)}$ satisfying the following condtions.

i) $\widetilde{H}(i\lambda)=\operatorname*{l.i.m.}_{\sigma\to 0+}H(\sigma+i\lambda)$,

ii) the inverse Fourier transform $\tilde{h}(t)$ of $\widetilde{H}(i\lambda)$ is in $L^2_{(R)}$ and $\mathfrak{L}\tilde{h}(t)=\widetilde{H}(s)$ in $\mathrm{Re}\,s>0$,

iii) $\widetilde{H}(i\lambda)=\lim_{\sigma\to 0+}H(\sigma+i\lambda)$, a.a. λ.

Since $H(\sigma+i\lambda)$ is continuous in $\sigma\geqslant 0$, we have $H(i\lambda)=\widetilde{H}(i\lambda)$, a.a. λ. Therefore, the inverse Fourier transform $h(t)$ of $H(i\lambda)=\dfrac{1}{a_0'+a_1'(i\lambda)+c(i\lambda)G_1(i\lambda)}$ belongs to $L^2_{(R)}$ and $\mathfrak{L}h=H(s)$ in $\mathrm{Re}\,s>0$. Define k_1 by $k_1=\delta^{(1)}*h$. Then, k_1 belongs to $\mathfrak{D}'_{(R)}$ and Laplace transformable. We, thus, have

$$\mathfrak{L}\{k_1\}=sH(s),\qquad \mathrm{Re}\,s>0.$$

Since $K_1(s)=K_0(s)$ in $\mathrm{Re}\,s>0$, we obtain

$$\{1+cG(s)\}K_1(s)=\{1+cG(s)\}K_0(s)=1,\qquad \mathrm{Re}\,s>0.$$

This implies

$$\mathfrak{L}\{(\delta+cg)*k_1-\delta\}=0,\qquad \mathrm{Re}\,s>0,$$

and by the uniqueness theorem of the Laplace transformation, we get

$$\{\delta+cg\}*\{\delta^{(1)}*h\}=\delta,$$

where $h(t)\in L^2_{(R)}$ and $\mathfrak{F}h=\dfrac{1}{a_0'+a_1'(i\lambda)+c(i\lambda)G_1(i\lambda)}$, and part (A) is proved.

(B) is a direct consequence of Theorem 1. See also Remark 1 of Theorem 1.

Proof of (C). We only deal with the case in which $a_0\neq 0$. The case in which $a_0=0$ can be treated in a similar way.

Define $H_0(s)$ by

$$H_0(s)=\frac{1}{1+cG(s)}, \qquad \text{Re}\, s>0,$$

and define $H(s)$ by

$$(7) \qquad H(s)=\frac{s}{\sum_{k=0}^{n} a_k' s^k+csG_1(s)}, \qquad \text{Re}\, s\geqslant 0,$$

where $a_0'=ca_0$, $a_1'=1+ca_1$, $a_i'=ca_i$ for $2\leqslant i\leqslant n$. Following the procedure given in the proof of (A), we may show that there exists a constant M such that for any s with $|s|>M$, the following inequality holds.

$$|H(s)|\leqslant\frac{4}{|a_n'|}\frac{1}{|s|^{n-1}}\leqslant\frac{N}{M^{n-1}}.$$

Since $H(s)$ is continuous in $\text{Re}\, s\geqslant 0$, we can verify that $H(s)$ belongs to H^2 in $\sigma>0$. Therefore, the inverse Fourier transform $h(t)$ of $H(i\lambda)$ belongs to $L^2_{(R)}$ and $\mathfrak{L}h=H(s)=H_0(s)$ in $\text{Re}\, s>0$. Consequently, we have

$$(\delta+cg)*h=\delta,$$

where $h\in L^2_{(R)}$ and $\mathfrak{F}h=\dfrac{(i\lambda)}{\sum_{k=0}^{n} a_k'(i\lambda)^k+c(i\lambda)G_1(i\lambda)}$. This completes the proof of part (C).

What we have shown in Theorem 2 is that if g satisfies the condition of Theorem 2 then an inverse convolution operator of $(\delta+cg)$ exists and unique in $\mathfrak{D}'_{(R)}$. Furthermore, this inverse has one of the three forms indicated in Theorem 2. We shall write this inverse by $(\delta+cg)^{-1}$.

A direct consequence of Theorem 2 is

Corollary 2-1. *Let g be in $\mathfrak{T}_n'$ and satisfy either* (3) *or* (4) *of Theorerem* 2. *Then the following results hold.*

i) *If $n=1$, $a_0\neq 0$, then there exist a nonzero number $\hat{d}$ and a function $\hat{h}$ in $L^2_{(R)}$ such that*

$$(\delta+cg)^{-1}*g=\hat{d}\delta+\delta^{(1)}*\hat{h}.$$

ii) *If $n=1$, $a_0=0$, $a_1\neq 0$, then there exist a nonzero number $\hat{d}$ and a function $\hat{h}$ in $L^1_{(R)}$ such that*

$$(\delta+cg)^{-1}*g=\hat{d}\delta+\hat{h}.$$

iii) *If* $n=1$, $a_0=0$, $a_1=0$, *then there exists a function* $\hat{h}$ *in* $L^1_{(R)}$ *such that*

$$(\delta+cg)^{-1}*g=\hat{h}.$$

iv) *If* $n\geqslant 2$, *then there exist a nonzero number* $\hat{d}$ *and a function* $\hat{h}$ *in* $L^2_{(R)}$ *such that*

$$(\delta+cg)^{-1}*g=\hat{d}\delta+\hat{h}.$$

Proof. If $n\geqslant 2$ and $a_0=0$, then we rewrite $\mathfrak{F}\{(\delta+cg)^{-1}*g\}$ in the following form and apply Theorem 2 (C).

$$\mathfrak{F}\{(\delta+cg)^{-1}*g\}=\frac{G(i\lambda)}{1+cG(i\lambda)}=\frac{1}{c}\left\{1-\frac{1}{1+cG(i\lambda)}\right\}.$$

When $n=1$ and $a_0=0$, $a_1\neq 0$, we may assume $1+a_1c\neq 0$, without any loss of generality. Hence, we have

$$\mathfrak{F}\{(\delta+cg)^{-1}*g\}=\frac{a_1}{1+ca_1}-\frac{1}{c(1+ca_1)}\,\frac{\tilde{G}_1(i\lambda)}{1+\tilde{G}_1(i\lambda)}.$$

where $\tilde{G}_1(i\lambda)=(c/1+ca_1)G_1(i\lambda)$. By Theorem 1, we see that there exists a function h in $L^1_{(R)}$ such that $\mathfrak{F}h=\dfrac{1}{c(1+ca_1)}\,\dfrac{\tilde{G}_1(i\lambda)}{1+\tilde{G}_1(i\lambda)}$. When $n=1$, $a_1=0$, $a_0=0$, then the corollary obviously holds from Theorem 2 (B). Now, when $n\geqslant 2$ and $a_0\neq 0$, the relation (7) of Lemma 2 (C) states that the Fourier transform of h is given by

$$\mathfrak{F}h=(i\lambda)\left\{\sum_{k=0}^{n}a'_k(i\lambda)^k+c(i\lambda)G_1(i\lambda)\right\}^{-1}.$$

Therefore, if g is defined as in the case (i) or (iv) with $a_0\neq 0$, then we may assume that $(\delta+cg)^{-1}$ has the representation $(\delta+cg)^{-1}=\delta^{(1)}*q$, where $q(t)\in L^2_{(R)}$ and $\mathfrak{F}q=\dfrac{1}{\sum_{k=0}^{n}a'_k(i\lambda)^k+c(i\lambda)G_1(i\lambda)}$, $n\geqslant 1$. Since the convolution $\delta^{(1)}*q*1_+$ is associative and commutative operation in $\mathfrak{D}'_{(R)}$, we have

$$(\delta+cg)^{-1}*g=\sum_{k=0}^{n}a_k\delta^{(k)}*q+\delta^{(1)}*q*g_1 \qquad (n\geqslant 1).$$

Noting that $q(t)\in L^2_{(R)}$ and $g_1(t)\in L^1_{(R)}$, from (1) we have

$$\mathfrak{F}\{(\delta+cg)^{-1}*g\}=\frac{\sum_{k=0}^{n}a_k(i\lambda)^k+(i\lambda)G_1(i\lambda)}{\sum_{k=0}^{n}a'_k(i\lambda)^k+c(i\lambda)G_1(i\lambda)}. \tag{8}$$

Recalling the definitions of a'_i, $0\leqslant i\leqslant n$, (8) can be rewritten by

$$\mathfrak{F}[(\delta+cg)^{-1}*g]=\frac{1}{c}\left[1-\frac{i\lambda}{\sum_{k=0}^{n}a'_k(i\lambda)^k+c(i\lambda)G_1(i\lambda)}\right], \qquad n\geqslant 1.$$

If we define $\hat{h}$ by $\hat{h}=-\frac{1}{c}\mathfrak{F}^{-1}\left[\frac{i\lambda}{\sum_{k=0}^{n} a'_k(i\lambda)^k+c(i\lambda)G_1(i\lambda)}\right]$ when $n\geqq 2$, $a_0\neq 0$ and $\hat{h}=-\frac{1}{c}\mathfrak{F}^{-1}\left[\frac{1}{a'_0+(i\lambda)[a'_1+cG_1(i\lambda)]}\right]$ when $n=1$, $a_0\neq 0$, then we obtain the statement of this corollary.

The proof of Corollary 2-1 shows that the corollary can be restated as in the following.

Corollary 2-2. Let g be in $\mathfrak{T}'_n$ and satisfy either (3) or (4) of Theorem 2. Then $\hat{g}$ is given by

$$\hat{g}=\frac{1}{c}\{\delta-(\delta+cg)^{-1}\}.$$

4. Integrability

The functions that belong to the class $L^1_{(R)}\cap L^2_{(R)}$ are easy to handle. This is because; i) the Fourier transform of a function in $L^1_{(R)}\cap L^2_{(R)}$ is continuous, bounded and satisfies the Parceval relation, ii) the Laplace transform exists in $\mathrm{Re}\, s>0$. Since $\mathfrak{F}\hat{g}$ and $\mathfrak{F}(\delta+cg)^{-1}$ are bounded, continuous and both $\hat{g}$ and $(\delta+cg)^{-1}$ are Laplace transformable, it is natural to study the conditions which assure the existence of h and $\hat{h}$ in $L^1_{(R)}\cap L^2_{(R)}$.

First we shall show

Theorem 3. *Let g be in $\mathfrak{T}'_1$ and satisfy either* (3) *or* (4) *of Theorem* 2. *If $g_1(t)$ belongs to $L^1_{(R)}\cap L^2_{(R)}$ and satisfies $\frac{ca_0}{1+ca_1}>0$ when $a_0\neq 0$, then there exist a function $h\in L^1_{(R)}\cap L^2_{(R)}$ and a constant $d\neq 0$ such that*

$$(\delta+cg)^{-1}=d\delta+h.$$

Proof. If $a_0=0$, then Theorem 3 is a special case of Theorem 1. Therefore, we only deal with the case in which $a_0\neq 0$.

From the assumption $b=\frac{ca_0}{1+ca_1}>0$, we have

$$H(s)=\frac{s}{ca_0+(1+ca_1)s+csG_1(s)}$$

$$=\frac{\eta s}{s+b}\cdot\frac{1}{1+\frac{\gamma sG_1(s)}{s+b}},$$

where $\eta=\frac{1}{1+ca_1}$, $\gamma=c\eta$. The inverse Laplace transform of $\frac{s}{s+b}$ is given by

$$\mathfrak{L}^{-1}\left\{\frac{s}{s+b}\right\}=\delta(t)-b1_{+}(t)e^{-|b|t}=\delta(t)+\hat{k}(t).$$

Thus, we have

$$\mathfrak{L}^{-1}\left\{\frac{\gamma sG_1(s)}{s+b}\right\}=\gamma g_1+\gamma \hat{k}*g_1=\tilde{k}$$

and $\tilde{k}$ belongs to $L^1_{(R)}\cap L^2_{(R)}$. From Theorem 1, there exist a nonzero constant $\tilde{d}\neq 0$ and a function $\tilde{h}$ in $L^1_{(R)}\cap L^2_{(R)}$ such that

$$\mathfrak{L}\{\tilde{d}\delta+\tilde{h}\}=\frac{1}{1+\dfrac{\gamma sG_1(s)}{s+b}}, \qquad \operatorname{Re} s\geqslant 0.$$

Consequently, we obtain

$$\mathfrak{L}^{-1}\{H(s)\}=\gamma(\tilde{d}\delta+\tilde{h}+\tilde{d}\hat{k}+\hat{k}*\tilde{h})$$

so that the proof of the theorem is complete.

REMARK 2. Note that the assumption $g_1\in L^1_{(R)}\cap L^2_{(R)}$ can not be replaced by the condition $g_1\in L^1_{(R)}$ in Theorem 3.

When $n\geqslant 2$, we already know from Theorem 2 that h belongs to $L^2_{(R)}$. Therefore, the problem is reduced to the following; given a function h in $L^2_{(R)}$ whose Fourier transform is bounded and continuous, find the condition for h to be in $L^1_{(R)}$. The known results to this problem have difficulties in applying to our cases unless $g_1(t)$ is restricted to a special class of functions. (HILL, TAMARKIN, 1933) Instead, we shall show

Theorem 4. *Let g be in $\mathfrak{T}'_n$ and $n\geqslant 2$. Suppose that g satisfies one of the following two conditions for some nonzero constant c;*

(9) $|\mathfrak{L}\{\delta+cg\}|\neq 0$ *and* $\left|\mathfrak{L}\left\{\sum_{k=1}^{n} a'_k\delta^{(k-1)}\right\}\right|\neq 0$,
in $\operatorname{Re} s\geqslant 0$ *when* $a_0=0$,

(10) $|\mathfrak{L}\{\delta^{(1)}+c\delta^{(1)}*g\}|\neq 0$ *and* $\left|\mathfrak{L}\left\{\sum_{k=0}^{n} a'_k\delta^{(k)}\right\}\right|\neq 0$,
in $\operatorname{Re} s>0$ *when* $a_0\neq 0$,

where $a'_0=ca_0$, $a'_1=1+ca_1$, $a'_i=ca_i$ $(i=2, 3, \cdots, n)$. *Then, there exists a function h in* $\bigcap_{p=1}^{\infty} L^p_{(R)}$ *such that*

$$(\delta+cg)^{-1}=h.$$

Proof. We only show the result when $a_0\neq 0$. Define $H(s)$ by

$$H(s)=\frac{s}{\sum_{k=0}^{n} a'_k s^k+csG_1(s)}, \qquad \operatorname{Re} s\geqslant 0.$$

Obviously, $H(s)$ is continuous in $\operatorname{Re} s\geqslant 0$, analytic in $\operatorname{Re} s>0$ and satisfies $H(s)=\dfrac{1}{1+cG(s)}$ in $\operatorname{Re} s>0$. Since $\left|\sum_{k=0}^{n} a'_k s^k\right|\neq 0$ in $\operatorname{Re} s\geqslant 0$, we have

$$H(s)=Q(s)\frac{1}{1+cQ(s)G_1(s)},$$

where

$$Q(s)=\frac{s}{\sum_{k=0}^{n} a'_k s^k}.$$

Noting that $n \geqslant 2$ and by the partial fraction expansion, we get

$$Q(s)=\sum_{i=1}^{n}\sum_{j=1}^{m_i} A_{ij}\frac{1}{(s-b_i)^{m_i-j+1}}, \qquad \left(\sum_{i=1}^{r} m_i=n\right)$$

and

$$q(t)=\sum_{i=1}^{r}\sum_{j=1}^{m_i} a_{ij}t^{j-1}1_{+}(t)e^{b_i t}.$$

Since $\mathrm{Re}\, b_i<0$ for $i=1, 2, \cdots, r$, $q(t)$ belongs to $L^p_{(R)}$, where $1 \leqslant p<\infty$. This implies that $cq*g_1$ belongs to $L^p_{(R)}$, $1 \leqslant p<\infty$, by Hausdorff-Young inequality and $\mathfrak{L}\{cq*g_1\}=\frac{csG_1(s)}{\sum_{k=0}^{n} a'_k s^k}$. Since $|1+cQ(s)G_1(s)| \neq 0$ in $\mathrm{Re}\, s \geqslant 0$, there exists a function $\hat{k}$ in $L^p_{(R)}$, $1 \leqslant p<\infty$, such that $\mathfrak{L}\{\delta+\hat{k}\}=\frac{1}{1+cQ(s)G_1(s)}$. Hence, we have $H(s)=\mathfrak{L}\{q*(\delta+\hat{k})\}$ in $\mathrm{Re}\, s \geqslant 0$. This completes the proof of Theorem 4.

REMARK 3. We note that condition $\frac{ca_0}{1+ca_1}>0$ in Theorem 2 is equivalent to $\left|\mathfrak{L}\left\{\sum_{k=0}^{1} a'_k \delta^{(k)}\right\}\right| \neq 0$, in $\mathrm{Re}\, s \geqslant 0$ and $1+ca_1 \neq 0$. (See also Corollary 2-2 of KAWASHIMA, 1973-a)

5. Differentiability and Support of the Operator

In this section we shall mainly examine the problem of the differentiability of the operator $(\delta+cg)^{-1}$. For this aim we introduce some notations and concepts related to the Sobolev Space $\mathfrak{D}^m_{L^2}$.

Let $\mathfrak{D}^m_{L^2}$ denote the set of distribution k such that all derivatives in the sense of distributions of order $\leqslant m$ of k belong to $L^2_{(-\infty,\infty)}$. $\mathfrak{D}^m_{L^2}$ is normed by

$$||k||_{m,L^2}\equiv\left(\sum_{i=0}^{m} ||\delta^{(i)}*k||^2\right)^{1/2},$$

where $||\cdot||$ denotes the L^2 norm. With this norm $\mathfrak{D}^m_{L^2}$ becomes a Hilbert space.

Let $\mathfrak{B}^m$ be the class of functions k such that all derivatives (in the ordinary sense) of order $\leqslant m$ of k exist and are bounded and continuous. The Space $\mathfrak{B}^m$ is equipped with the norm

$$|k(x)|_m=\sum_{i=0}^{m} \sup_{x\in R^1}\left|\frac{d^i k(x)}{dx^i}\right|.$$

Obviously, $\mathfrak{B}^m$ becomes a Banach space with this norm.

The following lemma is well known.

Lemma 5. *Let k be in $\mathfrak{D}^m_{L^2}$. Then k belongs to $\mathfrak{B}^{m-1}$ and its norm satisfies*

$$|k(x)|_{m-1} \leqslant c\|k\|_{m,L^2}.$$

Proof. See TREVES 1967, p. 331.

Lemma 5 tells us that a distribution in $\mathfrak{D}^m_{L^2}$ can be identified with the function having bounded and continuous derivatives up to the order $m-1$.

By a direct application of Lemma 5, we have

Theorem 5. *Suppose* g *in* $\mathfrak{T}'_n$, $n \geqslant 2$, *satisfies either* (9) *or* (10) *of Theorem* 4. *And suppose* $\frac{ca_0}{1+ca_1} > 0$ *when* $n=1$, $a_0 \neq 0$ *and* $1+ca_1 \neq 0$ *when* $n=1$, $a_0=0$. *If* $g_1(t) \in L^1_{(R)}$ *has derivatives* $g_1^{(j)}(t)$ *up to the order* m *and* $g_1^{(j)}(t) \in L^1_{(R)}$, $1 \leqslant j \leqslant m$, *then the following results hold.*

(A) *If* $n=1$, *then there exist a real number* d *and a function* h *in* $\mathfrak{B}^{m-2}$ *such that*

$$(\delta + cg)^{-1} = d\delta + h.$$

(B) *If* $n \geqslant 2$, *then there exists a function* h *in* $\mathfrak{B}^{2(n-2):m}$ *such that*

$$(\delta + c\delta)^{-1} = h.$$

Proof of (A). If $a_0=0$, then from Theorem 2-B, we have

$$\mathfrak{F}\{(\delta + cg)^{-1}\} = \frac{1}{1+ca_1}\left[1 - \frac{eG_1(i\lambda)}{1+eG_1(i\lambda)}\right] = \mathfrak{F}\{d\delta + h\},$$

where $e = \frac{1}{1+ca_1}$ provided that $1+ca_1 \neq 0$. Since $g_1^{(j)}(t) \in L^1_{(R)}$, $1 \leqslant j \leqslant m$, the Fourier transform of g_1 satisfies the inequality

$$|G_1(i\lambda)| \leqslant M\left|\frac{1}{\lambda}\right|^m$$

for all λ. Therefore, for any j we obtain

$$|\mathfrak{F}\{\delta^{(j)} * h\}| = \left|\frac{e}{1+ca_1}\right| \left|\frac{(i\lambda)^j G_1(i\lambda)}{1+eG_1(i\lambda)}\right|$$
$$\leqslant M_2 \left|\frac{1}{\lambda}\right|^{m-j}.$$

This shows that $\delta^{(j)} * h$ can be identified with a function in $L^2_{(R)}$ for $1 \leqslant j \leqslant m-1$. Thus, we have $h \in \mathfrak{D}^{m-1}_{L^2(R)}$.

Now if $a_0 \neq 0$, then we define two auxiliary functions defined by

$$L(s) = \frac{s}{(1+ca_1)s + ca_0 + csG_1(s)}, \qquad \text{in} \quad \operatorname{Re} s \geqslant 0,$$

and

$$K(s) = \frac{s}{(1+ca_1)s + ca_0} = \frac{1}{1+ca_1}\left(1 - \frac{b}{s+b}\right), \qquad \operatorname{Re} s \geqslant 0,$$

where $b = \frac{ca_0}{1+ca_1} > 0$ by the assumption. Then we have, in $\operatorname{Re} s > 0$, the equality

$\frac{1}{1+cG(s)}=L(s)$ and

$$L(s)=K(s)\frac{1}{1+cK(s)G_1(s)}=K(s)[1-\hat{K}(s)], \qquad \mathrm{Re}\, s\geqslant 0,$$

where $\hat{K}(s)=\frac{cK(s)G_1(s)}{1+cK(s)G_1(s)}$. By Theorem 3, we see that the Laplace transform $H(s)$ of h is given by

$$H(s)=\frac{b}{s+b}\hat{K}(s)-\frac{b}{s+b}-\hat{K}(s), \qquad \mathrm{Re}\, s\geqslant 0. \tag{11}$$

Now, from the definitions of $K(i\lambda)$ and $G_1(i\lambda)$, we have $|\hat{K}(s)|=\left|\frac{cK(i\lambda)}{1+cK(i\lambda)G_1(i\lambda)}\right|\leqslant M_1$ for all λ. Therefore, for any j

$$|\mathfrak{F}\{\delta^{(j)}*\hat{k}\}|\leqslant M_1|(i\lambda)^j G_1(i\lambda)|\leqslant M_2\left|\frac{1}{\lambda}\right|^{m-j}, \tag{12}$$

where $\hat{k}$ is the inverse Fourier transform of $\hat{K}(i\lambda)$. If $j\leqslant m-1$ then $\delta^{(j)}*\hat{k}$ can be identified with a function in $L^2_{(R)}$. Thus, we see that $\hat{k}$ belongs to $\mathfrak{D}^{m-1}_{L^2(R)}$. Define $\tilde{k}(t)$ by $\tilde{k}(t)=\mathfrak{F}^{-1}\left\{\frac{b}{b+i\lambda}\hat{K}(i\lambda)\right\}$. Then, we have

$$|\mathfrak{F}\{\delta^{(j)}*\tilde{k}(t)\}|\leqslant b|(i\lambda)^{j-1}\hat{K}(i\lambda)|\leqslant bM_2\left|\frac{1}{\lambda}\right|^{m-j+1}. \tag{13}$$

From (11), (12) and (13) we see that $h(t)$ belongs to $\mathfrak{D}^{m-1}_{L^2(R)}$.

Proof of (B). We only deal with the case in which $a_0\neq 0$. The proof for the case in which $a_0=0$ follows in a similar fashion. Define $L(s)$ by

$$L(s)=\frac{s}{\sum_{k=1}^{n} a'_k s^k+csG_1(s)}, \qquad \text{in } \mathrm{Re}\, s\geqslant 0,$$

and define $K(s)$ by

$$K(s)=\frac{s}{\sum_{k=1}^{n} a'_k s^k}, \qquad \text{in } \mathrm{Re}\, s\geqslant 0.$$

Then we have $\frac{1}{1+cG(s)}=L(s)$, in $\mathrm{Re}\, s>0$ and

$$L(s)=K(s)\frac{1}{1+cK(s)G(s)}=K(s)\left[1-\frac{cK(s)G(s)}{1+cK(s)G(s)}\right], \qquad \text{in } \mathrm{Re}\, s\geqslant 0.$$

From the definitions of $K(i\lambda)$ we get

$$\left|\frac{(i\lambda)^{2i-3+j}K^2(i\lambda)G(i\lambda)}{1+cK(i\lambda)G(i\lambda)}\right|\leqslant M\left|\frac{1}{\lambda}\right|^{2i-3+m-j}, \quad 2\leqslant i\leqslant n, \quad 1\leqslant j\leqslant m.$$

Noting that $K(i\lambda)$ is the Fourier transform of an infinitely differentiable function, above inequalities imply that $h=\mathfrak{F}^{-1}\{L(i\lambda)\}=\mathfrak{F}^{-1}\left\{\frac{1}{1+cG(i\lambda)}\right\}$ belongs to $\mathfrak{D}^{2n-3+m}_{L^2}$, as was asserted.

Combining the results of Corollary 2-2 and Theorem 5 we have

Lemma 6. *If* g *in* $\mathfrak{T}'_n$ *satisfies the conditions of Theorem* 5 *and* $g_1(t) \in L^1_{(R)}$ *has derivatives* $g_1^{(j)}(t)$ *up to the order* m *and* $g_1^{(j)}(t) \in L^1_{(R)}$, $1 \leqslant j \leqslant m$, *then the following results hold.*

(A) *If* $n=1$, *then there exist a real number* $\hat{d}$ *and a function* $\hat{h}$ *in* $\mathfrak{B}^{m-2}$ *such that*

$$(\delta + cg)^{-1} * g = \hat{d}\delta + \hat{h}.$$

(B) *If* $n \geqslant 2$, *then there exist a nonzero number* $\hat{d}$ *and a function* $\hat{h}$ *in* $\mathfrak{B}^{2(n-2)+m}$ *such that*

$$(\delta + cg)^{-1} * g = \hat{d}\delta + \hat{h}.$$

It is well known that if $H(s)$ is an entire function and of exponential type, then its boundary function $H(i\lambda)$ is the Fourier transform of a function with bounded support. The converse is also true. According to the representation in Corollary 2-1 it is natural to study the condition which assures boundedness of the support of $h(t)$. Howeverr, we do not have an affirmative or even a negative answer to this problem.

Acknowledgement

The author would like to appreciate Professor Yoshio HAYASHI and Professor Tatsuo KAWATA of Keio University for their valuable comments on his study.

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