Title	On the gentral limit theorem for a class of general linear process
Sub Title	
Author	本田, 郁二(Honda, Ikuji)
Publisher	慶応義塾大学工学部
Publication year	1973
Jtitle	Keio engineering reports Vol.26, No.3 (1973.) ,p.27- 36
JaLC DOI	
Abstract	The Central Limit Theorem for a class of general linear process, defined by T. KAWATA (1972), is studied in this paper.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00260003-0027

KEIO ENGINEERING REPORTS
VOL. 26 No. 3 YOKOHAMA 1973

ON THE GENTRAL LIMIT THEOREM FOR A CLASS OF GENERAL LINEAR PROCESS

BY

IKUJI HONDA

FACULTY OF ENGINEERING
KEIO UNIVERSITY

KEIO ENGINEERING REPORTS
VOL. 26, NO. 3, pp. 27-36, 1973

ON THE CENTRAL LIMIT THEOREM FOR A CLASS OF GENERAL LINEAR PROCESS

IKUJI HONDA

Dept. of Electrical Engineering, Keio University, Yokohama 223, Japan

(Received June 26, 1973)

ABSTRACT

The Central Limit Theorem for a class of general linear process, defined by T. KAWATA (1972), is studied in this paper.

1. Introduction

Let $\xi(S)=\xi(S, \omega)$ be a real valued random measure, that $E[\xi(S)]^2<\infty$ for all bounded Borel set S and $\xi(S)$ is σ-additive in the sense that $\xi(S)=\underset{n\to\infty}{\text{l.i.m.}} \sum_{k=1}^{n}\xi(S_k)$ for $S=\bigcup_{k=1}^{\infty} S_k$, S_k being disjoint Borel sets. Define the set function $F(S)=E[\xi(S)]^2$ and the nondecreasing function $F(t)$ by $F(t)-F(s)=F([s, t))$, $s<t$. Then $F(t)$ is left continuous, moreover for convinience, we modify the function $F(t)$ to be

$$F(t)=\frac{1}{2}[F(t+0)+F(t-0)].$$

Suppose that $a(t)$ is a real valued function such that $a(t)\in L^2(-\infty, \infty)$ and

$$\int_{\alpha}^{\beta} a^2(t-\lambda)dF(\lambda)<\infty,$$

for any finite α, β $(\alpha<\beta)$, and for any t. We can define stochastic integral

$$\int_{\alpha}^{\beta} a(t-\lambda)\xi(d\lambda)$$

in mean square sense, in an ordinary way (c.f. DOOB (1953)).

Now if a stochastic process $X(t)=X(t, \omega)$, $-\infty<t<\infty$, of the second order such that

$$\int_I E\left|\int_\alpha^\beta a(t-\lambda)\xi(d\lambda)-X(t)\right|^2 dt \to 0, \tag{1-1}$$

as $\alpha\to-\infty$, $\beta\to\infty$, for any finite interval I, then $X(t)$ is called a general linear process which was defined by T. KAWATA (1972), (1973).

In this paper, we aim at studying the central limit theorem of such a general linear process.

2. Preliminaries

In order to study the central limit theorem of a general linear process, we restrict ourselves to the *random measure* $\xi(S)$ *whose distribution is infinitely divisible and* $\xi(S_1)$ *and* $\xi(S_2)$ *are independent each other for any disjoint Borel sets* S_1, S_2.

Since $E\xi(S)=0$, and $E[\xi(S)]^2<\infty$ for bounded Borel set S, the characteristic function of $\xi(S)$ has Kolmogorov's canonical formula such that, (c.f. DOOB (1953))

$$E[\exp(iu\xi(S))]=\exp\left\{\int_S\int_{-\infty}^{\infty}[e^{iux}-iux-1]\frac{1}{x^2}d^2G(x,\lambda)\right\}, \tag{2-1}$$

where $G(x,\lambda)$ is nondecreasing for x and λ, and $G(-\infty,\lambda)=0$. $G(\cdot,\lambda)$ is bounded for each λ but $G(x,\cdot)$ is not necessarily bounded. This integrand is defined by continuity at $x=0$ where it assumes the value $-u^2/2$. The measure $F(S)$ is determined uniquely by

$$F(S)=E[\xi(S)]^2$$

$$=-\left\{\frac{d^2}{du^2}\log E[\exp(iu\xi(S))]\right\}_{u=0}.$$

Then

$$F(S)=\int_S\int_{-\infty}^{\infty}d^2G(x,\lambda). \tag{2-2}$$

Moreover we assume the condition

$$F(t+\lambda)-F(t)=O(\lambda), \tag{2-3}$$

for large λ uniformly for $-\infty<t<\infty$.

Lemma 1.

Define

$$\Phi(T,\lambda)\equiv\int_{-T}^{T}a(t-\lambda)dt, \tag{2-4}$$

then $\Phi(T,\lambda)$ is of $L^2(F)$ over $(-\infty,\infty)$ for each $T>0$.

Proof.

For any $\alpha<\beta$,

$$\int_{\alpha}^{\beta}\Phi^2(T,\lambda)dF(\lambda)$$

$$=\int_{\alpha}^{\beta}dF(\lambda)\left[\int_{-T}^{T}a(t-\lambda)dt\right]^2$$

$$\leqq\int_{\alpha}^{\beta}dF(\lambda)\int_{-T}^{T}a^2(t-\lambda)dt\cdot(2T)$$

$$=(2T)\cdot\left[\int_{-T-\alpha}^{T-\alpha}a^2(u)du\int_{\alpha}^{T-u}dF(\lambda)\right.$$

$$\left.+\int_{T-\beta}^{T-\alpha}a^2(u)du\int_{-T-u}^{T-u}dF(\lambda)+\int_{-T-\beta}^{T-\beta}a^2(u)du\int_{-T-u}^{\beta}dF(\lambda)\right]$$

$$=J_1+J_2+J_3.$$

By using the assumption of $F(\lambda)$ (2-3), there exists a constant K_1 such that $|F(t)-F(s)|\leqq K_1(t-s)$. Hence the integral J_1 is

$$J_1\leqq(2T)\int_{-T-\alpha}^{T-\alpha}a^2(u)duK_1(T-u-\alpha)$$

$$\leqq K_1(2T)^2\int_{-T-\alpha}^{T-\alpha}a^2(u)du.$$

Then $J_1\to0$ as $\alpha\to-\infty$ because of $a(u)\in L^2(-\infty,\infty)$. Also it can be shown $J_3\to0$ as $\beta\to\infty$, and

$$\limsup_{\substack{\alpha\to-\infty\\ \beta\to\infty}} J_2\leqq K_1(2T)^2\limsup_{\substack{\alpha\to-\infty\\ \beta\to\infty}}\int_{T-\beta}^{T-\alpha}a^2(u)du$$

$$=K_1(2T)^2\int_{-\infty}^{\infty}|a(u)|^2du<\infty.$$

Now in order to study the central limit theorem, we define the process $Z(T)$ such that

$$Z(T)=C(T)\int_{-T}^{T}X(t)dt, \tag{2-5}$$

where

$$C(T)=\left\{E\left[\int_{-T}^{T}X(t)dt\right]^2\right\}^{-\frac{1}{2}}, \tag{2-6}$$

which is assumed not to be zero. If $C(T)=0$, $Z(T)$ is defined to be zero almost surely.

Lemma 2.

Let $\{X_n(t)\}$ be a sequence which converges to $X(t)$ in the sense that $\int_I E|X_n(t)-X(t)|^2dt\to0$ as $n\to\infty$, for any finite interval I, and define the process $Z_n(T)$ by

$$Z_n(T)=C(T)\int_{-T}^{T} X_n(t)dt, \tag{2-7}$$

then there exists $Z(t)$ uniquely, almost surely, such that $E|Z(T)-Z_n(T)|^2\to 0$ as $n\to\infty$, and

$$Z(T)=C(T)\int_{-T}^{T} X(t)dt$$

Proof.

It is sufficient to show $E|Z_n(T)-Z_m(T)|^2\to 0$, as $m, n\to\infty$, because of the completeness of convergence in mean,

$$\begin{aligned}
&E|Z_n(T)-Z_m(T)|^2\\
&=E\left|C(T)\int_{-T}^{T}[X_n(t)-X_m(t)]dt\right|^2\\
&\leqq |C(T)|^2 E\int_{-T}^{T}|X_n(t)-X_m(t)|^2 dt\\
&=|C(T)|^2\int_{-T}^{T}E|X_n(t)-X_m(t)|^2 dt\to 0,
\end{aligned}$$

as $n, m\to\infty$.

Obviously,

$$E|Z(T)|^2=1. \tag{2-8}$$

Lemma 3.

$$E|Z(T)|^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[xC(T)\Phi(T,\lambda)]^2\frac{1}{x^2}d^2G(x,\lambda) \tag{2-9}$$

Proof.

We define $\{X_n(t)\}$ which converges to a general linear process $X(t)$ such that

$$X_n(t)=\int_{\alpha_n}^{\beta_n} a(t-\lambda)\xi(d\lambda). \tag{2-10}$$

Then we have

$$\begin{aligned}
E|Z_n(T)|^2&=E\left[C(T)\int_{-T}^{T}dt\int_{\alpha_n}^{\beta_n}a(t-\lambda)\xi(d\lambda)\right]^2\\
&=E\left[C(T)\int_{\alpha_n}^{\beta_n}\xi(d\lambda)\int_{-T}^{T}a(t-\lambda)dt\right]^2 \qquad (2\text{-}11)\\
&=C^2(T)\int_{\alpha_n}^{\beta_n}\Phi^2(T,\lambda)dF(\lambda).
\end{aligned}$$

By using *Lemma* 1 and the representation of the nondecreasing function $G(x, \lambda)$ (2-2), $E|Z(T)|^2$ exists and

$$E|Z(T)|^2 = C^2(T)\int_{-\infty}^{\infty} \Phi^2(T, \lambda)dF(\lambda)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [xC(T)\Phi(T, \lambda)]^2 \frac{1}{x^2} d^2G(x, \lambda).$$

3. Characteristic function

To find the characteristic function of $X(t)$ itself is not so simple, because $a(\lambda)$ is not necessarily of $L^2(F)$ over $(-\infty, \infty)$. But we can rather easily derive the form of the characteristic function of $Z(T)$ by using the method of R. LUGANNANI (1967).

Lemma 4.

$$\varphi(T, u) = E[\exp iuZ(T)]$$

$$= \exp\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [e^{iuxC(T)\Phi(T, \lambda)} - iuxC(T)\Phi(T, \lambda) - 1]\right\} \frac{1}{x^2} d^2G(x, \lambda) \tag{3-1}$$

Proof.

We shall show this lemma using the method of R. LUGANNANI (1967). Let the function $\Psi(\lambda)$ be a simple function such that

$$\Psi(\lambda) = \sum_{i=1}^{k} \Psi_i \chi(S_i), \qquad \bigcup_{i=1}^{k} S_i = [\alpha_n, \beta_n],$$

where $\chi(S_i)$ is the indicator and Ψ_i, $(i=1, \cdots, k)$, are values of $\Psi(\lambda)$ on $\lambda \in S_i$, $(i=1, \cdots, k)$. Z_n is represented by

$$Z_n = \int_{\alpha_n}^{\beta_n} \Psi(\lambda)\xi(d\lambda)$$

$$= \sum_{i=1}^{k} \Psi_i \xi(S_i).$$

Now, since $\xi(S_i)$ and $\xi(S_j)$ $(S_i \cap S_j = \phi,\ i \neq j,\ i, j = 1, \cdots, k)$, are independent, we have

$$\varphi_n(u) \equiv E[\exp iuZ_n]$$

$$= \prod_{i=1}^{k} E[\exp iu\Psi_i \xi(S_i)]$$

$$= \prod_{i=1}^{k} \exp\left\{\int_{S_i}\int_{-\infty}^{\infty} [e^{iux\Psi_i} - iux\Psi_i - 1] \frac{1}{x^2} d^2G(x, \lambda)\right\}$$

$$=\exp\left\{\int_{\alpha_n}^{\beta_n}\int_{-\infty}^{\infty}[e^{iux\Psi(\lambda)}-iux\Psi(\lambda)-1]\frac{1}{x^2}d^2G(x,\lambda)\right\}$$

If $\Psi(\lambda)$ is a function of $L^2(F)$, there exists a simple function sequence $\{\Psi_k(\lambda)\}$ which converges to $\Psi(\lambda)$ in $L^2(F)$ mean, and

$$\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[e^{iux\Psi(\lambda)}-iux\Psi(\lambda)-1]\frac{1}{x^2}d^2G(x,\lambda)\right.$$

$$\left.-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[e^{iux\Psi_k(\lambda)}-iux\Psi_k(\lambda)-1]\frac{1}{x^2}d^2G(x,\lambda)\right|$$

$$=\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[e^{iux\Psi(\lambda)}-e^{iux\Psi_k(\lambda)}-iux\Psi(\lambda)+iux\Psi_k(\lambda)]\frac{1}{x^2}d^2G(x,\lambda)\right|$$

$$\leqq K_2u^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Psi(\lambda)-\Psi_k(\lambda)|^2d^2G(x,\lambda)$$

$$=K_2u^2\int_{-\infty}^{\infty}|\Psi(\lambda)-\Psi_k(\lambda)|^2dF(\lambda)\to 0 \qquad \text{as} \quad k\to\infty.$$

Taking $\Psi(\lambda)=C(T)\Phi(T,\lambda)$, we obtain the characteristic function (3-1) of

$$Z(T)=\int_{-\infty}^{\infty}\left[C(T)\int_{-T}^{T}x(t-\lambda)\xi(d\lambda)\right]dt.$$

4. The Central Limit Theorem

Now we are going to give the central limit theorem for $Z(T)$ by using the method of R. LUGANNANI and J. B. THOMAS (1968), which was also used by Y. ENDOW (1972), for the general random noise process. (See also T. KAWATA (1973),) We shall show that $\varphi(T,u)\to\exp(-u^2/2)$ as $T\to\infty$. By using *Lemma* 3 and *Lemma* 4 we have

$$J(T)=\left|\log\varphi(T,u)+\frac{u^2}{2}\right|$$

$$=\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[e^{iuxC(T)\Phi(T,\lambda)}-iuxC(T)\Phi(T,\lambda)-1]\frac{1}{x^2}d^2G(x,\lambda)-\frac{(iu)^2}{2}\right| \tag{4-1}$$

$$=\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{e^{iuxC(T)\Phi(T,\lambda)}-\frac{1}{2}[iuxC(T)\Phi(T,\lambda)]^2\right.\right.$$

$$\left.\left.-iuxC(T)\Phi(T,\lambda)-1\right\}\frac{1}{x^2}d^2G(x,\lambda)\right|,$$

which we need for the proof of the following theorems.

Theorem 1.

Let $X(t)$ be a general linear process defined in **1**, and $\xi(d\lambda)$ be a random measure which satisfies the conditions in **2**. Suppose the functions $G(x, \lambda)$ and $F(\lambda)$ defined in **2**, satisfying the condition (2-3). Let $Z(T)$ be the random variable defined in **2**. If

$$\iint_{|xC(T)\Phi(T,\lambda)|\geq\varepsilon} |C(T)\Phi(T, \lambda)|^2 d^2G(x, \lambda)\to 0, \tag{4-2}$$

as $T\to\infty$, where $\Phi(T, \lambda)\equiv\int_{-T}^{T} a(t-\lambda)dt$, then the distribution of $Z(T)$ converges to normal distribution $N(0, 1)$ as $T\to\infty$.

Proof

We use the property that

$$\left|\exp(ix)-\frac{1}{2}(ix)^2-(ix)-1\right|\leqq\frac{K_3|x|^3}{1+|x|}, \tag{4-3}$$

$-\infty<x<\infty$, and K_3 is some constant. We have from (4-1),

$$\begin{aligned} J(T)&\leqq K_3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |uC(T)\Phi(T, \lambda)|^2 \frac{|xuC(T)\Phi(T, \lambda)|}{1+|xuC(T)\Phi(T, \lambda)|} d^2G(x, \lambda) \\ &\leqq K_3\varepsilon u^3\iint_{|xC(T)\Phi(T,\lambda)|<\varepsilon} |C(T)\Phi(T, \lambda)|^2 d^2G(x, \lambda) \\ &\quad + K_3u^2\iint_{|xC(T)\Phi(T,\lambda)|\geqq\varepsilon} |C(T)\Phi(T, \lambda)|^2 d^2G(x, \lambda) \\ &= K_3[\varepsilon u^3 J_1(T)+u^2 J_2(T)]. \end{aligned} \tag{4-4}$$

By using the fact $E|Z(T)|^2=1$ and *Lemma* 3, we get

$$\begin{aligned} J_1(T)&\leqq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |C(T)\Phi(T, \lambda)|^2 d^2G(x, \lambda) \\ &= |C(T)|^2\int_{-\infty}^{\infty} |\Phi(T, \lambda)|^2 dF(\lambda) \\ &= 1. \end{aligned}$$

By the assumption (4-2), $J_2(T)\to 0$ and this proves the theorem.

Now we restrict ourselves to the general linear process of special type, which still includes the general random noise process considered by Y. ENDOW and some class of pulse train process studied by R. LUGANNANI (1971).

Lemma 5.

If $X(t)$ is a general linear process and the nondecreasing function $F(\lambda)$ defined **1**, **2**, satisfies $F(t+\lambda)-F(t)=O(\lambda)$ (2-3), for large λ uniformly for t, and moreover the function $a(\lambda)$ is of $L(-\infty, \infty)$ as well as of $L^2(-\infty, \infty)$ and nonnegative, then we have

$$C(T)=O(T^{-\frac{1}{2}}).$$

Proof.

$$[(2T)^{\frac{1}{2}}C(T)]^{-2}=\lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}}\frac{1}{2T}E\left[\int_{-T}^{T}dt\int_{\alpha}^{\beta}a(t-\lambda)\xi(d\lambda)\right]^2$$

$$=\lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}}\frac{1}{2T}\int_{-T}^{T}dt\int_{-T}^{T}du\int_{\alpha}^{\beta}a(t-\lambda)a(u-\lambda)dF(\lambda)$$

Now specially we select the value of α, β, such that $\alpha=-T/2$, $\beta=T/2$ then

$$[(2T)^{\frac{1}{2}}C(T)]^{-2}\geqq\frac{1}{2T}\int_{-T}^{T}dt\int_{-T}^{T}du\int_{-T/2}^{T/2}a(t-\lambda)a(u-\lambda)dF(\lambda)$$

$$=\frac{1}{2T}\int_{-T}^{T}dt\int_{-T/2}^{T/2}a(t-\lambda)\left[\int_{-T}^{T}a(u-\lambda)du\right]dF(\lambda)$$

$$\geqq\frac{K_4}{2T}\int_{-T}^{T}dt\int_{-T/2}^{T/2}a(t-\lambda)dF(\lambda)$$

$$=\frac{K_4}{2T}\int_{-T/2}^{T/2}dF(\lambda)\int_{-T-\lambda}^{T-\lambda}a(v)dv$$

$$\geqq\frac{K_4^2}{2T}\int_{-T/2}^{T/2}dF(\lambda)$$

$$=K_4^2\left[\frac{F(T/2)-F(-T/2)}{T}\right]\longrightarrow \text{constant},$$

as $T\to\infty$. Hence we obtain the conclusion of Lemma 5.

Theorem 2.

Let $X(t)$ be a general linear process defined in **1.** Let $\xi(d\lambda)$ be a random measure which satisfies the conditions in **2.** Suppose that the nondecreasing function $F(\lambda)$ satisfies (2-3), and that the nondecreasing function $G(x, \lambda)$ is represented as $G(x, \lambda)=G_1(x)G_2(\lambda)$, where $G_1(x)$ is a bounded nondecreasing function and $G_2(\lambda)$ is a nondecreasing function not necessarily bounded, and moreover the function $a(\lambda)$ is of $L(-\infty, \infty)$ as well as of $L^2(-\infty, \infty)$ and nonnegative.

Then the distribution function of $Z(T)$ converges to the standard normal distribution $N(0, 1)$, as $T\to\infty$.

Proof

From *Lemma* 5, we see that $C(T)^{-1}=O(T^{\frac{1}{2}})$ and $\Phi(T, \lambda)$ increases to some constant K_5 as $T\to\infty$, which is independent for λ. Noting that

$$\begin{aligned}\{x:\ |xC(T)\Phi(T,\lambda)|\geqq\varepsilon\}&=\{x:\ |x|\geqq\varepsilon/|C(T)\Phi(T,\lambda)|\}\\&\subset\{x:\ |x|\geqq\varepsilon K_6\sqrt{T}\},\end{aligned}\tag{4-6}$$

where K_6 is some constant. We have

$$\begin{aligned}J_2(T)&=\iint_{|C(T)\Phi(T,\lambda)|\geqq\varepsilon}|C(T)\Phi(T,\lambda)|^2d^2G(x,\lambda)\\&\leqq|C(T)|^2\int_{-\infty}^{\infty}|\Phi(T,\lambda)|^2dG_2(\lambda)\int_{\{x:\ |xC(T)\Phi(T,\lambda)|\geqq\varepsilon\}}dG_1(x)\\&=|C(T)|^2\int_{-\infty}^{\infty}|\Phi(T,\lambda)|^2dF(\lambda)\frac{\int_{\{x:\ |xC(T)\Phi(T,\lambda)|\geqq\varepsilon\}}dG_1(x)}{\int_{-\infty}^{\infty}dG_1(x)}\\&\leqq\frac{1}{K_7}\int_{\{x:\ |x|\geqq\varepsilon K_6\sqrt{T}\}}dG_1(x),\end{aligned}$$

which converges to zero as $T\to\infty$, since $G_1(x)$ is bounded, where $K_7=G_1(\infty)-G_1(-\infty)$. The proof is now complete.

5. Application

The class of processes which satisfy the conditions of *Theorem* 2 includes the general random noise process whose random measure is generated by nonhomogenious compound poisson process, and even a little more general one. Another example of the process studied in *Theorem* 2, is a pulse train process defined by R. LUGANNANI (1971) to be

$$X(t)=\sum_{n=-\infty}^{\infty}\alpha_n a(t-nT),\tag{5-1}$$

where T is a fixed positive constant, and α_n $(n=0, \pm1, \pm2, \cdots)$ are real valued random variables with $E\alpha_n=0$, $E\alpha_n^2=\sigma^2<\infty$, $E\alpha_n\alpha_m=0$, $n\neq m$, and $a(t)$ is real valued function of $L^2(-\infty,\infty)$.

If moreover the pulse train process $X(t)$ satisfies that $a(t)$ is nonnegative and of $L(-\infty,\infty)$, and α_n and α_m are independent each other $(m\neq n)$, and the distribution of α_n is infinitely divisible, then *Theorem* 2 can be applied. Then the central limit theorem holds.

Acknowledgement

The author wishes to express his gratitude for the guidance and encouragement received from Prof. T. KAWATA and Prof. H. FUJITA of Keio University.

REFERENCES

DOOB, J. L. (1953): Stochastic Processes, John Wiley, 417–424, 425–433.

ENDOW, Y. (1972): A central limit theorem for the random noise process, Keio Eng. Rep. Vol. 25–3, 19–26.

KAWATA, T. (1955): On the stochastic process of random noise, Kodai Math. Sem. Rep. Vol. 7–2, 33–42.

KAWATA, T. (1972): On a class of linear process, Proc. Second Japan-U.S.S.R. Symposium on Probability Theory and Statistics Lecture Notes in Mathematics 330 Springer, 193–212.

KAWATA, T. (1973): Nonstationary stochastic Process, Keio Math. Sem. Rep. No. 1, 1–23.

LUGANNANI, B. and THOMAS, J. B. (1967): On a class of stochastic process which are closed under linear transformations, Information and Controll, **10**, 1–21.

LUGANNANI, B. and THOMAS, J. B. (1968): The central limit theorem for a class of stochastic process, J. Math. appl., Vol. 24, 25–38.

LUGANNANI, R. (1971): Convergence properties of the sample mean and sample correlation for a class of pulse trains, SIAM J. Appl. Math. **21**, 1–12.