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SOME CONVERGENCE PROPERTIES FOR A CLASS OF GENERAL LINEAR PROCESSES

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ABSTRACT

Let $X(t)$ be *a general linear process* defined by T. KAWATA (1972). In this paper the law of large numbers for $X(t)X(t+\tau)$ is studied.

1. Introduction

Let $m(S)$ be a real valued signed measure on Borel sets, generated from a function $m(t)$ of bounded variation over every finite interval. Let $\xi(S)=\xi(S, \omega)$ be a real valued random measure on Borel sets S with $E[\xi(S)]^2<\infty$ for any bounded Borel set such that,

(1) $$E\xi(S)=0.$$

Consider a set function $F(S)$ with property that

(2) $$E\xi(S_1)\xi(S_2)=F(S_1\cap S_2),$$

for any Borel sets S_1, S_2.

We define a nondecreasing function $F(\lambda)$ by

$$F([s, t))=F(t)-F(s), \qquad t>s.$$

For convenience, we modify the function $F(\lambda)$ and $m(\lambda)$ to be

$$F(\lambda)=\frac{1}{2}[F(\lambda+0)+F(\lambda-0)],$$

$$m(\lambda)=\frac{1}{2}[m(\lambda+0)+m(\lambda-0)].$$

Furthermore define

$$\eta(S, \omega)=m(S)+\xi(S, \omega),$$

for any Borel set S.

Suppose that $a(t)$ is a real valued function such that

$$\int_{\alpha}^{\beta} a^2(t-\lambda)dF(\lambda)<\infty$$

$$\int_{\alpha}^{\beta} |a(t-\lambda)||dm(\lambda)|<\infty.$$

for any finite α, $\beta(\alpha<\beta)$.

Then the stochastic integral

$$\int_{\alpha}^{\beta} a(t-\lambda)\eta(d\lambda)$$

can be defined in an ordinary way. (c.f. DOOB (1953)).

Now if a stochastic process $X(t)=X(t, \omega)$, $-\infty<t<\infty$, of the second order is such that

$$\int_I E\left|\int_{\alpha}^{\beta} a(t-\lambda)\eta(d\lambda)-X(t)\right|^2 dt \longrightarrow 0,$$

as $\alpha\to-\infty$, $\beta\to\infty$, for any finite interval I, then $X(t)$ is called a general linear process. This defined by T. KAWATA (1972), (1973) who discussed mainly about weak and strong laws of large numbers.

In this paper, we aim at studying the convergence properties of a sample covariance function of the general linear process.

2. Preliminaries

In order to treat the second order convergence properties of the sample covariance of a general linear process $X(t)$, we restrict $X(t)$ to the process of fourth order. Let us suppose $X(t)$ is the fourth order general linear process such that

$$\int_I E\left|\int_{\alpha}^{\beta} a(t-\lambda)\xi(d\lambda)-X(t)\right|^4 dt \longrightarrow 0,$$

$\alpha\to-\infty$, $\beta\to\infty$, in which the random measure $\xi(S)$ is $E[\xi(S)]^4<\infty$ for any bounded Borel set. Furthermore we suppose the following conditions: there is a set function $G(S)$ on Borel sets such that

$$\begin{aligned} (3) \qquad & E\xi(S_1)\xi(S_2)\xi(S_3)\xi(S_4) \\ &=G(S_1\cap S_2\cap S_3\cap S_4)+F(S_1\cap S_2)F(S_3\cap S_4)+F(S_1\cap S_3)F(S_2\cap S_4) \\ &\quad +F(S_1\cap S_4)F(S_2\cap S_3). \end{aligned}$$

Then for a Borel set S we have

$$(3)' \qquad E[\xi(S)]^4 = G(S) + 3[F(S)]^2.$$

It can be shown that $G(S)$ is σ-additive, if $E[\xi(S)]^4 < \infty$, and $\xi(S)$ is σ-additive in the fourth mean. We restrict ourselves to the case $G(S) \geqq 0$ so that we can define the nondecreasing function $G(\lambda)$:

$$G([s, t)) = G(t) - G(s), \qquad t > s.$$

We modify the function $G(\lambda)$ as

$$G(\lambda) = \frac{1}{2}[G(\lambda+0) + G(\lambda-0)].$$

For finite α, $\beta(\alpha < \beta)$, suppose

$$\int_\alpha^\beta a^4(t-\lambda)dG(\lambda) < \infty$$

and $EX^4(t) < \infty$. Moreover suppose that $X(t)$ is fourth mean continuous.

3. Convergence Properties of Sample Covariance Function (I)

Convergence properties of sample covariance functions, which we are going to discuss, have close connections with the weak and strong laws of large numbers of the second moment of $X(t)$. These properties play important roles in the communication theory. A special case of pulse train process was discussed by R. LUGANNANI (1971).

Theorem 1.

Let $X(t)$ be a general linear process of fourth order defined in **2**. Suppose that

$$G(t+\lambda) - G(t) = O(\lambda) \tag{3-1}$$

$$F(t+\lambda) - F(t) = v_0\lambda + O(1) \tag{3-2}$$

for large $|\lambda|$ uniformly for $-\infty < t < \infty$, and that $a(t)$ *is bounded, squarely integrable,* (which implies that $a(t) \in L^p$, $p \geq 2$), *absolutely continuous with* $a'(t) \in L^2(-\infty, \infty)$ *and such that* $a(t) \to 0$ *as* $|t| \to \infty$. Then

$$E\left\{\frac{1}{2A}\int_{-A}^{A} X(t)X(t+\tau) - EX(t)X(t+\tau)]dt\right\}^2 \longrightarrow 0$$

as $A \to \infty$, uniformly for τ.

Proof.

$$E\left\{\frac{1}{2A}\int_{-A}^{A} [X(t)X(t+\tau) - EX(t)X(t+\tau)]dt\right\}^2$$

$$= \lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}} \left[\frac{1}{(2A)^2}\int_{-A}^{A} dt \int_{-A}^{A} ds \int_{\alpha}^{\beta} a(t-\lambda)a(t+\tau-\lambda)a(s-\lambda)a(s+\tau-\lambda)\, dG(\lambda) \right.$$

$$+\frac{1}{(2A)^2}\int_{-A}^{A} dt \int_{-A}^{A} ds \int_{\alpha}^{\beta} a(t-\lambda)a(s-\lambda)\, dF(\lambda) \int_{\alpha}^{\beta} a(t+\tau-\mu)a(s+\tau-\mu)\, dF(\mu)$$

$$\left. +\frac{1}{(2A)^2}\int_{-A}^{A} dt \int_{-A}^{A} ds \int_{\alpha}^{\beta} a(t-\lambda)a(s+\tau-\lambda)\, dF(\lambda) \int_{\alpha}^{\beta} a(t+\tau-\mu)a(s-\mu)\, dF(\mu) \right]$$

$$= \lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}} [I+J+L].$$

Setting the function $f(\tau, \lambda)=\int_{-A}^{A} a(s+\tau-\lambda)a(s-\lambda)\, ds$ and by changing the order of integration we have

$$I=\frac{1}{(2A)^2}\int_{-A}^{A} dt \int_{\alpha}^{\beta} a(t-\lambda)a(t+\tau-\lambda) f(\tau, \lambda)\, dG(\lambda)$$

$$=\frac{1}{(2A)^2}\int_{-A-\beta}^{A-\beta} a(u+\tau)a(u)\, du \int_{-A-u}^{\beta} f(\tau, \lambda)\, dG(\lambda)$$

$$+\frac{1}{(2A)^2}\int_{A-\beta}^{-A-\alpha}\int_{-A-u}^{A-u} + \frac{1}{(2A)^2}\int_{-A-\alpha}^{A-\alpha}\int_{\alpha}^{A-v}$$

$$=I_1+I_2+I_3. \tag{3-4}$$

In I_1 again changing the order of integration, we have

$$\int_{-A-u}^{\beta} f(\tau, \lambda)\, dG(\lambda)$$

$$=\int_{-A}^{A} ds \int_{-A-u}^{\beta} a(s+\tau-\lambda)a(s-\lambda)\, dG(\lambda)$$

$$=\int_{-A-\beta}^{u} a(v+\tau)a(v)\, dv \int_{-A-v}^{\beta} dG(\lambda)$$

$$+\int_{u}^{A-\beta}\int_{-A-u}^{\beta} + \int_{A-\beta}^{u+2A}\int_{-A-u}^{A-v} \tag{3-5}$$

and hence

$$I_1=\frac{1}{(2A)^2}\int_{-A-\beta}^{A-\beta} a(u+\tau)a(u)\, du \int_{A-\beta}^{u} a(v+\tau)a(v)\, dv \int_{-A-v}^{\beta} dG(\lambda)$$

$$+\frac{1}{(2A)^2}\int_{-A-\beta}^{A-\beta}\int_{u}^{A-\beta}\int_{-A-u}^{\beta} + \frac{1}{(2A)^2}\int_{-A-\beta}^{A-\beta}\int_{A-\beta}^{u+2A}\int_{-A-u}^{A-v}$$

$$=I_{11}+I_{12}+I_{13}. \tag{3-6}$$

Since we are supposing that $G(t+\lambda)-G(t)=O(\lambda)$ for large $|\lambda|$, there exists a constant K_1, such that $G(\lambda-\mu)\leqq K_1(\lambda-\mu)$, $\lambda\geqq\mu$. Hence the integral I_{11} is

$$|I_{11}| \leq \frac{K_1}{(2A)^2} \int_{-A-\beta}^{A-\beta} |a(u+\tau)a(u)|\, du \int_{-A-\beta}^{u} |a(v+\tau)a(v)|(\beta+A+v) dv. \tag{3-7}$$

Since $0<\beta+A+v<2A$, $-A-\beta<u<A-\beta$, and $a(t)\in L^2(-\infty, \infty)$ we have $|I_{11}|\to 0$ as $\beta\to\infty$. In the similar way it can be shown that $|I_{12}|\to 0$, $|I_{13}|\to 0$, as $\beta\to\infty$. So we have $|I_1|\to 0$ as $\beta\to\infty$ and similarly, $|I_3|\to 0$ as $\alpha\to-\infty$. Now again using the assumption of $G(\lambda)$, we obtain

$$\begin{aligned}
&\lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}} I = \lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}} I_2 \\
&= \frac{1}{(2A)^2} \int_{-\infty}^{\infty} a(u+\tau)a(u)\, du \int_{-A-u}^{A-u} dG(\lambda) \int_{-A}^{A} a(s+\tau-\lambda)a(s-\lambda)\, ds \\
&= \frac{1}{(2A)^2} \left[\int_{-\infty}^{\infty} a(u+\tau)a(u)\, du\right]^2 \cdot O(A) \\
&= O(A^{-1}).
\end{aligned} \tag{3-8}$$

Next we treat the integral J. $F(\lambda)$ can be written by $F(\lambda)=v_0\lambda+\varepsilon(\lambda)$ where $\varepsilon(\lambda)$ is bounded. By using integration by parts and the condition that $a(t)\to 0$ boundedly as $|t|\to\infty$, we have

$$\begin{aligned}
J &= \frac{v_0^2}{(2A)^2} \int_{-A}^{A} dt \int_{-A}^{A} ds \int_{\alpha}^{\beta} a(t-\lambda)a(s-\lambda)\, d\lambda \int_{\alpha}^{\beta} a(t+\tau-\mu)a(s+\tau-\mu)\, d\mu \\
&+ \frac{v_0}{(2A)^2} \int_{-A}^{A} dt \int_{-A}^{A} ds \int_{\alpha}^{\beta} a(t-\lambda)a(s-\lambda) d\lambda \int_{\alpha}^{\beta} \{a(t+\tau-\mu)a(s+\tau-\mu)\}'\varepsilon(\mu)\, d\mu \\
&+ \frac{v_0}{(2A)^2} \int_{-A}^{A} dt \int_{-A}^{A} ds \int_{\alpha}^{\beta} \{a(t-\lambda)a(s-\lambda)\}'\varepsilon(\lambda)\, d\lambda \int_{\alpha}^{\beta} a(t+\tau-\mu)a(s+\tau-\mu)\, d\mu \\
&+ \frac{1}{(2A)^2} \int_{-A}^{A} dt \int_{-A}^{A} ds \int_{\alpha}^{\beta} \{a(t-\lambda)a(s-\lambda)\}'\varepsilon(\lambda)\, d\lambda \int_{\alpha}^{\beta} \{a(t+\tau-\mu)a(s+\tau-\mu)\}'\varepsilon(\mu) d\mu + o(1) \\
&= J_1+J_2+J_3+J_4+o(1),
\end{aligned} \tag{3-9}$$

where $o(1)$ is a term which tends to zero as $\alpha\to-\infty$, $\beta\to\infty$. For J_1 by changing variables we get

$$J_1 = \frac{v_0^2}{(2A)^2} \int_{-A}^{A} dt \int_{-A-t}^{A-t} du \int_{t-\beta}^{t-\alpha} a(v)a(v+u)\, dv \int_{t-\beta}^{t-\alpha} a(w)a(w+u)\, dw,$$

from which we have, using Parseval relation,

$$\begin{aligned}
\lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}} J_1 &= \frac{v_0^2}{(2A)^2} \int_{-A}^{A} dt \int_{-A-t}^{A-t} du \left[\int_{-\infty}^{\infty} a(v)a(v+u)\, dv\right]^2 \\
&= \frac{v_0^2}{(2A)^2} \int_{0}^{2A} du \int_{-A}^{A-u} dt \left[\int_{-\infty}^{\infty} |\hat{a}(x)|^2 e^{-ixu}\, dx\right]^2 \\
&+ \frac{v_0^2}{(2A)^2} \int_{-2A}^{0} du \int_{-A-u}^{A} dt \left[\int_{-\infty}^{\infty} |\hat{a}(x)|^2 e^{-ixu}\, dx\right]^2
\end{aligned}$$

$$=J_{11}+J_{12},$$

where $\hat{a}(x)$ is the Fourier transform of $a(t)\in L^2(-\infty, \infty)$. J_{11} can be written

$$\frac{v_0^2}{(2A)^2}\int_0^{2A}\left[1-\frac{u}{2A}\right]\left[\int_{-\infty}^{\infty}|\hat{a}(x)|^2 e^{ixu}\,dx\right]^2 du$$

Since $\int_{-\infty}^{\infty}|\hat{a}(x)|^2 e^{-ixu}\,dx$ converges to zero as $|u|\to\infty$, we obtain the result that $|J_{11}|\to 0$ as $A\to\infty$. In the similar way it is shown that $|J_{12}|\to 0$ as $A\to\infty$. Hence we have

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to\ \infty}} |J_1| \longrightarrow 0, \qquad \text{as } A \longrightarrow \infty.$$

For J_2, J_3 and J_4 we can apply the same argument. Since $a(t)$ is absolutely continuous, $a'(t)\in L^2(-\infty, \infty)$, and $\varepsilon(\lambda)$ is bounded, we have $J_2=o(1)$, $J_3=o(1)$, $J_4=o(1)$, as, $\alpha\to-\infty$, $\beta\to\infty$, Putting the above altogether, we have

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to\ \infty}} |J| \longrightarrow 0, \qquad \text{as } A \longrightarrow \infty.$$

L has the similar form which J so we can apply the same argument to obtain

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to\ \infty}} |L| \longrightarrow 0, \qquad \text{as } A \longrightarrow \infty.$$

This complets the proof.

4. Convergence Properties of Sample Covariance Function (II)

In this section we discuss about the almost sure convergence of $\frac{1}{2A}\times\int_{-A}^{A}[X(t)X(t+\tau)-EX(t)X(t+\tau)]\,dt$. This property corresponds to the strong law of large numbers of the product moment. The idea of the proof of it is to regard the process of product moments of a general linear process $X(t)$ as another second order process and appeal to the strong law of large numbers for a second order process, which was by T. KAWATA (1972), (1973), generalizing the way of I. N. VERBITSKAYA (1964), (1966).

Lemma (Thm. 6.1, 6.2, 6.3, of T. KAWATA (1972))

Let $Y(t)$ be a real valued stochastic process of the second order with continuous covariance function $\rho(s,t)=EY(s)Y(t)$, where $EY(t)=0$, $-\infty<t<\infty$, is assumed. Suppose

$$\text{(i)} \qquad \left|\int_n^{n+1} E[Y(t)]^2\,dt\right| \leqq K, \tag{4-1}$$

for all $-\infty<n<\infty$, K being a constant, independent of n, and

(ii) there is a nonnegative even function $g(u)$, $-\infty<u<\infty$, and a function $h(u)$, $-\infty<u<\infty$ such that

$$\left|\int_0^x\int_0^x \rho(u+\tau, v+\tau)\,dudv\right|$$

$$\leqq\left|\int_0^x g(u)\,du\right|+\left|\int_0^x dt\int_0^t h(u)\,du\right|, \tag{4-2}$$

where
$$\int_1^\infty \frac{\log^2 x}{x^2} g(x)\,dx<\infty, \tag{4-3}$$

and $h(u)$ is the Fourier Stieltjes transform $\int_{-\infty}^{\infty} e^{iu\lambda}\,dH(\lambda)$ of some bounded nondecreasing function $H(\lambda)$, with the property that

$$\int_1^\infty \frac{\log^2 x}{x^2}\,dx\int_0^x h(u)du$$

converges. Then we obtain

$$\frac{1}{A}\int_0^A Y(t)\,dt \longrightarrow 0, \qquad \text{as} \quad A \longrightarrow \infty. \quad \text{(almost surely)}$$

Theorem 2.

Let $X(t)$ be the general linear process defined in **2**. Suppose that

$$G(t+\lambda)-G(t)=\mu_0\lambda+O(1),$$

$$F(t+\lambda)-F(t)=v_0\lambda+O(1),$$

for large $|\lambda|$ uniformly for $-\infty<t<\infty$, where μ_0 and v_0 are constants, and that $a(t)$ *is of* $L^2(-\infty,\infty)$, *bounded and absolutely continuous with* $a'(t)\in L^2(-\infty,\infty)$, *and* $a(t)\to 0$ as $|t|\to\infty$.

Furthermore, suppose that

(1) $$\int_1^\infty \frac{\log^2 x}{x^2}\,dx\int_0^x dv\int_{-\infty}^{\infty} a(u+v+\tau)a(u)\,du\int_{-\infty}^{\infty} a(w+v+\tau)a(w)dw$$

(2) $$\int_1^\infty \frac{\log^2 x}{x^2}\,dx\int_0^x dv\int_{-\infty}^{\infty} |\{a(u+v+\tau)a(u)\}'|du\int_{-\infty}^{\infty} |a(w+v+\tau)a(w)|dw$$

(3) $$\int_1^\infty \frac{\log^2 x}{x^2}\,dx\int_0^x dv\int_{-\infty}^{\infty} |\{a(u+v+\tau)a(u)\}'|du\int_{-\infty}^{\infty} |\{a(w+v-\tau)a(w)\}'|dw$$

converge for $-\infty<\tau<\infty$.

Then we obtain

$$\frac{1}{A}\int_0^A [X(t)X(t+\tau)-EX(t)X(t+\tau)]\,dt \longrightarrow 0$$

as $A\to\infty$. (almost surely)

Corollary

Let $X(t)$ be a general linear process defined in **2**. Suppose $G(\lambda)$ and $F(\lambda)$ are the same as in Theorem 2, and $a(t)$ *is of* $L(-\infty,\infty)$, *bounded, absolutely conti-*

nuous and is such that $a'(t)\in L^1\cap L^2(-\infty,\infty)$ *and* $a(t)\to 0$ *as* $|t|\to\infty$. Then *the result of Theorem* 2, *holds.*

Proof of Theorem 2.

Take $Y(t)=X(t+\tau)X(t)-EX(t+\tau)X(t)$, τ being a fixed constant. Then $Y(t)$ becomes a second order process. Consider

$$S=\int_n^{n+1}E[Y(t)]^2dt$$

$$=\int_n^{n+1}E[X(t+\tau)X(t)-EX(t+\tau)X(t)]^2\,dt$$

$$=\lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}}\Bigg[\int_n^{n+1}dt\int_\alpha^\beta a^2(t+\tau-\lambda)a^2(t-\lambda)\,dG(\lambda)$$

$$+\int_n^{n+1}dt\int_\alpha^\beta a^2(t+\tau-\lambda)\,dF(\lambda)\int_\alpha^\beta a^2(t-\mu)\,dF(\mu)$$

$$+\int_n^{n+1}dt\left\{\int_\alpha^\beta a(t+\tau-\lambda)a(t-\lambda)\,dF(\lambda)\right\}^2\Bigg]$$

$$=\lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}}[S_1+S_2+S_3] \tag{4-5}$$

Putting $t-\lambda=u$, interchanging the order of integration and again changing the variable, we easily see, as we did before, that

$$S=\int_{n-\beta}^{n+1-\beta}a^2(u+\tau)a^2(u)\,du\int_{n-u}^{\beta}dG(\lambda)+\int_{n+1-\beta}^{\lambda-\alpha}\int_{n-\alpha}^{n+1-u}+\int_{n-\alpha}^{n+1-\alpha}\int_{\alpha}^{n+1-u}\cdot\;\cdot$$

It is easily verified that $|S_1|<\infty$, owing to the assumption $a(t)\in L^4(-\infty,\infty)$, $G(\lambda-\mu)\leqq K_1(\lambda-\mu)$ and the fact $|\beta-n+u|\leqq 1$, $|n+1-u-\alpha|\leqq 1$. $|S_2|$ and $|S_3|$ can be also verified easily to be finite by integration by parts. The conditions that $a(t)$ is absolutely continuous, $a(t)\to 0$ as $|t|\to\infty$ and $a'(t)\in L^2(-\infty,\infty)$, and $F(\lambda)=v_0\lambda+\varepsilon(\lambda)$ where $\varepsilon(\lambda)$ is bounded, are used (c.f. (3-3), (3-9)). Hence S satisfies the condition (i) of Lemma (4-1).

Now taking $\rho(u+\tau,v+\tau)$ for the covariance of $Y(u+\tau)$ and $Y(v+\tau)$, we have

$$\int_0^x\int_0^x\rho(u+\tau,\,v+\tau)\,du\,dv$$

$$=\int_0^x\int_0^x E[X(u+t+\tau)X(u+t)-EX(u+t+\tau)X(u+t)]$$

$$[X(v+t+\tau)X(v+t)-EX(v+t+\tau)X(v+t)]\,du\,dv$$

$$=\lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}}\Bigg[\int_0^x\int_0^x du\,dv\int_\alpha^\beta a(u+t+\tau-\lambda)a(u+t-\lambda)a(v+t+\tau-\lambda)a(v+t-\lambda)\,dG(\lambda)$$

$$+\int_0^x\int_0^x du\,dv\int_\alpha^\beta a(u+t+\tau-\lambda)a(v+t-\lambda)\,dF(\lambda)\int_\alpha^\beta a(u+t-\mu)a(v+t+\tau-\mu)\,dF(\mu)$$

$$+\int_0^x\int_0^x du\,dv\int_\alpha^\beta a(u+t+\tau-\lambda)a(v+t+\tau-\lambda)\,dF(\lambda)\int_\alpha^\beta a(u+t-\mu)a(v+t-\mu)\,dF(\mu)\Big]$$

$$=\lim_{\substack{\alpha\to-\infty\\ \beta\to\ \infty}}[L_1+L_2+L_3].$$

Because of the assumption $G(\lambda)=\mu_0\lambda+O(1)$, we can verify that $\lim\limits_{\substack{\alpha\to-\infty\\ \beta\to\ \infty}} L_1$ exists, by using the same manner in Thm. 1 (3-5)$\sim$(3-8). And we obtain

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to\ \infty}} L_1$$

$$=\lim_{\substack{\alpha\to-\infty\\ \beta\to\ \infty}}\int_\alpha^\beta dG(\lambda)\int_{-\lambda}^{x-\lambda}\Big[\int_0^x a(v+t+\tau-\lambda)a(v+t-\lambda)\,dv\Big]a(u+t+\tau)a(u+t)\,du$$

$$=\int_{-\infty}^{\infty} a(u+t+\tau)a(u+t)\,du\int_{-x+u}^{u} a(v+t+\tau)a(v+t)\,dv\int_{-v}^{x-u} dG(\lambda)$$

$$+\int_{-\infty}^{\infty} a(u+t+\tau)a(u+t)\,du\int_{u}^{x+u} a(v+t+\tau)a(v+t)\,dv\int_{-u}^{x-v} dG(\lambda)$$

$$=L_{11}+L_{12}.$$

Again, by using the assumption $G(\lambda)=\mu_0\lambda+O(1)$, we have

$$L_{11}=\int_{-\infty}^{\infty} a(u+t+\tau)a(u+t)\,du\int_{-x}^{0} a(w+u+t+\tau)a(w+u+t)[G(x-u+t)-G(v-u+t)]\,du$$

$$\leqq\mu_0\int_0^x (x-v)\,dv\int_{-\infty}^{\infty} a(u+\tau)a(u)a(u-v+\tau)a(u-v)\,du$$

$$+O(1)\int_0^x dv\int_{-\infty}^{\infty} |a(u+\tau)a(u)a(u-v+\tau)a(u-v)|\,du$$

$$=\mu_0\int_0^x dw\int_0^w dv\int_{-\infty}^{\infty} a(u+\tau)a(u)a(u-v+\tau)a(u-v)\,du$$

$$+O(1)\int_0^x dv\int_{-\infty}^{\infty} |a(u+\tau)a(u)a(u-v+\tau)a(u-v)|\,du.$$

For L_{12}, we have a similar inequality in the same manner. Therefore we obtain

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to\ \infty}} L_1$$

$$\leqq 2\mu_0\int_0^x dw\int_0^w dv\int_{-\infty}^{\infty} a(u+\tau)a(u)a(u+v+\tau)a(u+v)\,du$$

$$+O(1)\int_0^u dv\int_{-\infty}^{\infty} a(u+\tau)a(u)a(u+v+\tau)a(u+v)|\,du,$$

and from the fact that $a(t)\in L^2\cap L^4(-\infty,\infty)$, the condition (ii) of Lemma is satisfied, where $h(v)$ and $g(v)$ in Lemma are

$$h(v)=\int_{-\infty}^{\infty} a(u+\tau)a(u)a(u+v+\tau)a(u+v)\,du$$

and

$$g(v)=\int_{-\infty}^{\infty} |a(u+\tau)a(u)a(u+v+\tau)a(u+v)|\,du$$

in which

$$\begin{aligned}
&\int_1^\infty \frac{\log^2 x}{x^2}\,dx\int_0^x h(v)\,dv \\
&\leqq \int_1^\infty \frac{\log^2 x}{x^2}\,dx\int_{-\infty}^{\infty} |a(u+v+\tau)a(u+v)|\,dv\int_{-\infty}^{\infty} |a(u+\tau)a(u)|\,du \\
&\leqq K_2\int_1^\infty \frac{\log^2 x}{x^2}\,dx<\infty,
\end{aligned}$$

and $|g(v)|<\infty$.

In the case of L_2, we see that $a(t)$ is absolutely continuous and $a(t)\to 0$ as $|t|\to\infty$ which can be follows from the assumption on $F(\lambda)$.

For L_2 we use the assumption $F(\lambda)=v_0\lambda+\varepsilon(\lambda)$ where $\varepsilon(\lambda)$ is bounded. Then integration by parts shows that

$$\begin{aligned}
&\lim_{\substack{\alpha\to-\infty\\ \beta\to\ \infty}} L_2 \\
&=v_0^2\int_0^x\int_0^x du\,dv\int_{-\infty}^{\infty} a(u+\tau+\lambda)a(v+\lambda)\,d\lambda\int_{-\infty}^{\infty} a(u+\mu)a(v+\tau+\mu)\,d\mu \\
&\quad+v_0\int_0^x\int_0^x du\,dv\int_{-\infty}^{\infty} \{a(u+\tau+\lambda)a(v+\lambda)\}'\varepsilon(t-\lambda)\,d\lambda\int_{-\infty}^{\infty} a(u+\mu)a(v+\tau+\mu)\,d\mu \\
&\quad+v_0\int_0^x\int_0^x du\,dv\int_{-\infty}^{\infty} a(u+\tau+\lambda)a(v+\lambda)\,d\lambda\int_{-\infty}^{\infty} \{a(u+\mu)a(v+\tau+\mu)\}'\varepsilon(t-\mu)\,d\mu \\
&\quad+\int_0^x\int_0^x du\,dv\int_{-\infty}^{\infty} \{a(u+\tau+\lambda)a(v+\lambda)\}'\varepsilon(t-\lambda)\,d\lambda\int_{-\infty}^{\infty} \{a(u+\mu)a(v+\tau+\mu)\}'\varepsilon(t-\mu)\,d\mu \\
&=L_{21}+L_{22}+L_{23}+L_{24} \qquad \text{say.}
\end{aligned}$$

We here consider L_{22}. We have

$$L_{22}=v_0\int_0^x\int_{-v}^{x-v} dw\int_{-\infty}^{\infty} \{a(w+\tau+\lambda)a(\lambda)\}'\varepsilon(t+v-\lambda)\,d\lambda\int_{-\infty}^{\infty} a(w-\tau+\mu)a(\mu)\,d\mu.$$

Since $\varepsilon(\mu)$ is bounded,

$$\begin{aligned}
|L_{22}|\leqq v_0\int_0^x dv\int_{-v}^{x-v} dw\int_{-\infty}^{\infty} &\{a(w+\tau+\lambda)a(\lambda)\}'\,|\,|\varepsilon(t+v-\lambda)|\,d\lambda \\
&\int_{-\infty}^{\infty} |a(w-\tau+\mu)a(\mu)|\,d\mu
\end{aligned}$$

$$\leqq K_3 \int_0^x du \int_0^u dw \int_{-\infty}^{\infty} |\{a(w+\tau+\lambda)a(\lambda)\}'| \, d\lambda$$

$$\int_{-\infty}^{\infty} |a(w-\tau+\mu)a(\mu)| \, d\mu,$$

where K_3 is a constant. Similar inequalities are obtained for other L's. These inequalities we see that satisfy the condition (ii) of Lemma, where for example, in handling L_{23},

$$h(w)=\int_{-\infty}^{\infty} |\{a(w+\tau+\lambda)a(\lambda)\}'| \, d\lambda \int_{-\infty}^{\infty} |a(w-\tau+\mu)a(\mu)| \, d\mu.$$

Also for L_3 the condition (ii) of Lemma is satisfied true. Hence putting the above considerations, we see that the conditions of Lemma are satisfied for the process $Y(t)$. Moreover in view of Lemma, we reach the conclusion of Theorem 2.

The Corollary is easily proved since we see that the conditions on $a(t)$ show the validity of the condition (1)~(3) of Theorem 2.

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