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BY

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ABSTRACT

In this paper the existence and uniqueness theorems are derived for the equation describing a feedback system consisting of a nonlinear element f and a linear element g which involves the Heaviside unit step function and the delta functional and its derivatives. The conditions of the theorems look involved, but are simply expressed in the frequency domain, the locus defined by the Fourier transform of the linear element being used. The essential part of the method in proving the theorems is to transform the distributional equation to an ordinary nonlinear Volterra type integral equation and apply the known theorem on integral equations to the equation obtained by the transformation.

1. Introduction

There have been a number of interesting results on the analysis of nonlinear feedback systems shown in Fig. 1, where f is a memoryless nonlinear element and g is a certain operator from a normed vector space into another. (WILLEMS, 1969; ZAMES, 1966-a, b) It is obvious that the system shown in Fig. 1 can be described

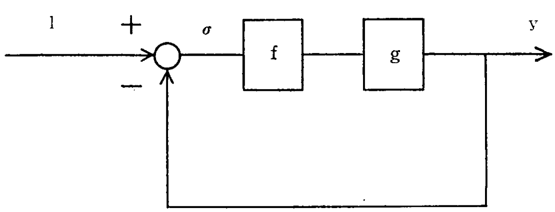

Fig. 1. A Nonlinear Feedback System.

by a nonlinear functional equation or an operator equation of σ. In order to show that the equation which describes the system has some meaning, we must examine whether the equation has a solution in some class of functions or distributions. Therefore, the problem of existence and uniqueness is one of the basic subjects in the analysis of nonlinear feedback systems. However, there have been only few papers that have dealt with this problem. (ZAMES, 1964; WILLEMS, 1969)

In this paper we give two existence and uniqueness theorems related to the equation which is an abstract formulation of a nonlinear feedback system. Our point of departure is slightly restricted than that of WILLEMS (1971), that is, we only deal with the case in which g is defined by a convolution operator. With this formulation, we realize that the theory of distribution is more appropriate to treat the problem than the operator theory in Banach spaces. (See KRASNOSEL'SKII (1963) for the operator theoretic approach). For example, the Heaviside unit step function and the delta functional and its derivatives, which are common in automatic control systems, can be easily defined as an element of the linear part of the system.

In § 2 we take up the problem in terms of the distribution theory and review some known results and definitions of the distributional Fourier and Laplace transforms. In § 3 we shall show that the feedback equation can be transformed to an ordinary Volterra type equation. The inversion of certain convolution operators play an essential role in this transformation. In § 4 we give two theorems on the existence and uniqueness of the solution of the equation concerning the system. In § 5 we shall discuss the relationship between the conditions of two theorems we obtained and give in the frequency domain, the geometrical interpretation of the conditions shown in the theorems. Finally, we make some concluding remarks in § 6.

We add that some of the results in this paper have been announced and shown in KAWASHIMA (1973-a) and this paper is actually a refinement of that paper.

2. Notations and Definitions

This section is provided for definitions of certain class of functions and distributions. At the end of this section the feedback systems that we are going to study are described.

Let $L^p_{(R)}$, where $p=1,2$, denote the space of all measurable functions $x(\cdot)$ which vanish for negative arguments and satisfy

$$\int_0^\infty |x(t)|^p dt < \infty.$$

The spaces $L^p_{(-\infty,\infty)}$ on the real axis $(-\infty,\infty)$ are the ordinary ones. We shall also define the spaces $L^p_{loc(R)}$, $p=1,2$, of all measurable functions $x(\cdot)$ which vanish for negative arguments and satisfy

$$\int_0^T |x(t)|^p dt < \infty \qquad \text{for any finite } T.$$

Let $\mathfrak{D}$ denote the space of testing functions of bounded supports and $\mathfrak{D}'$ as its dual space. Moreover, let $\mathfrak{D}'_{(R)}$ be the space of distributions whose supports contained in the non-negative real axis.

The space of testing functions of rapid descent is denoted by $\mathfrak{S}$ and the dual space of $\mathfrak{S}$ or the space of temperate distribution is denoted by $\mathfrak{S}'$.

Throughout this paper, the notation $\mathfrak{F}k$ is understood to be either of the followings.

i) k belongs to $L^1_{(-\infty,\infty)}$ and $\mathfrak{F}k=\int_{-\infty}^{\infty} k(t)e^{-i\lambda t}dt$;

ii) k belongs to $L^2_{(-\infty,\infty)}$ and $\mathfrak{F}k$ is the Fourier transform in $L^2_{(-\infty,\infty)}$ of k;

iii) k belongs to $\mathfrak{S}'$ and $\mathfrak{F}k$ is in $L^1{}_{loc(-\infty,\infty)}$, that is a regular distribution defined by $\langle\mathfrak{F}k, \phi\rangle=\langle k, \mathfrak{F}\phi\rangle$ for any $\phi\in\mathfrak{S}$.

If k belongs to $\mathfrak{D}'_{(R)}$ and if $e^{-\gamma t}k(t)\in\mathfrak{S}'$ for $\gamma>\gamma'$, then the Laplace transform of $k(t)$ is given by

$$K(s)=\mathfrak{L}k=\mathfrak{F}\{e^{-\gamma t}k(t)\},\ \gamma>\gamma',\ s=\gamma+i\lambda.$$

$K(s)$ is an analytic function in its region of convergence $\mathrm{Re}\, s>\gamma'$.

We shall moreover denote the convolution of distributions f and g by $f*g$ whenever the convolution is defined.

Suppose the distributions u and v are defined by

$$u(t)=\sum_{k=0}^{n}\delta^{(k)}*p_k(t),\qquad p_k(t)\in L^p_{(-\infty,\infty)},\ 0\leqslant n<\infty,$$

and

$$v(t)=\sum_{k=0}^{m}\delta^{(k)}*q_k(t),\qquad q_k(t)\in L^{p'}_{(-\infty,\infty)},\ 0\leqslant m<\infty,$$

where $1/p+1/p'-1>0, p=1,2$ and $p'=1,2$. Then, we have

$$\mathfrak{F}\{u*v\}=\mathfrak{F}\{u\}\cdot\mathfrak{F}\{v\}=\sum_{k=0}^{n}(i\lambda)^kP_k(i\lambda)\sum_{k=0}^{m}(i\lambda)^kQ_k(i\lambda), \tag{1}$$

where $P_k(i\lambda)=\mathfrak{F}p_k$ and $Q_k(i\lambda)=\mathfrak{F}q_k$. (Schwartz, 1966, p. 270)

Now we shall define a class of functions and a class of distributions specific to this paper, that is $\mathfrak{N}(\alpha,\beta)$ and $\mathfrak{T}'_n$, respectively.

A real-valued continuous function $f(\cdot)$ is said to be of class $\mathfrak{N}(\alpha,\beta)$ if,

i) $f(0)\equiv 0$,

ii) there are two real numbers α,β such that

$$\alpha\leqslant\frac{f(\sigma_1)-f(\sigma_2)}{\sigma_1-\sigma_2}\leqslant\beta,\qquad(\alpha<\beta,\ \beta<0)\quad\text{for all }\sigma_1,\sigma_2\text{ with }\sigma_1\neq\sigma_2.$$

We note that if $\alpha\geqslant 0$, then $f(\sigma)$ is a monotone function of σ with a Lipschitz constant β and if $\alpha<0$ then $f(\sigma)$ is a function with a Lipschitz constant $\max\{|\alpha|,\beta\}$.

A real-valued distribution g is said to be of class $\mathfrak{T}'_n$ if it has the following form;

$$g(t)=\sum_{k=1}^{n}a_k\delta^{(k-1)}(t)+a_0 1_+(t)+g_1(t),\qquad(1\leqslant n<\infty)$$

where

i) $g_1(t)$ is a real-valued function in $L^1_{(R)}$,

ii) a_k is a real constant and $a_n \neq 0$,

iii) $\delta^{(k)}(t)$ is the k-th derivative of the delta functional,

iv) $1_+(t)$ is the Heaviside unit step function.

Now we are ready to define the feedback system. Let σ be a real-valued function in $L^1{}_{loc(R)}$. σ is called *an error* or *a solution* of a nonlinear feedback system shown in Fig. 1, if and only if σ satisfies the following relation.

$$\langle \sigma, \phi \rangle = \langle l, \phi \rangle - \langle g * f(\sigma), \phi \rangle \tag{2}$$

that is

$$\langle \sigma, \phi \rangle = \langle l, \phi \rangle - \sum_{k=1}^{n} a_k \langle f(\sigma), (-1)^{k-1} \phi^{(k-1)} \rangle$$
$$- \langle a_0 1_+ * f(\sigma), \phi \rangle - \langle g_1 * f(\sigma), \phi \rangle,$$

for any ϕ in $\mathfrak{D}$ and some real-valued distribution l in $\mathfrak{D}'_{(R)}$.

3. Transformation of the Equation

In this section we shall show that Eq. (2) can be transformed to an ordinary Volterra type equation.

The definition of $f(\sigma)$ implies that f is different from a linear form of σ by a function satisfying Lipschitz condition. To be more precise, we have

Lemma 1. *If* $f \in \mathfrak{N}(\alpha, \beta)$, *then for any* $c \in R^1$, *there exists a real-valued continuous function* $\hat{f}(\sigma)$ *such that*

i) $f(\sigma) = c\sigma + \hat{f}(\sigma)$,

ii) $|\hat{f}(\sigma_1) - \hat{f}(\sigma_2)| \leqslant k |\sigma_1 - \sigma_2|$; $k = \max\{|\beta - c|, |c - \alpha|\}$,

for real σ_1, σ_2 *with* $\sigma_1 \neq \sigma_2$.

Proof. Define $\hat{f}(\sigma)$ by $\hat{f}(\sigma) = f(\sigma) - c\sigma$. It is obvious that $\hat{f}(\sigma)$ is a real-valued continuous function for which $\hat{f}(0) \equiv 0$. From the definition of $\hat{f}$, we have

$$|\hat{f}(\sigma_1) - \hat{f}(\sigma_2)| = |\sigma_1 - \sigma_2| \left| \frac{f(\sigma_1) - f(\sigma_2)}{\sigma_1 - \sigma_2} - c \right|.$$

Now define k by $k = \max\limits_{\sigma_1 \neq \sigma_2} \left\{ \left| \frac{f(\sigma_1) - f(\sigma_2)}{\sigma_1 - \sigma_2} - c \right| \right\}$. If $\beta > \alpha \geqslant c > 0$, then $k = \beta - c$. If $c > 0$ and $\beta \geqslant c > \alpha$, then $k = \max\{\beta - c, |\alpha - c|\}$. Furthermore, if $c \geqslant \beta > \alpha$, then $k = \max\{|\beta - c|, |\alpha - c|\}$. In this way we can find a constant k for other cases and obtain the statement of the lemma.

We, thus, have from Lemma 1 and Eq. (2)

$$\langle \sigma, \phi \rangle = \langle l, \phi \rangle - \langle g * [c\sigma + \hat{f}(\sigma)], \phi \rangle$$

or

$$\langle(\delta+cg)*\sigma, \phi\rangle=\langle l, \phi\rangle-\langle g*\hat{f}(\sigma), \phi\rangle.$$

This shows that if $(\delta+cg)$ has an inverse $(\delta+cg)^{-1}$ in $\mathfrak{D}'_{(R)}$ then the left multiplication by the convolution operator $(\delta+cg)^{-1}$ gives us

$$(3) \qquad \langle\sigma, \phi\rangle=\langle(\delta+cg)^{-1}*l, \phi\rangle-\langle\hat{g}*\hat{f}(\sigma), \phi\rangle$$

where $\hat{g}=(\delta+cg)^{-1}*g$. Note that (3) is well defined as a distribution since we assumed that $l, \sigma, (\delta+cg)^{-1}$ all are in $\mathfrak{D}'_{(R)}$.

We review some results on inverting the convolution operator $(\delta+cg)$ in $\mathfrak{D}'_{(R)}$. (KAWASHIMA, 1973-b).

Lemma 2. *Let g be in $\mathfrak{T}'_n$. Suppose g satisfies one of the following two conditions for some nonzero constant c;*

i) $|\mathfrak{L}\{\delta+cg\}|\neq 0$, in $\mathrm{Re}\, s\geqslant 0$ *when* $a_0=0$,

ii) $|\mathfrak{L}\{\delta^{(1)}+c\delta^{(1)}*g\}|\neq 0$, in $\mathrm{Re}\, s\geqslant 0$ *when* $a_0\neq 0$,

where $\mathfrak{L}$ means the Laplace transform operator. Then the following results hold.

(A) *If $n=1$, $a_0\neq 0$, then there exists a function h in $L^2_{(R)}$ such that*

$$(\delta+cg)*(\delta^{(1)}*h)=\delta.$$

(B) *If $n=1$, $a_0=0$, then there exist a function h in $L^1_{(R)}$ and a real number $d\neq 0$ such that*

$$(\delta+cg)*(d\delta+h)=\delta.$$

(C) *If $n\geqslant 2$, then there exists a function h in $L^2_{(R)}$ such that*

$$(\delta+cg)^{-1}*h=\delta.$$

We also have

Corollary 2-1. *Let g be in $\mathfrak{T}'_n$ and satisfy the conditions of* Lemma 2. *Then the following results hold.*

i) *If $n=1$, $a_0\neq 0$, then there exist a nonzero real number $\hat{d}$ and a function $\hat{h}$ in $L^2_{(R)}$ such that*

$$(\delta+cg)^{-1}*g=\hat{d}\delta+\delta^{(1)}*\hat{h}.$$

ii) *If $n=1$, $a_0=0$, $a_1\neq 0$, then there exist a nonzero real number $\hat{d}$ and a function $\hat{h}$ in $L^1_{(R)}$ such that*

$$(\delta+cg)^{-1}*g=\hat{d}\delta+\hat{h}.$$

iii) *If $n=1$, $a_0=0$, $a_1=0$, then there exists a function $\hat{h}$ in $L^1_{(R)}$ such that*

$$(\delta+cg)^{-1}*g=\hat{h}.$$

iv) *If $n\geqslant 2$, then there exist a nonzero number $\hat{d}$ and a function $\hat{h}$ in $L^2_{(R)}$ such that*

$$(\delta+cg)^{-1}*g=\hat{d}\delta+\hat{h}.$$

We now state a lemma which is of frequent use in later analysis.

Lemma 3. *Let $x(t)$ be a function in $L^2{}_{loc(R)}$. Let r be a distribution in $\mathfrak{S}' \cap \mathfrak{D}'_{(R)}$ such that the Fourier transform $R(i\lambda)$ of r is a function in $L^1{}_{loc(-\infty,\infty)}$ and satisfies*

$$\sup_{-\infty<\lambda<\infty} |R(i\lambda)| \equiv \mu < \infty$$

Then

$$\int_0^T |r * x(t)|^2 dt \leqslant \mu^2 \int_0^T |x(t)|^2 dt$$

for any $T<\infty$.

Proof. See KAWASHIMA. (1973-c)

4. Existence and Uniqueness

4-1. Method Based on Contraction.

Throughout this section we assume that f is in $\mathfrak{N}(\alpha, \beta)$ for some fixed α, β and $g \in \mathfrak{T}'_n$ satisfies one of the conditions of Lemma 2 for some $c \neq 0$. From Lemma 2 and Corollary 2-1, it is obvious that the Fourier transform of $\hat{g}$ and $(\delta + cg)^{-1}$ are continuous functions of λ and

$$(4) \qquad \sup_{-\infty<\lambda<\infty} |\mathfrak{F}\hat{g}| < \infty$$

$$\sup_{-\infty<\lambda<\infty} |\mathfrak{F}\{(\delta + cg)^{-1}\}| < \infty.$$

Now, in view of Lemma 2, Lemma 3 and (4), we can easily verify that if $l(t)$ belongs to $L^2{}_{loc(R)}$ then $\hat{l} \equiv (\delta + cg)^{-1} * l$ is also a function in $L^2{}_{loc(R)}$. In this case Eq. (3) is equivalent to

$$(5) \qquad \sigma(t) = \hat{l}(t) - \hat{g} * \hat{f}(\sigma)(t), \qquad t \geqslant 0.$$

Therefore, if σ in $L^2{}_{loc(R)} \subset L^1{}_{loc(R)}$ satisfies (5) for some $\hat{l}$ in $L^2{}_{loc(R)}$, then this σ satisfies (2), that is, σ is *a solution or an error* of a nonlinear feedback system. From this we see that it is sufficient to consider Eq. (5) for studying the existence and uniqueness of a solution described by (2) for certain inputs in $\mathfrak{D}'_{(R)}$.

REMARK 1. We note that if a distribution l in $\mathfrak{D}'_{(R)}$ and g in $\mathfrak{T}'_n$ satisfy certain conditions, then $\hat{l}$ becomes a function in $L^2{}_{loc(R)}$. In order to discuss the case different from the one in which $l \in L^2{}_{loc(R)}$, we further assume that l has the form $l(t) = \sum_{k=0}^{m} \delta^{(k)} * p_k(t)$, where $p_k \in L^2{}_{(R)}$. Suppose $n > m$ in the defining equation of g in $\mathfrak{T}'_n$. Then by a direct calculation we see that $\hat{l}$ belongs to $L^2{}_{(R)} \subset L^2{}_{loc(R)}$.

Next theorem is based on the contraction mapping theorem in Banach spaces.

We assume that g in $\mathfrak{T}'_n$ satisfies the conditions of Lemma 2 for some nonzero c. We then obtain

Theorem 1. *Let f be in $\mathfrak{N}(\alpha, \beta)$ and let $\hat{l}$ be in $L^2_{loc(R)}$. Moreover, suppose $\hat{g}$ satisfies*

$$k \sup_{-\infty<\lambda<\infty} |\mathfrak{F}\hat{g}|<1, \qquad k=\max\{|\beta-c|, |c-\alpha|\}.$$

Then there exists a unique solution of (5) *which belongs to $L^2_{loc(R)}$.*

Proof. See KAWASHIMA (1973-c).

4-2. Method Based on the Properties of a Nonlinear Function *f*.

We now give a different approach to the ploblem of the existence and uniqueness of a nonlinear feedback systems. We need three lemmas for our purpose.

Lemma 4. *Let $\hat{f}$ be a function in $\mathfrak{N}(-k, k)$ where k is a positive constant. For nonzero real number $\hat{d}$, define $\psi(\sigma)$ by*

$$\psi(\sigma)=\sigma+\hat{d}\hat{f}(\sigma). \tag{7}$$

If $|\hat{d}|<1/k$, then $\psi(\sigma)$ belongs to $\mathfrak{N}(1-|\hat{d}|k, 1+|\hat{d}|k)$.

Proof. Obviously $\psi(\sigma)$ is a real-valued continuous function with $\psi(0)\equiv 0$. Define $\Delta\sigma\Delta\psi$, $\Delta\sigma\Delta\hat{f}$ by

$$\Delta\sigma\Delta\psi=(\sigma_1-\sigma_2)(\psi(\sigma_1)-\psi(\sigma_2))$$

$$\Delta\sigma\Delta\hat{f}=(\sigma_1-\sigma_2)(\hat{f}(\sigma_1)-\hat{f}(\sigma_2))$$

for any σ_1, σ_2 with $\sigma_1\neq\sigma_2$. From (7) we have

$$\Delta\sigma\Delta\psi=(\Delta\sigma)^2+\hat{d}\Delta\sigma\Delta\hat{f}. \tag{8}$$

If $\hat{d}>0$ and $\Delta\sigma\Delta\hat{f}\geqslant 0$, then from (8) we have

$$\Delta\sigma\Delta\psi=(\Delta\sigma)^2+\hat{d}|\Delta\sigma||\Delta\hat{f}|\leqslant(1+\hat{d}k)(\Delta\sigma)^2. \tag{9}$$

If $\hat{d}>0$ and $\Delta\sigma\Delta\hat{f}\leqslant 0$, then from (8) we have

$$\Delta\sigma\Delta\psi=(\Delta\sigma)^2-\hat{d}|\Delta\sigma||\Delta\hat{f}|\geqslant(1-\hat{d}k)(\Delta\sigma)^2. \tag{10}$$

Now, from the assumption of the lemma and (9), (10) we see that $\Delta\sigma\Delta\psi>0$ for any σ_1, σ_2 with $\sigma_1\neq\sigma_2$. This shows that $\psi(\sigma)$ is a monotone function of σ for $\hat{d}>0$. The monotonicity of $\psi(\sigma)$ for $\hat{d}<0$ follows in a similar way. Thus, we see that $\psi(\sigma)$ is in $\mathfrak{N}(1-|\hat{d}|k, 1+|\hat{d}|k)$.

Next we have

Lemma 5. *Let $\hat{f}$ belong to $\mathfrak{N}(-k, k)$ and let $\hat{d}$ be a nonzero real number such that $|\hat{d}|<1/k$. Define $\chi\equiv\psi(\sigma)=\sigma+\hat{d}\hat{f}(\sigma)$. Then, there exists a monotone function Φ which belongs to $\mathfrak{N}(1/1+|\hat{d}|k, 1/1-|\hat{d}|k)$ and satisfies,*

$$\Phi(\psi(\sigma))=\sigma,\ \psi(\Phi(\chi))=\chi. \tag{11}$$

Proof. Since $|\hat{d}|<1/k$, $\psi(\sigma)$ is a continuous monotone function. Therefore, there exists a monotone continuous function $\Phi(\chi)$ such that $\Phi(\chi)$ is an inverse function of $\psi(\sigma)$. We shall next show that Φ belongs to $\mathfrak{N}(1/1+|\hat{d}|k,\ 1/1-|\hat{d}|k)$. From (11) we have

$$\begin{aligned}\Delta\Phi\Delta\chi&=\{\Phi(\chi_1)-\Phi(\chi_2)\}\{\chi_1-\chi_2\}\\&=\{\sigma_1-\sigma_2\}\{\psi(\sigma_1)-\psi(\sigma_2)\}=\Delta\sigma\Delta\psi\end{aligned} \tag{12}$$

where $\chi_1\neq\chi_2$. From (9), (10), (11) and (12) we see

$$(1-|\hat{d}|k)(\Delta\Phi)^2\leqslant\Delta\Phi\Delta\chi\leqslant(1+|\hat{d}|k)(\Delta\Phi)^2$$

or

$$(1-|\hat{d}|k)\leqslant\frac{\Delta\chi}{\Delta\Phi}\leqslant(1+|\hat{d}|k).$$

This shows that Φ belongs to $\mathfrak{N}(1/1+|\hat{d}|k,\ 1/1-|\hat{d}|k)$.

Finally, we have

Lemma 6. *Consider the integral equation* (13).

$$\chi(t)=\hat{l}(t)-\int_0^t\hat{h}(t-\tau)\tilde{f}(\chi(\tau))d\tau. \tag{13}$$

Let $\hat{l}$ and $\hat{h}$ be functions in $L^2_{loc(R)}$ and let $\tilde{f}$ be a function that satisfies the Lipschitz condition and $\tilde{f}(0)=0$. Then, there exists a unique solution $\chi(t)$ of (13) *which belongs to $L^2_{loc(R)}$.*

Proof. We can easily verify that if $\tilde{f},\hat{h},\hat{l}$ satisfy the assumptions of the lemma, then $\tilde{f},\hat{h},\hat{l}$ satisfy the sufficient conditions of the existence and uniqueness theorem of the nonlinear Volterra integral equations given by TRICOMI (1957, p. 42). Uniqueness in $L^2_{loc(R)}$ follows from the assumption that $\hat{h}$ vanishes on negative real axis.

In Lemma 6 the assumption $\hat{h}\in L^2_{loc(R)}$ is most essential. However, the results of Corollary 2-1 show that this assumption does not hold when $n=1$. Therefore we need the following corollary.

Corollary 2-2. *Let g be in $\mathfrak{T}_1'$ and satisfy the conditions of Lemma* 2. *Suppose, in addition, $g_1(t)$ belongs to $L^1_{(R)}\cap L^2_{(R)}$. Then there exist a real constant $\hat{d}$ and a function $\hat{h}\in L^2_{(R)}$ such that*

$$(\delta+cg)^{-1}*g=\hat{d}\delta+\hat{h}.$$

Proof. If $a_0=0$, then we have $\hat{h}=\mathfrak{F}^{-1}\left\{\frac{1}{c(1+ca_1)}\frac{\tilde{G}_1(i\lambda)}{1+\tilde{G}_1(i\lambda)}\right\}$ where $\tilde{G}_1(i\lambda)=\frac{c}{1+ca_1}\times G_1(i\lambda)$ and $\hat{d}=a_1/1+ca_1$. From Theorem 1 of KAWASHIMA (1973-b), we can easily show that $\hat{h}$ actually belongs to $L^1_{(R)}\cap L^2_{(R)}$. On the other hand, when $a_0\neq0$, we shall give different proofs according to the value of a_1.

i) $a_0 \neq 0$, $a_1 = 0$.

Since the convolution $\delta^{(1)} * h * 1_+$ is an associative and a commutative operation in $\mathfrak{D}'_{(R)}$, we have by Lemma 2(A) that

$$(\delta + cg)^{-1} * g = (\delta^{(1)} * h) * (a_0 1_+ + g_1) = a_0 h + \delta^{(1)} * h * g_1.$$

The Fourier transform of a distribution $\delta^{(1)} * h * g_1 \in \mathfrak{S}' \cap \mathfrak{D}'_{(R)}$ is given by $(i\lambda)H(i\lambda)G_1(i\lambda)$ from the relation (1), since $h(t) \in L^2_{(R)}$ and $g_1(t) \in L^1_{(R)} \cap L^2_{(R)}$. We further have

$$\int_{-\infty}^{\infty} |(i\lambda)H(i\lambda)G_1(i\lambda)|^2 d\lambda \leqslant N^2 \int_{-\infty}^{\infty} |G_1(i\lambda)|^2 d\lambda < \infty$$

where

$$N = \sup_{-\infty < \lambda < \infty} |(i\lambda)H(i\lambda)| = \sup_{-\infty < \lambda < \infty} \left| \frac{i\lambda}{a_0' + (i\lambda)[a_1' + cG_1(i\lambda)]} \right|$$

and $a_0' = ca_0$, $a_1' = 1 + ca_1$. This shows that a distribution $\delta^{(1)} * h * g_1$ can be identified with a function in $L^2_{(R)}$.

ii) $a_0 \neq 0$, $a_1 \neq 0$.

In this case we have

$$(\delta + cg)^{-1} * g = a_0 h + a_1 \delta^{(1)} * h + \delta^{(1)} * h * g_1. \tag{14}$$

We shall show that $\delta^{(1)} * h$ can be decomposed into the form $\delta^{(1)} * h = e\delta + q$, where $q(t) \in L^2_{(R)}$ and $e \neq 0$. In particular, we have

$$\mathfrak{F}\{\delta^{(1)} * h\} = \frac{1}{a_1'} \left\{ 1 - \frac{a_0' + c(i\lambda)G_1(i\lambda)}{a_0' + (i\lambda)[a_1' + cG_1(i\lambda)]} \right\}$$

$$= e + Q(i\lambda).$$

Noting the definition of $H(i\lambda)$, we rewrite $Q(i\lambda)$ by

$$Q(i\lambda) = \frac{a_0'}{a_1'} H(i\lambda) + \frac{c}{a_1'} (i\lambda)H(i\lambda)G_1(i\lambda).$$

Thus $Q(i\lambda)$ is the Fourier transform of a function in $L^2_{(R)}$. Using the relation $\delta^{(1)} * h = e\delta + q$ in (14), we obtain

$$(\delta + cg)^{-1} * g = a_1 e\delta + a_0 h + a_1 q * h + eg_1 + q * g_1.$$

This shows that the statement of the corollary holds.

Now we are ready to show the main theorem of this paper. We assume that g in $\mathfrak{T}'_n$ satisfies the conditions of Lemma 2 for some nonzero c.

Theorem 2. *Let f be in $\mathfrak{N}(\alpha, \beta)$, $\hat{l}$ be in $L^2_{loc(R)}$. When $n=1$, suppose g_1 of g satisfies the conditions of Corollary* 2-2 *so that for any n, $\hat{g}$ has the form $\hat{g} = \hat{d}\delta + \hat{h}$, where $\hat{h} \in L^2_{(R)}$. Moreover, suppose $\hat{d}$ satisfies the inequality $|\hat{d}| < 1/k$, where $k = \max\{|\beta - c|, |c - \alpha|\}$. Then, there exists a unique solution of* (5) *which belongs to $L^2_{loc(R)}$.*

Proof. If $\hat{d} = 0$, then the proof is trivial from Lemma 5. Hence we only deal with the case $\hat{d} \neq 0$.

From the assumptions, (5) can be rewritten by

$$\sigma(t)=\hat{l}(t)-\hat{d}\hat{f}(\sigma(t))-\int_0^t \hat{h}(t-\tau)\hat{f}(\sigma(\tau))d\tau, \qquad t\geqslant 0, \tag{15}$$

where $\hat{f}$ belongs to $\mathfrak{N}(-k, k)$ and $k=\max\{|\beta-c|, |c-\alpha|\}$. Define $\varphi(\sigma)$ by (7) and form $\Phi(\chi)$. Also define $\tilde{f}(\chi)$ by $\hat{f}(\sigma)=\hat{f}(\Phi(\chi))\equiv\tilde{f}(\chi)$.

Now consider the following Volterra type integral equation.

$$\chi(t)=\hat{l}(t)-\hat{h}*\tilde{f}(\chi)(t), \qquad t\geqslant 0. \tag{16}$$

Since the Lipschitz constant of $\tilde{f}$ is given by $k/1-|\hat{d}|k$ and $\tilde{f}(0)=0$, we easily verify from Lemma 5 that (16) has a unique solution χ in $L^2{}_{(0,T)}$ for any $T<\infty$. This implies that for an arbitrary fixed T, we have $|\chi(t)|<\infty$, a.a. t, $t\in[0, T]$. Operate the function Φ to χ for such $t=t^*$. Then we have

$$\sigma(t^*)=\Phi(\chi(t^*))=\chi(t^*)-\hat{d}\tilde{f}(\chi(t^*)). \tag{17}$$

Moreover, since Φ is a continuous function, $\sigma(t)=\Phi(\chi(t))$ becomes a Lebesgue measurable function of t in $[0, T]$. Disregarding the values of σ defined on null sets, $\sigma(t)$ is a uniquely defined measurable function on $[0, T]$. We have from (17)

$$\int_0^T |\sigma(t)|^2 dt\leqslant 2\left\{1+\left(\frac{|\hat{d}|k}{1-|\hat{d}|k}\right)^2\right\}\int_0^T |\chi(t)|^2 dt<\infty,$$

for some fixed finite T.

Suppose there are two inputs $\hat{l}_1(t)$ and $\hat{l}_2(t)$ such that

$$\hat{l}_1(t)=\hat{l}_2(t), \qquad 0\leqslant t\leqslant T;$$

$$\hat{l}_1(t)\neq\hat{l}_2(t), \qquad T<t\leqslant T+A, \qquad A>0.$$

Let χ_1 and χ_2 be the solutions of (16) respectively with respect to $\hat{l}_1$ and $\hat{l}_2$ and let σ_1 and σ_2 be the functions defined by (17) respectively with respect to χ_1 and χ_2. Since Lemma 6 asserts that $\chi_1(t)=\chi_2(t)$, a.a. t, $t\in[0, T]$, we have $\sigma_1(t)=\sigma_2(t)$, a.a. t, $t\in[0, T]$. Therefore, $\sigma(t)$ is unique in $L^2{}_{loc(R)}$.

Now, noting the equality (17) and the relation $\hat{f}(\sigma)\equiv\tilde{f}(\chi)$, we see that (15) holds in $L^2{}_{loc(R)}$. Thus, if (16) has a solution in $L^2{}_{loc(R)}$ then (15) also has a solution in $L^2{}_{loc(R)}$. This completes the proof.

Remark 2. We have shown in Remark 1 that if $l=\sum_{k=1}^{m}\delta^{(k)}*p_k(t), p_k(t)\in L^2{}_{(R)}$, $m<n$, then $\hat{l}$ becomes a function in $L^2{}_{(R)}$. Therefore, if $\hat{g}$ satisfies the conditions of Theorem 2, then the solution σ exists in $L^2{}_{loc(R)}$. However this does not mean that the output $y=g*f(\sigma)$ becomes a function in $L^2{}_{loc(R)}$. On the other hand, suppose $\hat{g}$ satisfies the conditions of Theorem 1 and l is given by $l=\sum_{k=1}^{m}\delta^{(k)}*p_k(t), p_k(t)\in L^2{}_{(R)}$, $m<n$. Then, by Minkowski inequality and Lemma 3, we see that the solution σ actually belongs to $L^2{}_{(R)}$. By a direct application of Lemma 2, we have

$$y(t)=c\hat{g}*l(t)+\hat{g}*\hat{f}(\sigma)(t), \qquad t\geqslant 0. \tag{18}$$

This equality and the results of Corollary 2–1 show that $y(t)$ has the following representation.

$$y(t)=\sum_{k=1}^{m'}\delta^{(k)}*q_k(t),\qquad q_k(t)\in L^2{}_{(R)},\qquad m'\leqslant m+1.$$

We also note that if l belongs to $L^2{}_{loc(R)}$ then (18) and the existence of σ in $L^2{}_{loc(R)}$ implies $y(t)\in L^2{}_{loc(R)}$.

5. Geometrical Interpretation of Theorems

In this section we shall discuss the relation between conditions of the theorems which we have obtained in the previous sections. In general, Theorem 2 gives a much better condition than Theorem 1 does under the assumption that Corollary 2–2 holds.

Case a: $n=1, a_0\neq 0$.

Clearly we can apply Theorem 1 to a more general class of distributions than in Theorem 2. However, if g satisfies the conditions of Corollary 2–2, then Theorem 2 is more general than Theorem 1. This is so, because $\lim_{|\lambda|\to\infty}|\mathfrak{F}\hat{g}|=\left|\frac{a_1}{1+a_1c}\right|$, $\lim_{|\lambda|\to\infty}|\mathfrak{F}\hat{g}|=|\hat{d}|=\left|\frac{1}{c}\right|$ and $\sup_{-\infty<\lambda<\infty}|\mathfrak{F}\hat{g}|<\frac{1}{k}$ implies $\left|\frac{a_1}{1+a_1c}\right|<\frac{1}{k}$ and $\left|\frac{1}{c}\right|<\frac{1}{k}$. Moreover, if $c=(\alpha+\beta)/2$, then α must be positive in Theorem 1. On the other hand, since the Fourier transform of g is a distribution in the frequency domain, the locus of $\mathfrak{F}g$ does not exist.

Case b: $n=1, a_0=0$.

In this case, if $\hat{h}\in L^1{}_{(R)}$ then we have only to apply Theorem 1 to $\hat{g}$. If $g\in\mathfrak{D}'_1$ satisfies the condition of Corollary 2–2, then $\hat{h}$ belongs to $L^2{}_{(R)}$ and we can apply both theorems to $\hat{g}$. Since $\sup_{-\infty<\lambda<\infty}|\mathfrak{F}\hat{g}|<\frac{1}{k}$ implies $|\hat{d}|=\left|\frac{a_1}{1+a_1c}\right|<\frac{1}{k}$, we have that Theorem 2 is more general than Theorem 1. We also note that if $c=(\alpha+\beta)/2$ then α can be nonnegative in both theorems.

Next we shall show an interesting example which is a special case of Case b, that is $a_1\neq 0$ and $g_1\in L^1{}_{(R)}\cap L^2{}_{(R)}$. In this case, obviously we can apply both theorems to $g=a_1\delta+g_1$ and $\hat{g}=\hat{d}\delta+\hat{h}$. Moreover geometrical interpretation of various conditions for existence of the solution is possible. We shall only deal with the case in which $\alpha>0$, $c=(\alpha+\beta)/2$.

Now, if we apply Theorem 1 to g, then we obtain the condition $\sup_{-\infty<\lambda<\infty}|\mathfrak{F}g|=\sup_{-\infty<\lambda<\infty}|G(i\lambda)|<1/\beta$, where β is the Lipschitz constant of f in this case. This implies that the locus of $G(i\lambda)$ is contained in the circle C_1 shown in Fig. 2. Applying Theorem 2 to g, we see that a_1 is contained in the open interval $(-1/\beta, 1/\beta)$. On the other hand, an application of Theorem 1 to $\hat{g}$ leads to a conclusion that the locus of $G(i\lambda)$ for $\lambda\in(-\infty,\infty)$ lies the outside of C_2 shown in Fig. 3 and does not encircle it. This can be easily verified if the conditions, $|1+cG(s)|>0$, in Re

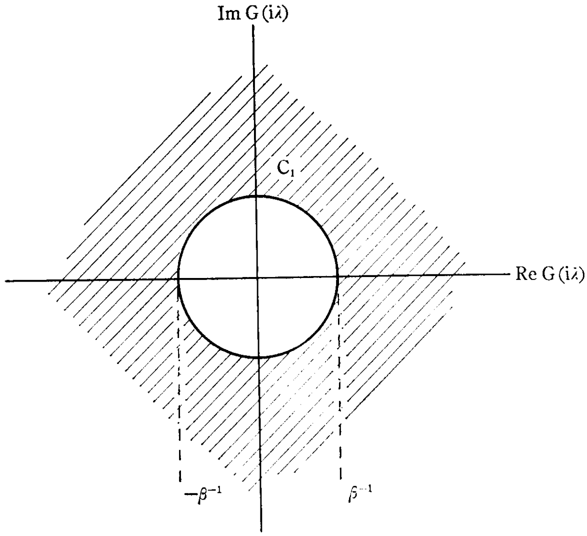

Fig. 2. Geometric Interprepation of the Existence Conditions; the Circle C_1.

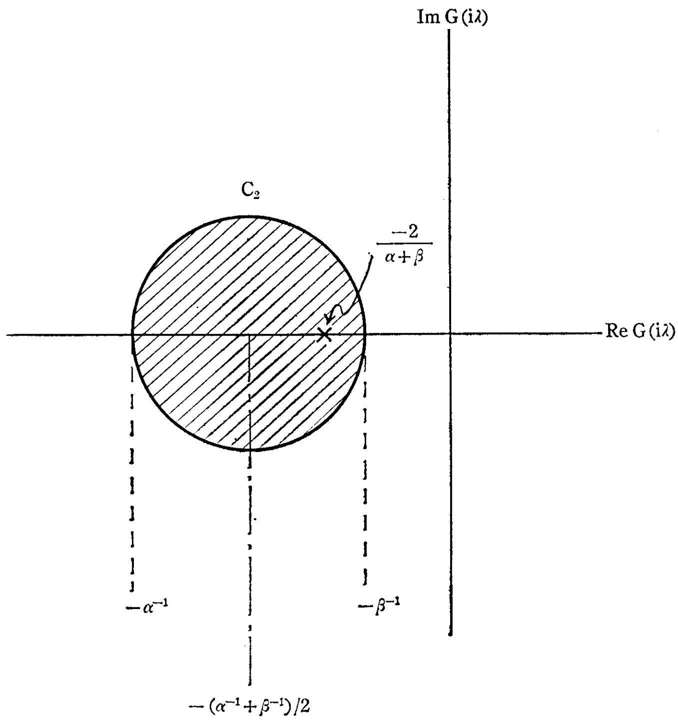

Fig. 3. Geometric Interpretation of the Existence Conditions; the Circle C_2.

$s \geqslant 0$, $\sup_{-\infty<\lambda<\infty} |\mathfrak{F}\hat{g}| < \frac{1}{k}$ are taken in mind and the principle of the arguments are applied. (HOLTZMAN 1970, p. 177). We remark that if the locus of $G(i\lambda)$ is contained in the circle C_1, then the locus of $G(i\lambda)$ obviously lies outside the circle C_2.

If we apply Theorem 2 to $\hat{g}$ then we can show that $a_1 \notin [-1/\alpha, -1/\beta]$ and locus of $G(i\lambda)$ does not go through and does not encircle the point $-2/(\alpha+\beta), 0$.

Case c: $n \geqslant 2$.

In this case $\hat{h}$ belongs to $L^2{}_{(R)}$. From the definition of $\hat{h}$, $\lim_{|\lambda|\to\infty} |\mathfrak{F}\hat{h}|=0$, when $a_0=0$ and $\lim_{|\lambda|\to\infty} |\mathfrak{F}\hat{h}|=0$, $\lim_{|\lambda|\to 0} |\mathfrak{F}\hat{h}|=0$, when $a_0 \neq 0$. Thus, $\sup_{-\infty<\lambda<\infty} |\mathfrak{F}\hat{g}|<1/k$ implies $|\hat{d}|=|1/c|<1/k$. This shows that Theorem 2 is more general than Theorem 1. If the constant c is given by $c=(\alpha+\beta)/2$, then $|1/c|<1/k$ implies $\alpha>0$. Moreover, if $a_0=0$, $\sup_{-\infty<\lambda<\infty} |\mathfrak{F}\hat{g}|<1/k=2/(\beta-\alpha)$ implies that the locus of $G(i\lambda)$ lies outside the circle C_2. However, we can not use the principle of arguments to check the conditions of Lemma 2 in this case, since $G(i\lambda)$ has a pole at $|\lambda|=\infty$. Now, if $a_0 \neq 0$, then g becomes a distribution in the frequency domain and the graphical interpretation is not possible.

6. Concluding Remarks

We have defined a nonlinear feedback system in terms of distribution theory. By this formulation, we can generalize the results of HOLTZMAN (1970) and ZAMES (1964) to the case in which the linear element involves certain distributions familiar in the theory of automatic control.

Our main result here is Theorem 2 which gives a very general condition on the existence and uniqueness of the solution concerning the system. We have shown some graphical interpretation of the theorems in the frequency domain.

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REFERENCES

HOLTZMAN, J. M. (1970): Nonlinear System Theory-A Functional Analysis Approach, Prentice-Hall.

KAWASHIMA, H. (1973-a): On the Existence and Uniqueness of the Solution Related to Nonlinear Feedback Systems, (in Japanese), Transaction of the Society of Instrument and Control Engineers, in preparation.

——— (1973-b): On the inversion of Certain Convolution Operators Related to Feedback Systems, Keio Engineering Reports, Vol. 26.

——— (1973-c): Nonlinear Feedback Systems and Weakly Stationary Stochastic Processes Information and Control, Vol. 22, No. 3, 283-295.

KRASNOSEL'SKII, M. A. (1963): Topological Methods in the Theory of Nonlinear Integral Equations, (translation from Russian), Pergamon Press.

SCHWARTZ, L. (1966): Théorie des Distributions, Hermann.

TRICOMI, F. G. (1957): Integral Equations, Interscience Publishers.

WILLEMS, J. C. (1969): Stability, Instability, Invertibility and Causality, SIAM Journal on Control, Vol. 7, No. 4, 645-671.

——— (1971): The Analysis of Feedback Systems, Research Monograph No. 62, The MIT Press.

ZAMES, G. (1966-a): On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems, Part 1: Conditions Derived Using Concepts of Loop Gain, Conicity, and Positivity, IEEE Transactions on Automatic Control, Vol. Ac-11, No. 2 228-238.

——— (1966-b): On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems, Part 2: Conditions Involving Circles in the Frequency Plane and Sector Nonlinearities, IEEE Transactions on Automatic Control, Vol. Ac-11, No. 3, 465-476.

——— (1964): Realizability Conditions for Nonlinear Feedback Systems, IEEE Transactions on Circuit Theory, Vol. CT-11, 189-194.