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CONCEPTS OF CONTROLLABILITY OF QUANTIZED CONTROL SYSTEMS

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ABSTRACT

Concepts of controllability of linear, time-invariant, discrete quantized control systems are clarified and the differences from those of the systems with continuous-level controls are pointed out. The relations between eigenvalues and controllability of quantized control systems are discussed, and it is shown that the *numerical* property of eigenvalues has an essential influence on controllability of such systems. Then, an index to measure the degree of controllability of quantized control systems is introduced and its characteristics are discussed in the case that every element of coefficient matrices in the transition equation is rational.

1. Introduction

Control systems whose controls can take only given discrete-level values are called quantized control systems. Some works have been made on those systems. Control problems for systems with saturation characteristics or relay controls have been studied by a lot of authors (see, e.g., FLÜGGE-LOTZ 1953; GIBSON 1963; PONTRYAGIN et al. 1962). LEWIS and TOU (1963) provided dynamic programming technique for the optimal control synthesis of linear, sampled-data quantized control systems with a single control input and distinct eigenvalues of the transition matrix. KIM and DJADJURI (1967) provided integer programming approach for the optimal control synthesis of quantized control systems. HAVIRA and LEWIS (1972) treated control problems with even and piecewise-linear cost functionals and found that the optimal control signals of those systems are necessarily quantized. They also provided differential dynamic programming technique for the synthesis of optimal controls of those systems.

As is seen above, most of the works on quantized control systems have been restricted to classical control theories and optimal control synthesis. So far as the auther knows, any basic studies on control system structures of them have not been made so far. For example, even in the fundamental problems such as reachability and controllability, the behavior of those systems may be quite different from the systems with continuous-level controls. In the sense of reachability and controllability of continuouslevel control systems, the state space is neither reachable nor controllable almost everywhere if the system is the finite-dimensional, linear, time-invariant, discrete quantized control system.

With the development of systems engineering, the applications of control theories to various fields other than the conventional automatic control processes have been rapidly stimulated. Many of them have, though it may not be recognized well, controllers with discrete-level controls. For example, capital budgeting problems for multi-stage economic systems with economies of scale (see, e.g., AOKI 1971), control problems for additive automata (ARBIB 1965; KALMAN et al. 1969) and linear sequential networks (COHN 1962; GILL 1966; TZAFESTAS 1972), programmed control problems for traffic control systems, schedule control problems (MITSUMORI 1969) and control problems for the systems with some parallel-connected on-off controllers are classified into this category.

Considering such directions of research development as above, we need some basic studies on control system structures of quantized control systems to answer questions such that 'what is the system we are to control?' or 'how good can we control the system?'.

Approaches may be different whether the continuous or discrete (multi-stage) systems are treated. We restrict our attention to the latter, especially, the simple finite-dimensional, linear, time-invariant, discrete quantized control systems as follows in this paper:

$$x(t+1)=Ax(t)+Bu(t), \qquad t=0,1,\dots$$

where a state, $x(t)$, is an element of n-dimensional real Euclidean space, E^n, and a control, $u(t)$, is an element of the set of r-dimensional integer vectors, $Z^r \subset E^r$. We call them quantized control systems for simplification in this paper.

The objectives of this paper are:

1. to clarify the controllability concept for the quantized control systems, where the *t-step controllable set* will be defined as:

$$M_t=\left\{x \mid x\in E^n,\ \exists\, u(\tau)\in Z^r, \tau=0,1,\cdots,t-1 \text{ such that } x=\sum_{\tau=0}^{t-1} A^{t-1-\tau}Bu(\tau)\right\},$$

2. to obtain conditions for the existence of finite t such that $M_{t'}=M_t$ for all $t'\geqslant t$ in the quantized control systems with distinct eigenvalues,

3. to introduce *t-step controllability index* which is an index to measure the degree of controllability of a quantized control system, discuss its characteristics, obtain its easily calcullable upper and lower bounds in the case that every element of matrices A and B is rational, and show that it is a physically natural index for controllability of quantized control systems.

Quantized control systems can also be classified as the systems with restrictions to controls. Some results on such systems have been obtained so far. For example, DESOER and WING (1961) and WING and DESOER (1963) proved that the necessary and sufficient condition for a single- or double-input, linear, time-invariant discrete system whose controls, $u_k(\tau)$, are bounded as $|u_k(\tau)| \leqslant 1$ for all τ to be completely controllable is that it is completely controllable without bounds on controls and $|\lambda_j| \leqslant 1$ for all eigenvalues, λ_j, of the system. BERTSEKAS (1972), RAGHAVAN (1971) and SAPERSTONE (1971) also obtained some results on controllability of the systems with bounded controls.

However, the restrictions on controls in the studies so far are mostly such that the set of admissible controls is compact and convex. But the set of admissible controls in quantized control systems is not convex, and the analytical approaches taken so far may be impractical. KALMAN and others have developed the algebraic approach to control theory (KALMAN et al. 1969; ZEIGER 1967), where z-transforms of inputs and outputs are treated as elements of the ring of polynomials, and input-output relations and the state space as homomorphisms and the module on that ring, respectively. Since the ring of integers which is the set of admissible control values of quantized control systems resembles the ring of polynomials very much, the same approach as theirs might be attractive to conquer the difficulty of discreteness of control values in quantized control systems.

But, because of the integral property of controls, more concrete discussions can be expected to be made in the studies of control structures of quantized control systems. The approach taken in this paper is essentially based on the theories of algebraic equations and geometry of numbers (see, e.g., TAKAGI 1971; TAKEKUMA 1972) which are among fundamental approaches in theories of integers and may be easily understood in engineering sense.

2. Controllability of Quantized Control Systems

We introduce some fundamental definitions and define the controllability of quantized control systems in this section.

Let Z be the ring of integers and let

$$Z^r = \{z \mid z = (z_1, \cdots, z_r) \in E^r, z_i \in Z, i = 1, \cdots, r\}.$$

Definition 1. A system defined on $T = \{\cdots -1, 0, 1, \cdots\}$:

$$x(t+1) = Ax(t) + Bu(t), \; x(t) \in E^n, \; u(t) \in Z^r \tag{1}$$

is said to be a (linear, time-invariant and discrete) *quantized control system*, and represented as S.

Note that any system with discrete-level controls such that $u_k(t)$ takes a value in $\{\Delta_k n_k \mid n_k \in Z\}$ where Δ_k is a fixed real number for all $k = 1, \cdots, r$ is equivalent to S by replacing B with $B\begin{pmatrix} \Delta_1 & & 0 \\ & \ddots & \\ 0 & & \Delta_r \end{pmatrix}$.

Let $\bar{S}$ be a system in which the restriction of $u(t)$ is relaxed from Z^r to E^r in S. $\bar{S}$ is a usual constant, linear discrete system, and we have natural definitions of controllability and reachability for $\bar{S}$ (KALMAN 1961), that is:

> x in E^n is reachable (controllable) if and only if there exist $t(\geqslant 1)$ and $u(\tau)$ in $E^r, \tau=0, 1, \cdots, t-1$, such that $x(0)=0$ $(x(0)=x)$ and $x(t)=x$ $(x(t)=0)$.

It is known that, in general, $\{x|x\in E^n, x\text{: controllable}\}\supset\{x|x\in E^n, x\text{: reachable}\}$, and two sets coinside if and only if A is nonsingular.

However, if we define controllability and reachability of S the same as above, such relation may not hold for S. Because, in S, if $\bar{x}$ is reachable, there exist $t(\geqslant 1)$ and $\bar{u}(\tau)\in Z^r, \tau=1, \cdots, t-1$ such that

$$\bar{x}=x(0)=A^{t-1}B\bar{u}(-t)+\cdots+AB\bar{u}(-2)+B\bar{u}(-1).$$

Then, starting from this initial state,

$$x(t')=A^{t'}\{A^{t-1}B\bar{u}(-t)+\cdots+B\bar{u}(-1)\}+A^{t'-1}Bu(0)+\cdots+Bu(t'-1) \qquad (2)$$

for any $t'(\geqslant 1)$. Since $u(\tau)$ belongs to Z^r for all $\tau=0, 1, \cdots, t'-1$, $t'(\geqslant 1)$ and $u(\tau)$ in $Z^r(\tau=0, 1, \cdots, t'-1)$ which make $x(t')=0$ in (2) do not necessarily exist. Hence, $\bar{x}$ may not be controllable.

Let us define controllability of S as follows:

Definition 2. Let x^1 and x^2 be states of S. Then, x^2 is said to be *quantized controllable from* x^1 if there exist $t(\geqslant 1)$ and $u(\tau)\in Z^r$ $(\tau=0, 1, \cdots, t-1)$ such that $x^2=A^t x^1+\sum_{\tau=0}^{t-1} A^{t-1-\tau}Bu(\tau)$. (x^1, x^2) is called a *controllable state pair.*

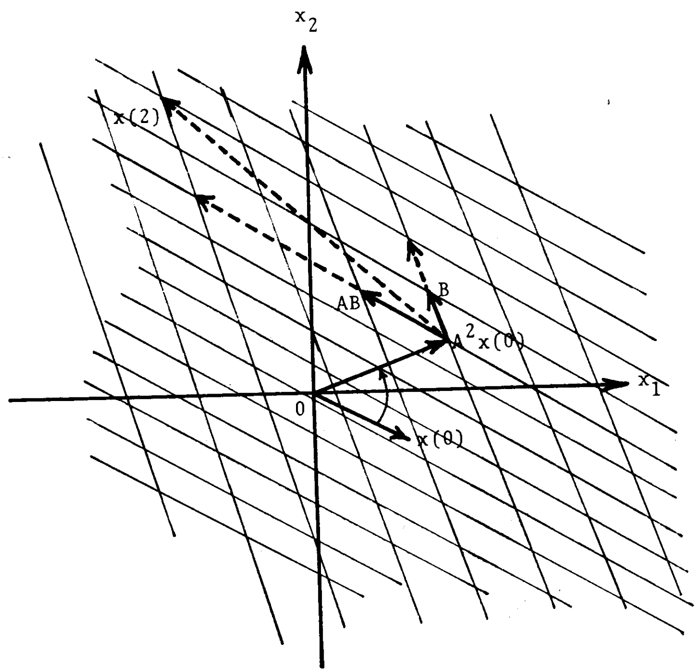

Fig. 1. Controllable state pair $(x(0), x(2))$, where $x(2)=A^2x(0)+3AB+2B$.

Definition 3. S is said to be *completely quantized controllable* if (x^1, x^2) is a controllable state pair for any $x^1, x^2 \in E^n$.

Definition 4. $R_t = \{(x^1, x^2) | x^1, x^2 \in E^n, \exists\, u(\tau) \in Z^r, \tau = 0, 1, \cdots, t-1,$ such that $x^2 = A^t x^1 + \sum_{\tau=0}^{t-1} A^{t-1-\tau} B u(\tau)\}$ is said to be the *set of t-step controllable state pairs.*

Let us define an equivalence relation, $\sim$, on R_t as follows:

$$(x^1, x^2) \sim (y^1, y^2) \text{ if and only if } x^2 - A^t x^1 = y^2 - A^t y^1.$$

Then, the coset $R_t/\sim$ is obviously isomorphic to $M_t = \{x | x \in E^n, \exists\, u(\tau) \in Z^r, \tau = 0, 1, \cdots, t-1,$ such that $x = \sum_{\tau=0}^{t-1} A^{t-1-\tau} B u(\tau)\}$. M_t is said to be the *t-step controllable set* of S.

Theorem 1. S is completely quantized controllable if and only if, for any x in E^n, there exists $t (\geqslant 1)$ such that $x \in M_t$.

Proof. Necessity: if there exists $\bar{x} \in E^n$ such that $\bar{x} \notin M_t$ for any $t(\geqslant 1)$, $(0, \bar{x})$ is not a controllable state pair by the definition of M_t. Hence S is not completely quantized controllable.

Sufficiency: for any states, x^1 and x^2, there exists $t(\geqslant 1)$ such that $x = x^2 - A^t x^1 \in M_t$ by the assumption. Hence, there exist $u(\tau) \in Z^r$ $(\tau = 0, 1, \cdots, t-1)$ such that $x^2 - A^t x^1 = \sum_{\tau=0}^{t-1} A^{t-1-\tau} B u(\tau)$. This implies that S is completely quantized controllable.

Corollary 1. No quantized control system is completely quantized controllable.

Proof. For any quantized control system, S, $M_\infty = \cup_{t=1}^{\infty} M_t$ is countable by the definition. Hence, there exists $x \in E^n$ such that $x \notin M_\infty$. Then, by Theorem 1, S is not completely quantized controllable.

Above discussions imply that we had rather study under what conditions there exists finite $t(\geqslant 1)$ such that $M_{t'} = M_t$ for all $t' \geqslant t$. We provide its partial solution in Section 3.

3. Expansion of M_t

It is known that $V_t = V_n$ for all $t \geqslant n$ in a continuous-level control system, $\bar{S}$, where $V_t = \{x | x \in E^n, x$: a state reachable in t steps$\}$. However, in S, this fact does not necessarily hold and, in general, the ascending sequence of M_t such that

$$M_1 \subset \cdots \subset M_{t-1} \subset M_t \subset M_{t+1} \subset \cdots$$

is generated.

We assume that A has distinct eigenvalues, $d_1, \cdots, d_n$, in this section. Then, A can be factored into $A = T^{-1} D T$, where T is a nonsingular matrix and $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$.

Define sets

$$D(x, s, k) = \{u | u = (u^0, u^1, \cdots, u^{k-1}) \in Z^{kr},\ x \text{ satisfies}$$
$$x^k + (s^T u^{k-1}) x^{k-1} + \cdots + (s^T u^1) x + (s^T u^0) = 0\}$$

and

$$D'(x, s, k-1)=\{u|u=(u^0, u^1, \cdots, u^{k-1})\in Z^{kr},\ x \text{ satisfies}$$
$$(s^T u^{k-1})x^{k-1}+ \cdots +(s^T u^1)x+(s^T u^0)=0\}$$

where $x\in C^n$, $s\in C^r$, $C^r=\{c|c=(c_1, \cdots, c_r),\ c_i$ is a complex number$\}$ and k is a positive integer.

Furthermore, let $TB=(h_{ij})_{n\times r}$ and let $I=\{i|1\leqslant i\leqslant n\}$ and $I_j=\{i|i\in I,\ h_{ij}\neq 0\}$ $(j=1, \cdots, r)$.

Lemma 1. $M_{t'}=M_t$ for all $t'\geqslant t$ if and only if $M_{t+1}=M_t$.

Proof. Since $M_{t+2}\subset M_{t+1}$ if $M_{t+1}=M_t$, the proof is obvious.

Theorem 2. $M_{t'}=M_t$ for all $t'\geqslant t$ if and only if

$$\cap_{i\in I_j} D\left(d_i, -\frac{1}{h_{ij}}(h_{i1}, \cdots, h_{ir}), t\right) \cap_{i\in I-I_j} D'(d_i,(h_{i1}, \cdots, h_{ir}), t-1)=\phi \tag{3}$$

for all $j=1, \cdots, r$.

Proof. $M_{t'}=M_t$ for all $t'\geqslant t$ if and only if A^tB is represented as an integer liner combination of $B, AB, \cdots, A^{t-1}B$ by Lemma 1. That is, for any column, $b^j (j=1, \cdots, r)$, of B, there exist $u^\tau\in Z^r (\tau=0, 1, \cdots, t-1)$ such that

$$A^t b^j=A^{t-1}Bu^{t-1}+ \cdots +ABu^1+Bu^0. \tag{4}$$

Substituting $A=T^{-1}DT$ to (4) and multiplying T from the left, we obtain

$$D^t Tb^j=D^{t-1}TBu^{t-1}+ \cdots +DTBu^1+TBu^0,$$

or, in component form,

$$d_i^t h_{ij}=d_i^{t-1} \sum_{j=1}^{r} h_{ij}u_j^{t-1}+ \cdots +d_i \sum_{j=1}^{r} h_{ij}u_j^1+ \sum_{j=1}^{r} h_{ij}u_j^0, \qquad i=1, \cdots, n. \tag{5}$$

Hence, the condition (4) is obviously equivalent to the condition (2).

Though Theorem 2 is an intuitively trivial assertion, an important corollary for single-input quantized control systems can be derived from it:

Corollary 2. Assume that $r=1$, i.e., S is a single-input system. Then, there exists $t(\geqslant 1)$ such that $M_{t'}=M_t$ for all $t'\geqslant t$ if and only if d_i is an algebraic integer* for all $i\in I_1$.

Proof. Necessity: by Theorem 2, there exist $u_\tau\in Z (\tau=0, 1, \cdots, t-1)$ such that

$$d_i^t-u_{t-1}d_i^{t-1}- \cdots -u_1 d_i-u_0=0$$

* An algebraic integer is a solution of any finite order algebraic equation with integral coefficients whose highest order coefficient is 1, i.e., $x^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0=0$, $a_i\in Z$, $i=0, 1, \cdots, m-1$.

for all $i \in I_1$. This implies that d_i is an algebraic integer for all $i \in I_1$.

Sufficiency: by the assumption, there exist $t_i(\geqslant 1)$ and $u_{i\tau} \in Z(i \in I_1, \tau=0, \cdots, t_i-1)$ such that $d_i(i \in I_1)$ satisfies an equation

$$x^{t_i}+u_{it_i-1}x^{t_i-1}+\cdots+u_{i1}x+u_{i0}=f_i(x)=0.$$

Then, all $d_i(i \in I_1)$ satisfy an equation with integral coefficients

$$\prod_{i \in I_1} f_i(x)=x^{t^*}+u_{t^*-1}x^{t^*-1}+\cdots+u_1x+u_0=0,$$

$$u_\tau \in Z, \tau=0,1,\cdots,t^*-1. \tag{6}$$

And for any $i \in I-I_1$, it is obvious that for any $t(\geqslant 1)$, $Z^{tr} \subset D'(d_i, h_{i1}, t-1)$. This and (6) imply that the condition (2) in Theorem 2 holds. Hence, there exists $t(\geqslant 1)$ such that $M_{t'}=M_t$ for all $t' \geqslant t$ by Theorem 2.

Corollary 2 shows the relation between the controllability property and the eigenvalues of S when S has a single-input. The fact that the *numerical* property of eigenvalues heavily influences on controllability is essentially different from the results for $\bar{S}$, where the expansion property of the set of reachable states can be clearly described by using controllable subspaces in E^n.

For example, since any rational number which is not an integer can not be an algebraic integer, $M_{t+1} \supsetneqq M_t$ for any $t(\geqslant 1)$ if there exists $i \in I_1$ such that d_i is rational but not integral, that is, the expansion of M_t never terminates.

Extension of Corollary 2 to the general multi-input case can be obtained only as a sufficient condition.

Corollary 3. If d_i is an algebraic integer for all $i=1,\cdots,n$, then there exists $t(\geqslant 1)$ such that $M_{t'}=M_t$ for all $t' \geqslant t$.

Proof. Let $t_i(i=1,\cdots,n)$ be the order of d_i (the order of the algebraic equation which d_i satisfies), and let $t=\max_i t_i$. Then, all $d_i(i=1,\cdots,n)$ satisfy an equation

$$\prod_{i=1}^{n}(x^t+u_{it-1}x^{t-1}+\cdots+u_{i1}x+u_{i0})$$

$$=x^{t^*-1}+u_{t^*-2}x^{t^*-2}+\cdots+u_1x+u_0$$

$$=0, \qquad u_\tau \in Z, \tau=0,1,\cdots,t^*-2, \tag{8}$$

where $t^*=nt+1$ and d_i satisfies $x^t+u_{it-1}x^{t-1}+\cdots+u_{i1}+u_{i0}=0$ $(u_{i\tau} \in Z, \tau=0,1,\cdots,t-1)$ for $i=1,\cdots,n$.

Besides, multiplying x to (8), all $d_i(i=1,\cdots,n)$ satisfy

$$x^{t^*}+w_{t^*-1}x^{t^*-1}+\cdots+w_1x+w_0=0,$$

$$w_\tau \in Z, \tau=0,1,\cdots,t^*-1. \tag{9}$$

On the other hand, (5) can be rewritten as

$$h_{i1}\left(\sum_{\tau=0}^{t-1}(-u_1^\tau)d_i^\tau\right)+\cdots+h_{ij}\left(\sum_{\tau=0}^{t-1}(-u_j^\tau)d_i^\tau+d_i^t\right)+\cdots+h_{ir}\left(\sum_{\tau=0}^{t-1}(-u_r^\tau)d_i^\tau\right)=0$$

$$i=1, \cdots, n, j=1, \cdots, r. \tag{10}$$

Then, (8) and (9) imply that there exist $t(\geqslant 1)$ and $u_j^\tau \in Z$ ($\tau=0,1,\cdots,t-1, j=1,\cdots,r$) such that (10) holds for all $i=1,\cdots,n$ and $j=1,\cdots,r$. Hence, the corollary is proved since $M_{t'}=M_t$ for all $t'\geqslant t$ when (10) holds as is seen in Theorem 2.

We should note that the assertion of Corollary 3 is weak. Even if d_i is not an algebraic integer for some i, there may exist $t(\geqslant 1)$ such that $M_{t'}=M_t$ for all $t'\geqslant t$. But the above corollaries tell us that the *numerical* characteristics of eigenvalues of S highly influence on the controllability properties of S.

Numerical Example

Let us investigate the expansion property of M_t of the quantized control system:

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix} = \begin{pmatrix} (3-\sqrt{5})/2 & 0 \\ 2/3 & (3+\sqrt{5})/2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} u(t),$$

where $u(t)\in Z$ for all $t\geqslant 0$.

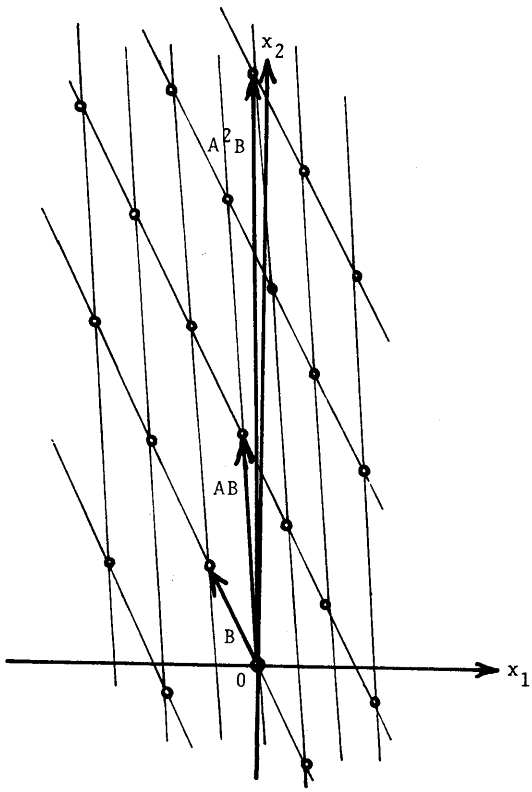

Fig. 2. M_2 in the numerical example.

The eigenvalues of the system is $(3\pm\sqrt{5})/2$, which are algebraic integers since they both satisfy the equation $x^2-3x+1=0$. Hence, there exists $t(\geqslant 1)$ such that $M_{t'}=M_t$ for all $t'\geqslant t$ by Corollary 2.

Actually, at $t=2$,

$$(B\ AB\ A^2B)=\begin{pmatrix} -1 & (-3+\sqrt{5})/2 & (-7+3\sqrt{5})/2 \\ 2 & (7+3\sqrt{5})/2 & 5+3\sqrt{5} \end{pmatrix}$$

and

$$A^2B=3AB-B$$

holds. Hence, $M_{t'}=M_t$ for all $t'\geqslant 2$ for the above system (Fig. 2).

4. t-step controllability index

4.1. q-mesh quantized controllability and t-step controllability index

As is seen in Sections 1 and 2, the concept of controllability of quantized control systems, (1), should be much different from that of systems with continuous-level controls. Quantized control systems are not completely controllable in the ordinary sense (Corollary 1). Hence, we had rather handle with the problem; ' to what extent is the system controllable ?'. We provide some physically natural index to measure the degree of controllability of quantized control systems. We assume

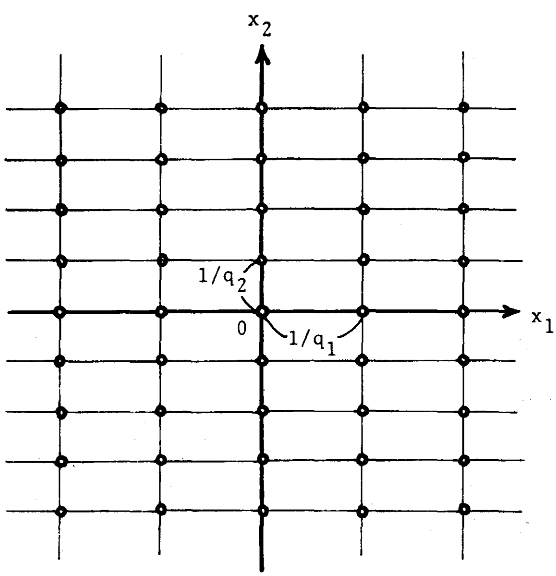

Fig. 3. A q-mesh set in E^2.

that every element of matrices A and B is rational.

First, we introduce the concept of mesh controllability.

Definition 5. Let $q=(q_1, \cdots, q_n)$ be a vector in E^n such that q_i is a positive rational number for all $i=1, \cdots, n$. Then,

$$Q_q=\{x \mid x=(x_1, \cdots, x_n) \in E^n, x_i=s_i/q_i, s_i \in Z, i=1, \cdots, n\}$$

is said to be the *q-mesh set* of E^n (Fig. 3).

Definition 6. A quantized control system, S, is said to be *q-mesh quantized controllable* if $Q_q \subset M_\infty$.

Define $G_t=[B\ AB \dots A^{t-1}B]$ and let $g_{ij}=n_{ij}/m_{ij}$ be the (i,j)-element of G_t where g_{ij} is rational, and n_{ij} and m_{ij} are relatively prime integers $(i=1, \cdots, n, j=1, \cdots, tr)$. Let $z_{ti}=\text{l.c.m.}\ (m_{i1}, \cdots, m_{itr})$, $z_{ti}=m'_{ij}m_{ij}$ $(j=1, \cdots, tr)$, $y_{ti}=\text{g.c.d.}\ (n_{i1}m'_{i1}, \cdots, n_{itr}m'_{itr})$ $(i=1, \cdots, n)$, $z^t=(y_{t1}/z_{t1}, \cdots, y_{tn}/z_{tn})$ and $P_t=\begin{pmatrix} z_{t1} & & 0 \\ & \ddots & \\ 0 & & z_{tn} \end{pmatrix}$.

Lemma 2. $M_t \subset Q_{z^t}$ for any $t(\geqslant 1)$.

Proof. Obvious.

Since $M_t \supset M_{\bar{t}}$ for any $\bar{t}$ such that $1 \leqslant \bar{t} \leqslant t$, Lemma 2 implies that every $x \notin Q_{z^t}$ can not be reached from the origin by $\bar{t}(\leqslant t)$ steps. Furthermore, it is obvious by the definition of z^t that Q_{z^t} is the finest mesh set which includes the states reachable by t steps.

Since M_∞ is countable in E^n, the degree of dispersion (in physical sense) of the points of M_t in E^n would be a physically natural index to measure the degree of controllability of S. Such an index should be independent of controllable state pairs and depend only on the *structure* of S. Considering those, we define:

Definition 7. $\varepsilon_t=\sup_{x \in E^n} \inf_{u \in Z^{tr}} ||x-G_t u||$ is said to be the *t-step controllability index* of S, where $||\cdot||$ is the Euclidean norm.

To begin with, we should make clear the relation of ε_t to the degree of controllability in the original state space where controllability pairs are defined. In that space, ε_t can be written as follows:

$$\begin{aligned} \varepsilon_t &= \sup_{x^1, x^2 \in E^n} \inf_{u \in Z^{tr}} ||(x^2-A^t x^1)-G_t u|| \\ &= \sup_{x^1, x^2 \in E^n} \inf_{u \in Z^{tr}} ||x^2-(A^t x^1+G_t u)||. \end{aligned}$$

That is, ε_t represents the degree of controllability when the worst state pair (x^1, x^2) is assigned as the initial and the terminal states. This implies that ε_t is the index also to measure the degree of controllability in the original state space, and this fact makes important the concept of ε_t.

Though it is not easy to calculate ε_t for each $t(\geqslant 1)$, usually we are not likely to need precise data of ε_t and we may need only some upper and lower bounds in most cases. By the definition, $\varepsilon_t=\infty$ if rank $G_t<n$. We develop the bounds of ε_t in the case that rank $G_t=n$.

Let g.c.d. (E) denote the greatest common devisor of n-order subdeterminants of $n \times n'$ $(n \leqslant n')$ matrix E.

Lemma 3. $c_{ti}=$g.c.d. $(P_tG_t)/$g.c.d. $(P_tG_t\,{:}\,e_i)$ is an integer for all $i=1,\cdots,$ n and all $t(\geqslant 1)$ such that rank $G_t=n$, where $(P_tG_t\,{:}\,e_i)$ is the augmented matrix of P_tG_t to which $e_i=(0,\cdots,\underset{(i)}{1},\cdots,0)$ is added.

Proof. There exists $\alpha\in Z$ such that g.c.d. $(P_tG_t)=\alpha$g.c.d. $(P_tG_t\,{:}\,e_i)$ since g.c.d. $(P_tG_t\,{:}\,e_i)=$g.c.d. (g.c.d. (P_tG_t), g.c.d. $(P_tG_t\,{:}\,e_i)$). But g.c.d. $(P_tG_t\,{:}\,e_i)\neq 0$ since rank $G_t=n$. Hence, c_{ti} is an integer.

Lemma 4. g.c.d. $(P_tG_t)=$g.c.d. $(P_tG_t\,{:}\,c_{ti}e_i)$ for all $i=1,\cdots,n$ and all $t(\geqslant 1)$ such that rank $G_t=n$. And c_{ti} is the smallest positive integer such that this equation holds.

Proof. c_{ti} is an integer by Lemma 3. Let g.c.d. $(P_tG_t\,{:}\,c_{ti}e_i)'$ be the greatest common devisor of n-order subdeterminants of $(P_tG_t\,{:}\,c_{ti}e_i)$ of which columns include $c_{ti}e_i$. Then,

$$\begin{aligned}&\text{g.c.d. } (P_tG_t\,{:}\,c_{ti}e_i)\\&=\text{g.c.d. (g.c.d. } (P_tG_t),\ \text{g.c.d. } (P_tG_t\,{:}\,c_{ti}e_i)')\\&=\text{g.c.d. } (c_{ti}\,\text{g.c.d. } (P_tG_t\,{:}\,e_i),\ c_{ti}\,\text{g.c.d. } (P_tG_t\,{:}\,e_i)')\\&=c_{ti}\,\text{g.c.d. } (P_tG_t\,{:}\,e_i)\end{aligned}$$

This proves the former part.

For any $\bar{c}_{ti}\in Z$ such that $1\leqslant\bar{c}_{ti}<c_{ti}$,

$$\begin{aligned}&\text{g.c.d. } (P_tG_t)\\&>\bar{c}_{ti}\text{g.c.d. } (P_tG_t\,{:}\,e_i)\\&=\bar{c}_{ti}\,\text{g.c.d. (g.c.d. } (P_tG_t),\ \text{g.c.d. } (P_tG_t\,{:}\,e_i)')\\&=\text{g.c.d. } (\bar{c}_{ti}\,\text{g.c.d. } (P_tG_t),\ \bar{c}_{ti}\,\text{g.c.d. } (P_tG_t\,{:}\,e_i)')\\&\geqslant\text{g.c.d. (g.c.d. } (P_tG_t),\ \text{g.c.d. } (P_tG_t\,{:}\,\bar{c}_{ti}e_i)')\\&=\text{g.c.d. } (P_tG_t\,{:}\,\bar{c}_{ti}e_i).\end{aligned}$$

This proves the latter part.

Lemma 5. Let $Ax=b$ be a linear simultaneous equation where A is an $n\times n'$ $(n\leqslant n')$ matrix and b is an n-dimensional vector with all elements integral. Then, the equation has an integral solution, $x\in Z^{n'}$, if and only if g.c.d. $(A)=$g.c.d. $(A\,{:}\,b)$.

Proof. See (SAATY 1970).

Theorem 3. For any $t(\geqslant 1)$ such that rank $G_t=n$, let $q^t=(z_{t1}/c_{t1},\cdots,z_{tn}/c_{tn})$. Then, $Q_{q^t}\subset M_t$, and S is q^t-mesh quantized controllable.

Proof. For any such t, there exist $u^i\in Z^{tr}(i=1,\cdots,n)$ such that $G_tu^i=P_t^{-1}C_te_i$ $(i=1,\cdots,n)$ by Lemmas 4 and 5, where $C_t=\begin{pmatrix}c_{t1} & & 0\\ & \ddots & \\ 0 & & c_{tn}\end{pmatrix}$. For any $s\in Q_{q^t}$ (q^t-mesh set), there exist $s_i \in Z$ $(i=1, \cdots, n)$ such that $s=\sum_{i=1}^{n} P_t^{-1} C_t s_i e_i$. Hence, $s=G_t u$ where $u=\sum_{i=1}^{n} s_i u^i \in Z^{tr}$. This implies that $s \in M_t$, which means that $Q_{qt} \subset M_t \subset M_\infty$.

Corollary 4. S is z^t-mesh quantized controllable if $c_{ti}=1$ $(i=1, \cdots, n)$ for some $t(\geqslant 1)$ such that rank $G_t=n$. If $Q_{zt}=M_t$ for some $t(\geqslant 1)$ such that rank $G_t=n$, then $c_{ti}=1$ for all $i=1, \cdots, n$.

Proof. The former part is obvious. The latter part is also easily verified by using Lemma 5.

Theorem 4. Let q^t be a vector defined in Theorem 3, and let $q'=(q_1', \cdots, q_n')$ be any vector such that components are rational and positive and $q_i^t < q_i'$ for some i $(1 \leqslant i \leqslant n)$. Then, $Q_{q'} \not\subset M_{t'}$ for any $t' \leqslant t$.

Proof. It is sufficient to prove that $Q_{q'} \not\subset M_t$. For any q' satisfying the assumption, assume that $q_k^t < q_k'$, and let $c_i'=z_{ti}/q_i'$ and $C'=\begin{pmatrix} c_1' & & 0 \\ & \ddots & \\ 0 & & c_n' \end{pmatrix}$.

Suppose that $Q_{q'} \subset M_t$. Then, there exists $u^k \in Z^{tr}$ such that $G_t u^k=P_t^{-1} C' e_k$ since $P_t^{-1} C' e_k \in Q_{q'} \subset M_t$. This implies that $P_t G_t x=c_k' e_k$ has an integral solution, from which g.c.d. $(P_t G_t)=$g.c.d. $(P_t G_t : c_k' e_k)$ holds by Lemma 5. But since $c_k' < c_{tk}$ by the assumption, it contradicts Lemma 4. Hence, $Q_{q'} \not\subset M_t$.

Theorems 3 and 4 show that Q_{qt} is the finest mesh set reachable by t steps in any quantized control system. Furthermore, Q_{qt} is the only such mesh set except subsets of Q_{qt} which are mesh sets:

Theorem 5. For any $t(\geqslant 1)$ such that rank $G_t=n$, let $q=(q_1, \cdots, q_n)$ be any vector such that $q_i=s_i c_{ti}/z_{ti}$ $(i=1, \cdots, n)$ where $s_i \in Z$ $(i=1, \cdots, n)$. Then, $Q_q \subset M_t$, and $Q_{q'} \not\subset M_t$ for any vector with all components rational and positive, $q'(\neq q)$.

Proof. Easily verified by using Lemma 4.

We characterized the finest mesh sets related to the mesh controllability in Lemma 1, Theorems 3, 4 and 5. These lead us to the introduction of easily calcullable upper and lower bounds of ε_t.

Let us define two parallelotopes as follows (Fig. 4):

$$K^1=\{x \mid x=(x_1, \cdots, x_n) \in E^n, 0 \leqslant x_i \leqslant y_{ti}/z_{ti}, i=1, \cdots, n\}$$

$$K^2=\{x \mid x=\{x_1, \cdots, x_n) \in E^n, 0 \leqslant x_i \leqslant c_{ti}/z_{ti}, i=1, \cdots, n\},$$

where they are defined for any $t(\geqslant 1)$ such that rank $G_t=n$. And let $extK^i$ $(i=1,2)$ be the set of vertices of $K^i (i=1,2)$.

Then, $M_t \cap (K^1 - extK^1)=\phi$ by Lemma 2. Hence, if we define $\underline{\varepsilon}_t=\max_{x \in K^1} \min_{v \in extK^1} \|x-v\|$, $\underline{\varepsilon}_t \leqslant \varepsilon_t$ holds. Since $\underline{\varepsilon}_t$ is obviously the Euclidean distance of the origin and the center of K^1,

$$\underline{\varepsilon}_t=\frac{1}{2}\left(\sum_{i=1}^{n}\left(\frac{y_{ti}}{z_{ti}}\right)^2\right)^{1/2} \tag{11}$$

On the other hand, $extK^2 \subset M_t$ by Theorem 3. Hence, if we define $\bar{\varepsilon}_t=\max_{x \in K^2} \min_{v \in extK^2} \|x-v\|$, $\varepsilon_t \leqslant \bar{\varepsilon}_t$ holds. As same as above,

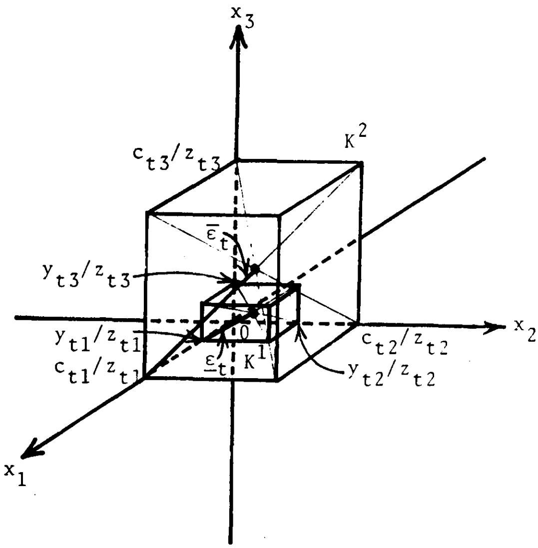

Fig. 4. Parallelotopes defining $\underline{\varepsilon}_t$ and $\bar{\varepsilon}_t$.

$$\bar{\varepsilon}_t=\frac{1}{2}\left(\sum_{i=1}^{n}\left(\frac{c_{ti}}{z_{ti}}\right)^2\right)^{1/2}. \tag{12}$$

So we have proved:

Theorem 6. $\underline{\varepsilon}_t \leqslant \varepsilon_t \leqslant \bar{\varepsilon}_t$ holds for any $t(\geqslant 1)$ satisfying rank $G_t=n$, where $\underline{\varepsilon}_t$ and $\bar{\varepsilon}_t$ are defined as (11) and (12), respectively.

Hence, $\bar{\varepsilon}_t$ and $\underline{\varepsilon}_t$ are upper and lower bounds of ε_t, respectively. It is relatively easy to obtain their values, though the value of ε_t itself is not easy to find*.

Numerical Example

Let us calculate upper and lower bounds of ε_t of the quantized control system:

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix}=\begin{pmatrix} -1/2 & 4/3 \\ 0 & -2/5 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}+\begin{pmatrix} 1/3 \\ 1 \end{pmatrix}u(t),\ u(t)\in Z.$$

1. $t=2$: $G_2=[B\ AB]=\begin{pmatrix} 1/3 & 7/6 \\ 2 & -2/5 \end{pmatrix}, P_2=\begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix},$

$$c_{21}=\frac{\text{g.c.d. }(P_2G_2)}{\text{g.c.d. }(P_2G_2 \vdots e_1)}=\frac{\text{g.c.d. }(-39)}{\text{g.c.d. }(-39,\ -5{,}2)}=39,$$

* The approximate value of ε_t in any precision can be obtained by the finite number of iterations of relaxation algorithm for nonlinear programming.

$$c_{22}=\frac{\text{g.c.d. }(P_2G_2)}{\text{g.c.d. }(P_2G_2:e_2)}=\frac{\text{g.c.d. }(-39)}{\text{g.c.d. }(-39,2,7)}=39,$$

$$\underline{\varepsilon}_2=1/2((1/6)^2+(1/5)^2)^{1/2}=0.0768,$$

$$\bar{\varepsilon}_2=1/2((39/6)^2+(39/5)^2)^{1/2}=5.08.$$

2. $t=3$: $G_3=\begin{pmatrix}1/3 & 7/6 & -67/60\\ 1 & -2/5 & 4/25\end{pmatrix}$, $P_3=\begin{pmatrix}60 & 0\\ 0 & 25\end{pmatrix}$,

$$c_{31}=195,\ c_{32}=195,$$

$$\underline{\varepsilon}_3=1/2((1/60)^2+(1/25)^2)^{1/2}=0.0216,$$

$$\bar{\varepsilon}_3=1/2((195/60)^2+(195/25)^2)^{1/2}=4.225.$$

3. $t=4$: $G_4=\begin{pmatrix}1/3 & 7/6 & -67/60 & 463/600\\ 1 & -2/5 & 4/25 & -8/125\end{pmatrix}$, $P_4=\begin{pmatrix}600 & 0\\ 0 & 125\end{pmatrix}$,

$$c_{41}=5,\ c_{42}=5,$$

$$\underline{\varepsilon}_4=1/2((1/600)^2+(1/125)^2)^{1/2}=0.00410,$$

$$\bar{\varepsilon}_4=1/2((5/600)^2+(5/125)^2)^{1/2}=0.0411.$$

4. $t=5$: $G_5=\begin{pmatrix}1/3 & 7/6 & -67/60 & 463/600 & -2827/6000\\ 1 & -2/5 & 4/25 & -8/125 & 16/625\end{pmatrix}$, $P_5=\begin{pmatrix}6000 & 0\\ 0 & 625\end{pmatrix}$,

$$c_{51}=5,\ c_{52}=5,$$

$$\underline{\varepsilon}_5=1/2((1/6000)^2+(1/625)^2)^{1/2}=0.000804,$$

$$\bar{\varepsilon}_5=1/2((5/6000)^2+(5/625)^2)^{1/2}=0.00409.$$

Corollary 5. $\underline{\varepsilon}_t=1/2(\sum_{i=1}^{n}(y_{ti}/z_{ti})^2)^{1/2}$ for $t(\geqslant 1)$ such that rank $G_t=n$ if $M_t=Q_{zt}$.

Proof. Easily verified by Corollary 4 and Theorem 6.

$\underline{\varepsilon}_t$ is monotone nonincreasing by the definition and the fact that $u=(u',0)\in Z^{(t+1)r}$ for any $u'\in Z^{tr}$. This property also holds for $\bar{\varepsilon}_t$:

Theorem 7. $\bar{\varepsilon}_t$ is monotone nonincreasing with respect to t.

Proof. Since Q_{qt} and Q_{qt+1} are the finest mesh sets included in M_t and M_{t+1}, respectively, by Theorems 3, 4 and 5, and $M_t\subset M_{t+1}$, $c_{t+1i}/z_{t+1i}\leqslant c_{ti}/z_{ti}$ for all $i=1,\cdots,n$ where q^t and q^{t+1} are vectors defined in Theorem 3. Then, the assertion is true by the definition of $\bar{\varepsilon}_t$, (12).

Lastly, we relate the convergence of $\underline{\varepsilon}_t$ to the infinite behavior of the expansion of M_t.

Theorem 8. M_∞ is dense in E^n if and only if $\lim_{t\to\infty} \varepsilon_t = 0$.

Proof. Necessity: assume that there exists $\delta > 0$ such that $\varepsilon_t = \sup_{x\in E^n} \inf_{u\in Z^{tr}} \|x - G_t u\| > \delta$ for any t satisfying rank $G_t = n$. Then there exists $\bar{x} \in E^n$ such that $\|\bar{x} - G_t u\| > \delta$ for any such t and any $u \in Z^{tr}$. This implies that M_∞ is not dense in E^n.

Sufficiency: if $\lim_{t\to\infty} \varepsilon_t = 0$, for any $\delta > 0$, there exists $\bar{t}$ such that $\sup_{x\in E^n} \inf_{u\in Z^{tr}} \|x - G_t u\| < \delta$ for any $t \geqslant \bar{t}$. Then, for any $x \in E^n$ and any $t \geqslant \bar{t}$, there exists $u \in Z^{tr}$ such that $\|x - G_t u\| > \delta$, which implies that M_∞ is dense in E^n.

By Theorems 6 and 8, M_∞ is dense in E^n if $\lim_{t\to\infty} c_{ti}/z_{ti} = 0$ for all $i = 1, \cdots, n$. If M_∞ is dense in E^n, we can reach a point sufficiency near any terminal state from any initial state, that is, S is almost completely quantized controllable. When the controls are restricted to discrete levels, this would be the best situation of controllability. The relation in Theorem 8 is also the basis of the physical significance of ε_t.

4.2 Relation to controllability of continuous-level control systems and controllability of quantized control systems with integral coefficients

It may be interesting to compare q-mesh quantized controllability and usual reachability of continuous-level control systems. We have the following relation between them:

Theorem 9. $\bar{S}$ is completely reachable if and only if there exists $q = (q_1, \cdots, q_n)$ where $q_i (i = 1, \cdots, n)$ is rational and positive such that S is q-mesh quantized controllable.

Proof. Necessity: if $\bar{S}$ is completely reachable, rank $G_n = n$ (KALMAN 1961). Then, by Theorem 3, S is q^n-mesh quantized controllable where $q^n = (z_{n1}/c_{n1}, \cdots, z_{nn}/c_{nn})$.

Sufficiency: by the assumption, there exist $t_i (\geqslant 1)$ and $u^i \in Z^{t_i r}$ such that $G_{t_i} u^i = (1/q_i) e_i \in M_{t_i}$ for all $i = 1, \cdots, n$. Then, by Lemma 5, rank $G_{t_i} =$ rank $(G_{t_i} \vdots (1/q_i) e_i)$ $(i = 1, \cdots, n)$. Hence, defining $t = \max_i t_i$, every e_i $(i = 1, \cdots, n)$ is linearly dependent to the columns of G_t, that is, rank $G_t = n$. This means that $\bar{S}$ is completely reachable (KALMAN 1961).

Complete reachability of $\bar{S}$ does not necessarily imply q-mesh quantized controllability of S for any q with all components rational and positive. For example, the following system is completely reachable as $\bar{S}$, but not **1**-mesh quantized controllable as S where $\mathbf{1} = (1, \cdots, 1)$:

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}. \tag{13}$$

In this system,

$$\{x \mid x = (x_1, x_2) \in E^2, x_1 \in Z, x_2 = 2s+1, s \in Z\} \cap M_t = \phi$$

for any $t (\geqslant 1)$. Hence, $M_t \not\supseteq Q_1$ for any $t (\geqslant 1)$ (Fig. 5).

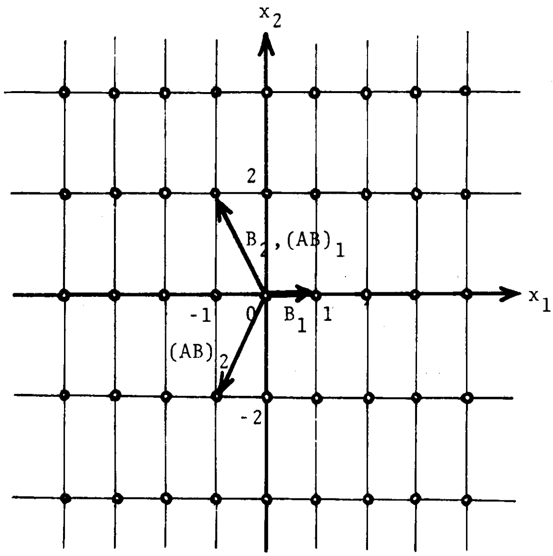

Fig. 5. M_∞ of the example system (13).

If all elements of A and B are integers, most of the above results become quite simple. We are only necessary to substitute $z_{t1}= \cdots =z_{tn}=1$ to the above results. For example:

Corollary 6. Assume that all elements of A and B are integers. Then, $M_t=Z^n$ and $\varepsilon_t=1/2(n)^{1/2}$ if $c_{ti}=1$ for all $i=1, \cdots, n$.

Proof. Easily verified by using Lemma 1, Theorems 3 and 6, and $z_{t1}= \cdots z_{tn}= c_{t1}= \cdots c_{tn}=1$.

Lemma 6. If the elements of A are all integers, A^tB can be represented as an integer linear combination of $B, AB, \cdots, A^{t-1}B$ for any $t(\geqslant n)$.

Proof. If all elements of A are integers, all coefficients of the characteristic equation of A are integers. Then, the assertion can be proved by induction using Cayley-Hamilton's theorem.

Lemma 6 tells us that, if the elements of A are integers, the expansion property of M_t is very much similar to that of a continuous-level control system, that is:

Theorem 10. If the elements of A are all integers, $M_t=M_n$ and $\varepsilon_t=\varepsilon_n$ for all $t(\geqslant n)$.

Proof. Obvious by Lemma 6.

Hence, if we interpret controllability of quantized control systems with integral coefficients as **1**-mesh quantized controllability, controllability properties of quantized and continuous-level control systems are quite similar.

A discriminating condition for S with all elements of matrices integral to be **1**-mesh quantized controllable can be obtained as follows:

Corollary 7. Assume that all elements of A are integers. Then, S is **1**-mesh quantized controllable if and only if g.c.d. $(P_nG_n)=$g.c.d. $(P_nG_n : P_ne_i)$ for all $i=1, \cdots, n$.

Proof. Necessity is obvious by Lemma 5 since $P_ne_i \in Q_1$ $(i=1, \cdots, n)$. For sufficiency, there exists $u^i \in Z^{tr}$ such that $G_tu^i=e_i$ for all $i=1, \cdots, n$ and for all $t(\geqslant n)$ if the assumption holds by Lemmas 5 and 6. This implies that S is **1**-mesh quantized controllable since $e_1, \cdots, e_n$ form a basis of Z^n.

For the 2-order system (13), g.c.d. $(G_2)=2$, g.c.d. $(G_2 : e_1)=2$ and g.c.d. $(G_2 : e_2)=1$. Hence, the system is not **1**-mesh quantized controllable by Corollary 7.

The condition in Corollary 7 is a necessary and sufficient condition for S to be quantized controllable in a sense of the interpretation stated above. In continuous-level control systems, the condition and the concept of controllability corresponding to those of S are rank $G_n=n$ and complete reachability.

5. Conclusion

No quantized control system is completely reachable in the ordinary sense. Furthermore, in quantized control systems, reachability and controllability have no such clear relations as in usual continuous-level control systems.

Quantized and q-mesh quantized controllabilities were introduced to make clear the concepts of controllability of quantized control systems. And it was shown that the expansion property of the t-step controllable set, M_t, with respect to t is deeply related with the numerical properties of eigenvalues of the system.

Then, the t-step controllability index, ε_t, representing the degree of controllability of quantized control systems was defined, and its upper and lower bounds were introduced by using the relation of M_t with q-mesh quantized controllability. The definition and characteristics of ε_t showed that ε_t is a physically natural index to represent the control structures of quantized control systems.

The relations of q-mesh quantized controllability to reachability of continuous-level systems were shown. Especially, the former was found to be a natural concept of controllability of quantized control systems with all coefficients integral. Furthermore, it was shown that the expansion propery of M_t in those systems is similar to that of continuous-level control systems.

Actually systems involving both quantized and continuous-level controllers are also important. Also for those systems, it is basically possible to extend the discussions in this paper to the subspace which can not be spanned by continuous-level controls.

The difficulty for the systems discussed in this paper arises from the fact that the state space and the set of admissible controls are defined on quite different algebraic systems, i.e., a field and a ring. In that sense they must be classified into a different category from any systems of which control structures have been studied so far.

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