Title	Force acting on an oscillating cylinder in incompressible viscous fluid
Sub Title	
Author	棚橋, 隆彦(Tanahashi, Takahiko)
Publisher	慶応義塾大学工学部
Publication year	1972
Jtitle	Keio engineering reports Vol.25, No.11 (1972.) ,p.129- 140
JaLC DOI	
Abstract	In this paper a new method is shown to derive the theoretical formula given by Stokes, which expresses the force acting on an oscillating cylinder in incompressible viscous fluid, making use of the Landau-Lifshitz method for a sphere.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00250011-0129

KEIO ENGINEERING REPORTS
VOL. 25 NO. 11

FORCE ACTING ON AN OSCILLATING CYLINDER IN INCOMPRESSIBLE VISCOUS FLUID

BY

TAKAHIKO TANAHASHI

FACULTY OF ENGINEERING
KEIO UNIVERSITY
YOKOHAMA 1972

KEIO ENGINEERING REPORTS
VOL. 25, NO. 11, pp. 129-140, 1972

FORCE ACTING ON AN OSCILLATING CYLINDER IN INCOMPRESSIBLE VISCOUS FLUID

TAKAHIKO TANAHASHI

Dept. of Mechanical Engineering, Keio University, Yokohama 223, Japan

(*Received Dec. 11, 1972*)

ABSTRACT

In this paper a new method is shown to derive the theoretical formula given by Stokes, which expresses the force acting on an oscillating cylinder in incompressible viscous fluid, making use of the Landau-Lifshitz method for a sphere.

1. Introduction

The problem of the calculation of the force acting on a cylinder is classical. When the flow is steady, Stokes' approximation (1851)

$$0=-\frac{1}{\rho}\,\boldsymbol{grad}\,p+\nu\nabla^2\boldsymbol{v}$$

has been proved to have no solution which satisfies the boundary conditions at the surface of a cylinder. On the other hand Oseen's equation (1910)

$$(\boldsymbol{u}_0\cdot\boldsymbol{grad})\boldsymbol{v}=-\frac{1}{\rho}\,\boldsymbol{grad}\,p+\nu\nabla^2\boldsymbol{v}$$

gives the drag per unit length of the cylinder;

$$F=\frac{4\pi\eta u_0}{\dfrac{1}{2}-\gamma-\log\left(\dfrac{u_0 a}{4\nu}\right)}$$

In the case of unsteady flow, using the equation

$$\frac{\partial\boldsymbol{v}}{\partial t}=-\frac{1}{\rho}\,\boldsymbol{grad}\,p+\nu\nabla^2\boldsymbol{v},$$

STOKES (1851) gave the following drag on the oscillating cylinder in the fluid at rest;

$$F=-i\omega u_0 M'\left\{1-\frac{4K_1(Z)}{K_1(Z)+ZK_1'(Z)}\right\}e^{i\omega t}$$

$$\text{where} \quad Z=(1+i)\frac{a}{\delta},\ \delta=\sqrt{\frac{2\nu}{\omega}} \ \text{and} \ M'=\pi\rho a^2$$

This formula has more recently been verified experimentally to have good agreement between theory and experiment for large a/δ by MARTIN (1925) and by STUART and WOODGATE (1955) but to have much difference for small a/δ. ROSENHEAD (1963) says about the pressure gradient "STOKES' work shows a small pressure variation due to the boundary layer; the consequent modification of the surface pressure can be shown to give a contribution to the damping force of the same order of magnitude as the contribution from the skin friction. It is important to note, therefore, that for a bluff body which oscillating at a high frequency with a small amplitude, the damping force, though small, is strongly dependant on the change of pressure due to viscosity." Thus we cannot neglect the change of pressure change, viz. must consider the convective term $(\boldsymbol{v}\cdot\boldsymbol{grad})\boldsymbol{v}$. Since the convective term is non-linear, taking it into consideration is too difficult to solve the problem analytically. So we also neglect the convective term. LANDAU and LIFSHITZ (1959), by the modern method, gave the following formula for the drag acting on the small oscillating sphere in incompressible fluid at rest;

$$F=6\pi\eta a\left(1+\frac{a}{\delta}\right)u+3\pi a^2\sqrt{\frac{2\eta\rho}{\omega}}\left(1+\frac{2}{9}\frac{a}{\delta}\right)\frac{du}{dt}$$

The author applies LANDAU-LIFSHITZ method to the cylinder and will have the formula

$$F=i\omega u_0 M'\left\{1-\frac{4H_1^{(1)}(Z)}{2H_1^{(1)}(Z)-ZH_2^{(1)}(Z)}\right\}e^{-i\omega t}$$

It will be shown that this formula is equivalent to STOKES' formula.

2. Nomenclature

a $=$ radius of cylinder or sphere
$H_p^{(1)}(Z)$, $H_p^{(2)}(Z)$ $=$ Hankel function of the first or second kind, of order p.
i $=\sqrt{-1}$
$K_p(Z)$ $=$ modified Bessel function of the second kind, order p
$k=(1+i)\frac{1}{\delta}$ $=$ complex wave number
$\boldsymbol{m}$ $=$ unit vector parallel to $\boldsymbol{u}_0$
M' $=\pi\rho a^2$ for cylinder or $(4/3)\pi\rho a^3$ for sphere
$\boldsymbol{n}$ $=$ unit vector normal to the surface

p, p_0 = pressure
$r=|\boldsymbol{r}|$ = modulus of position vector
t = time
$\boldsymbol{T}$ = stress tensor
$u_0=|\boldsymbol{u}_0|$ = modulus of oscillating velocity or speed of uniform flow
u $=u_0e^{-i\omega t}$
$\boldsymbol{v}=(v_r, v_\theta, 0)$ = velocity of fluid
$Z=ka$ $=(1+i)\dfrac{a}{\delta}$
γ = Euler's constant
$\delta=\sqrt{\dfrac{2\nu}{\omega}}$ = depth of penetration
δ = density
η = dynamic viscosity
$\nu=\dfrac{\eta}{\rho}$ = kinematic viscosity
ε = amplitude of oscillating cylinder or sphere
ω = frequency
$\sigma'_{rr}, \sigma'_{r\theta}$ = components of deviatric stress tensor

3. Fundamental Equations and Boundary Conditions

The unsteady flow of an incompressible viscous liquid for the motion of an infinite cylinder oscillating ($x=\varepsilon\cos\omega t$) rectilinearly along its center is mathematically expressed by the equation of continuity

$$div\,\boldsymbol{v}=0 \tag{1}$$

and the equation of momentum

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{grad})\,\boldsymbol{v}=-\frac{1}{\rho}\,\boldsymbol{grad}\,p+\nu\nabla^2\boldsymbol{v}. \tag{2}$$

There are two cases that the convective term $(\boldsymbol{v}\cdot\boldsymbol{grad})\boldsymbol{v}$ may be neglected (LANDAU and LIFSHITZ, 1959). One is in the case of $(\boldsymbol{v}\cdot\boldsymbol{grad})\boldsymbol{v} \ll \nu\nabla^2\boldsymbol{v}$ for $\delta\gg$ a;

$$a^2\omega\ll\nu \quad \text{and} \quad \omega\varepsilon a\ll\nu \tag{3}$$

and the other is in the case of $(\boldsymbol{v}\cdot\boldsymbol{grad})\boldsymbol{v}\ll\partial\boldsymbol{v}/\partial t$ for $\delta\ll a$;

$$a^2\omega\gg\nu \quad \text{and} \quad \varepsilon\ll a \tag{4}$$

In the latter the Reynolds number need not be small. Thus Eq. (2) becomes

$$\frac{\partial \boldsymbol{v}}{\partial t}=-\frac{1}{\rho}\,\boldsymbol{grad}\,p+\nu\nabla^2\boldsymbol{v} \tag{5}$$

taking $\boldsymbol{curl}$ of this equation as $\boldsymbol{\omega}=\boldsymbol{curl}\,\boldsymbol{v}$ gives the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t}=\nu \nabla^2 \boldsymbol{\omega} \tag{6}$$

div $\boldsymbol{v}=0$ means the existence of a vector potential $\boldsymbol{A}$ to be $\boldsymbol{v}=\boldsymbol{curl}\ \boldsymbol{A}$. Linearity of the equation of motion and the boundary conditions written by $\boldsymbol{v}=\boldsymbol{u}$ on $r=a$ and $\boldsymbol{v}=0$ at $r=\infty$ requires that $\boldsymbol{A}$ must be a linear function of $\boldsymbol{u}$. Since $\boldsymbol{v}$ is a polar vector, $\boldsymbol{A}$ must be an axial vector and depend only on the two dimensional radius vector $\boldsymbol{r}$ which is a polar vector. The only such axial vector which can be constructed for a two-dimensionally completely symmetrical body (the cylinder) from two polar vectors $\boldsymbol{r}$ and $\boldsymbol{u}$ is the vector product $\boldsymbol{r}\times\boldsymbol{u}$. Hence $\boldsymbol{A}$ must be of the form $(\boldsymbol{grad}\ f)\times\boldsymbol{u}$, where $f(r)$ is some scalar function of $\boldsymbol{r}$. Since $\boldsymbol{u}=\boldsymbol{u}_0e^{-i\omega t}$ and $\boldsymbol{u}_0$ is a constant vector, the vorticity $\boldsymbol{\omega}$ becomes

$$\begin{aligned}\boldsymbol{\omega}&=\boldsymbol{curl}\ \boldsymbol{curl}[\boldsymbol{grad}\ f\ \times\boldsymbol{u}]\\&=\boldsymbol{curl}\ \boldsymbol{curl}\ \boldsymbol{curl}(f\boldsymbol{u})\\&=(\boldsymbol{grad}\ div-\nabla^2)\ \boldsymbol{curl}(f\boldsymbol{u})\\&=-\nabla^2\boldsymbol{curl}(f\boldsymbol{u})\\&=-\nabla^2(\boldsymbol{grad}\ f\times\boldsymbol{u})\\&=-(\nabla^2\boldsymbol{grad}\ f)\times\boldsymbol{u}\end{aligned}$$

Substituting Eq. (7) into Eq. (6) yields

$$\left(\nabla^2+\frac{i\omega}{\nu}\right)\nabla^2 f=\text{constant}\ (=0) \tag{8}$$

It is easy to see that the constant must be zero, since velocity $\boldsymbol{v}$ must be vanish at infinity, viz. $\nabla^2 f\to 0$ as $r\to\infty$. So we can divide Eq. (8) into two parts

$$\nabla^2 f=F \quad \text{or} \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)=F \tag{9}$$

and

$$(\nabla^2+k^2)F=0 \quad \text{or} \quad \frac{d^2F}{dr^2}+\frac{1}{r}\frac{dF}{dr}+k^2F=0 \tag{10}$$

where $k=(1+i)1/\delta$ is the complex wave number and $\delta=\sqrt{(2\nu/\omega)}$ is the depth of penetration.

The general solution of Eq. (10) can be written as

$$F=AH_0^{(1)}(kr)+CH_0^{(2)}(kr) \tag{11}$$

where A and C are two arbitrary constants determined by the boundary conditions, and $H_0^{(1)}(kr)$ and $H_0^{(2)}(kr)$ are the Hankel functions of the first and second kinds, respectively, and they have the asymptotic behavior for large values of kr, since $k=(1+i)1/\delta$,

$$H_0^{(1)}=J_0(kr)+iY_0(kr)\sim\sqrt{\frac{2}{\pi kr}}\,e^{i(kr-\pi/4)}\longrightarrow 0 \quad \text{as} \quad \frac{r}{\delta}\longrightarrow\infty$$

$$H_0(kr)J_0(kr)-iY_0(kr)\sim\sqrt{\frac{2}{\pi kr}}\,e^{-i(kr-\pi/4)}\longrightarrow\infty \quad \text{as} \quad \frac{r}{\delta}\longrightarrow\infty$$

Therefore $H_0^{(2)}(kr)$ does not satisfy the condition that $\nabla^2 f\to 0$ as $r\to\infty$. Hence C must be zero. This means physically that $H_0^{(1)}(kr)e^{-i\omega t}\sim\sqrt{\frac{2}{\pi kr}}e^{i(kr-\omega t-\pi/4)}$ is the out-going wave from the cylinder and $H_0^{(2)}(kr)e^{-i\omega t}\sim\sqrt{\frac{2}{\pi kr}}e^{-i(kr+\omega t-\pi/4)}$ is the incoming wave towards the cylinder reflected by some obstacle. Now we have no obstacle in this infinite region except for the cylinder, so we have no reflected wave. Thus Eq. (9) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)=AH_0^{(1)}(kr). \tag{12}$$

The solution of this equation is given by

$$\frac{df}{dr}=\frac{A}{k}H_1^{(1)}(kr)+\frac{B}{r}$$

in the first derivative of f with respect to r because we do not need $f(r)$ on the calculation of velocities.

4. Velocity and Stress Components, Pressure

The boundary conditions on the cylinder is given by $\boldsymbol{v}=\boldsymbol{u}$. This means that the normal and tangential components of velocities are the same on $r=a$, that is, $\boldsymbol{v}_0\cdot\boldsymbol{n}=\boldsymbol{u}_0\cdot\boldsymbol{n}$ and $\boldsymbol{n}\times(\boldsymbol{v}_0\times\boldsymbol{n})=\boldsymbol{n}\times(\boldsymbol{u}_0\times\boldsymbol{n})$ in the expression of $\boldsymbol{v}=\boldsymbol{v}_0e^{-i\omega t}$ due to the linearity of equations and boundary conditions. Since div $\boldsymbol{r}=2$ for the two-dimensional vector $\boldsymbol{r}$,

$$\begin{aligned}
\boldsymbol{v}_0&=\boldsymbol{curl}[\boldsymbol{grad}\,f\times\boldsymbol{u}_0]\\
&=\boldsymbol{curl}\left[\frac{1}{r}\frac{df}{dr}\boldsymbol{r}\times\boldsymbol{u}_0\right]\\
&=\left(\boldsymbol{grad}\,\frac{1}{r}\frac{df}{dr}\right)\times[\boldsymbol{r}\times\boldsymbol{u}_0]+\frac{1}{r}\frac{df}{dr}\,\boldsymbol{curl}[\boldsymbol{r}\times\boldsymbol{u}_0]\\
&=r\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right)\boldsymbol{n}\times[\boldsymbol{n}\times\boldsymbol{u}_0]-\frac{1}{r}\frac{df}{dr}\boldsymbol{u}_0\\
&=r\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right)(\boldsymbol{n}\cdot\boldsymbol{u}_0)\boldsymbol{n}-\left[r\frac{d}{dr}\left(\frac{df}{dr}\right)+\frac{1}{r}\frac{df}{dr}\right]\boldsymbol{u}_0
\end{aligned} \tag{14}$$

Hence we obtain the two components

$$v_r=-\frac{1}{r}\frac{df}{dr}u_0\cos\theta e^{-i\omega t} \tag{15}$$

and

$$v_\theta = \frac{d^2 f}{dr^2} u_0 \sin \theta e^{-i\omega t}. \tag{16}$$

The boundary conditions on $r=a$ can be written as

$$-\frac{1}{r}\frac{df}{dr} = 1 \quad \text{for} \quad \boldsymbol{v}_0 \cdot \boldsymbol{n} = \boldsymbol{u}_0 \cdot \boldsymbol{n} \tag{17}$$

and, using Eq. (17),

$$\frac{d^2 f}{dr^2} = -1 \quad \text{for} \quad \boldsymbol{n} \times (\boldsymbol{v}_0 \times \boldsymbol{n}) = \boldsymbol{n} \times (\boldsymbol{u}_0 \times \boldsymbol{n}). \tag{18}$$

Substituting Eq. (13) into Eqs. (17) and (18) gives

$$A = \frac{2}{H_2^{(1)}(ka) - \dfrac{2}{ka} H_1^{(1)}(ka)} \tag{19}$$

$$B = \frac{-a^2 H_2^{(1)}(ka)}{H_2^{(1)}(ka) - \dfrac{2}{ka} H_1^{(1)}(ka)} \tag{20}$$

The deviatric stress components are calculated from

$$\sigma'_{rr} = 2\eta \frac{\partial v_r}{\partial r} = -2\eta \frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right) u_0 \cos \theta e^{-i\omega t} \tag{21}$$

and

$$\sigma'_{r\theta} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) = \eta\left(\frac{d^3 f}{dr^3} - \frac{1}{r}\frac{d^2 f}{dr^2} + \frac{1}{r^2}\frac{df}{dr}\right) u_0 \sin \theta e^{-i\omega t} \tag{22}$$

The pressure p is given by the following: From the equation of motion, as $\rho =$ const., $\boldsymbol{u}_0 =$const., and $\boldsymbol{v} = \boldsymbol{v}_0 e^{-i\omega t}$, we obtain

$$\begin{aligned}
\boldsymbol{grad}\, \frac{p}{\rho} &= (i\omega + \nu \nabla^2) \boldsymbol{v}_0 e^{-i\omega t} \\
&= (\mathrm{i}\omega + \nu \nabla^2) \boldsymbol{curl}\ \boldsymbol{curl}(f\boldsymbol{u}_0) e^{-i\omega t} \\
&= (i\omega + \nu \nabla^2)(\boldsymbol{grad}\, \mathrm{div} - \nabla^2)(f\boldsymbol{u}_0) e^{-i\omega t} \\
&= (i\omega + \nu \nabla^2) \boldsymbol{grad}\, \mathrm{div}(f\boldsymbol{u}_0) e^{-i\omega t}
\end{aligned}$$

$$\therefore \quad \frac{p}{\rho} = \frac{p_0}{\rho} + (i\omega + \nu \nabla^2) \mathrm{div}(f\boldsymbol{u}_0) e^{-i\omega t}$$

or

$$\begin{aligned}
p &= p_0 + [i\omega\rho\, \boldsymbol{u}_0 \cdot \boldsymbol{grad}\, f + \eta \boldsymbol{u}_0 \cdot \boldsymbol{grad}(\nabla^2 f)] e^{-i\omega t} \\
&= p_0 + \left[i\omega\rho \frac{df}{dr} u_0 \cos \theta + \eta u_0 \cos \theta \left(\frac{d^3 f}{dr^3} + \frac{1}{r}\frac{d^2 f}{dr^2} - \frac{1}{r}\frac{df}{dr}\right)\right] e^{-i\omega t}
\end{aligned} \tag{23}$$

5. Drag

The drag acting on the cylinder per unit length is given by

$$F=\int_0^{2\pi}\{\boldsymbol{m}\cdot\boldsymbol{T}\cdot\boldsymbol{n}\}_{r=a}ad\theta$$

$$=\int_0^{2\pi}\{(-p+\sigma'_{rr})\cos\theta-\sigma'_{r\theta}\sin\theta\}_{r=a}ad\theta \tag{24}$$

where $\boldsymbol{m}=(\cos\theta, -\sin\theta, 0)$ is the unit vector parallel to $\boldsymbol{u}_0$, and $\boldsymbol{T}$ is the stress tensor;

$$\boldsymbol{T}=-\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}+\begin{pmatrix} \sigma'_{rr} & \sigma'_{r\theta} & \sigma'_{rz} \\ \sigma'_{\theta r} & \sigma'_{\theta\theta} & \sigma'_{\theta z} \\ \sigma'_{zr} & \sigma'_{z\theta} & \sigma'_{zz} \end{pmatrix} \tag{25}$$

Since $\int_0^{2\pi} p_0\cos\theta d\theta=0$, substituting Eqs. (21), (22) and (23) into (24) gives

$$F=-\left[\pi\, au_0 i\omega\rho\left(\frac{df}{dr}\right)_{r=a}+2\pi\eta u_0 a\left(\frac{d^3f}{dr^3}+\frac{1}{r}\frac{d^2f}{dr^2}-\frac{1}{r}\frac{df}{dr}\right)_{r=a}\right]e^{-i\omega t}$$

$$=i\omega u_0 M'\left\{1-\frac{4H_1^{(1)}(Z)}{H_1^{(1)}(Z)+ZH_1^{(1)\prime}(Z)}\right\}e^{-i\omega t} \tag{26}$$

$$=i\omega u_0 M'\left\{1-\frac{4H_1^{(1)}(Z)}{2H_1^{(1)}(Z)-ZH_2^{(1)}(Z)}\right\}e^{-i\omega t} \tag{27}$$

where $u=u_0e^{-i\omega t}$, $u_0=|\boldsymbol{u}_0|$, $ka=Z=(1+i)a/\delta$, $M'=\pi\rho a^2$ is the mass of fluid equivalent to the volume of the cylinder per unit length. The following expression for F is useful;

$$F=i\omega u_0 M'\{q+iq'\}e^{-i\omega t} \tag{28}$$

The real part of this expression gives

$$F=M'u_0\omega\{q\sin\omega t-q'\cos\omega t\} \tag{29}$$

where q and q' are the real and imaginary parts of { }, respectively. The asymptotic expansion of the Hankel function of the first kind (Watson, 1958 is given by

$$H_p^{(1)}(Z)=\left(\frac{2}{\pi Z}\right)^{\frac{1}{2}}e^{i(Z-(p/2)\pi-(1/4)\pi)}\sum_{m=0}^{\infty}\frac{(p, m)}{(-2iZ)m} \tag{30}$$

where

$$(p, m)=\frac{\{4p^2-1\}\{4p^2-3\}\dots\{4p^2-(2m-1)\}}{2^{2m}m!} \tag{31}$$

$$(p, 0) \equiv 1$$

$$\therefore \quad \frac{H_2^{(1)}(Z)}{H_1^{(1)}(Z)} \sim -i\left(1-\frac{3}{2iZ}\right) \tag{32}$$

For large values of Z, viz. small values of (δ/a),

$$q+iq' = 1-\frac{4}{2+iZ\left(1-\dfrac{3}{2iZ}\right)} \tag{33}$$

$$q = 1-\frac{4\left(\dfrac{1}{2}-\dfrac{a}{\delta}\right)}{\left(\dfrac{1}{2}-\dfrac{a}{\delta}\right)^2+\left(\dfrac{a}{\delta}\right)^2} = 1+2\frac{\delta}{a} \tag{34}$$

$$q' = \frac{4\left(\dfrac{a}{\delta}\right)}{\left(\dfrac{1}{2}-\dfrac{a}{\delta}\right)^2+\left(\dfrac{a}{\delta}\right)^2} = 2\frac{\delta}{a}\left(1+\frac{1}{2}\frac{\delta}{a}\right) \tag{35}$$

Eqs. (34) and (35) are identically equal to ones given by STOKES. On the other hand, for small values of Z, viz. small values of a/δ, the behaviors of the Hankel function of the first kind (HILDEBRAND, 1962) are

$$H_n^{(1)}(Z) = J_n(Z) + iY_n(Z) \tag{36}$$

where

$$J_n(Z) = \sum_{k=0}^{\infty} \frac{(-1)^k\left(\dfrac{Z}{2}\right)^{2k+n}}{k!\,(k+n)!} \tag{37}$$

$$Y_n(Z) = \frac{2}{\pi}\left[\left(\log\frac{Z}{2}+\gamma\right)J_n(Z) - \frac{1}{2}\sum_{k=0}^{n-1}\frac{(n-k-1)!\left(\dfrac{Z}{2}\right)^{2k-n}}{k!}\right.$$
$$\left. + \frac{1}{2}\sum_{k=0}^{\infty}(-1)^{k+1}[\varphi(k)+\varphi(k+n)]\frac{\left(\dfrac{Z}{2}\right)^{2k+n}}{k!\,(k+n)!}\right] \tag{38}$$

and the abbreviation $\varphi(k)$ is

$$\varphi(k) = \sum_{m=1}^{k}\frac{1}{m} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k} \tag{39}$$

$$\varphi(0) \equiv 0.$$

γ is Euler's constant, defined by the relation

$$\gamma = \lim_{k\to\infty}[\varphi(k)-\log k] = 0.5772157 \tag{40}$$

In particular, for $n=1$ and 2, we have

$$H_1^{(1)}(Z) \sim J_1(Z) + \frac{2i}{\pi}\left[\left(\log\frac{Z}{2}+\gamma\right)J_1(Z) - \frac{1}{Z} - \frac{Z}{4}\right] \tag{41}$$

$$J_1(Z) \sim \frac{Z}{2}$$

and

$$H_2^{(1)}(Z) \sim J_2(Z) + \frac{2i}{\pi}\left[\left(\log\frac{Z}{2}+\gamma\right)J_2(Z) - \frac{2}{Z^2} - 4 - \frac{3}{32}Z^2\right] \tag{42}$$

$$J_2(Z) \sim \frac{Z^2}{8}$$

Hence

$$\frac{H_2^{(1)}(Z)}{H_1^{(1)}(Z)} \sim \frac{2}{Z}\left\{1+\frac{Z^2}{2}\left(\log\frac{Z}{2}+\gamma\right)\right\} \tag{43}$$

For small values of Z, viz. small values of (a/δ),

$$q+iq' = 1 + \frac{4}{Z^2\left(\log\frac{Z}{2}+\gamma\right)} \tag{44}$$

$$q = 1 - \frac{\frac{\pi}{4}}{\left(\frac{a^2}{2\delta^2}\right)\left\{\left(\log\frac{a}{\sqrt{2}\delta}+\gamma\right)^2+\left(\frac{\pi}{4}\right)^2\right\}} \tag{45}$$

$$q' = -\frac{\log\frac{a}{\sqrt{2}\delta}+\gamma}{\left(\frac{a^2}{2\delta^2}\right)\left\{\left(\log\frac{a}{\sqrt{2}\delta}+\gamma\right)^2+\left(\frac{\pi}{4}\right)^2\right\}} \tag{46}$$

But these q and q' are not valid for small (a/δ) because both q and q' become infinite as (a/δ) goes to zero. So we should use the Oseen's formula in this region. We can only use Eqs. (34) and (35) for small (δ/a), which covers almost all of actual engineering problems.

6. Comparison with Sphere

We now discuss the difference between the forces acting on an oscillating cylinder and an oscillating sphere in incompressible fluid. We will find much similarity. Dash means the corresponding equations in the case of a cylinder. Eqs. (1) through (8) are valid for the sphere, but Eq. (9) becomes

$$\nabla^2 f = F \quad \text{or} \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = F \tag{9$'$}$$

Hence

$$\frac{df}{dr}=\frac{1}{r^2}\left[Ae^{ikr}\left(r-\frac{1}{ik}\right)+B\right] \tag{13$'$}$$

where

$$A=-\frac{3a}{2ik}\,e^{-ika} \tag{19$'$}$$

$$B=-\frac{1}{2}a^3\left(1-\frac{3}{ika}-\frac{3}{k^2a^2}\right) \tag{20$'$}$$

Because the space is three-dimensional; div $\boldsymbol{r}=3$, so we have

$$\boldsymbol{v}_0=r\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right)\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{u}_0)-\frac{2}{r}\frac{df}{dr}\boldsymbol{u}_0$$

$$=r\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right)(\boldsymbol{n}\cdot\boldsymbol{u}_0)\boldsymbol{n}-\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right)+\frac{2}{r}\right)\boldsymbol{u}_0 \tag{14$'$}$$

and

$$v_r=-\frac{2}{r}\frac{df}{dr}u_0\cos\theta e^{-i\omega t} \tag{15$'$}$$

$$v_\theta=\left(\frac{d^2f}{dr^2}+\frac{1}{r}\right)u_0\sin\theta e^{-i\omega t} \tag{16$'$}$$

with the boundary conditions

$$-\frac{2}{r}\frac{df}{dr}=1 \quad \text{for} \quad \boldsymbol{v}_0\cdot\boldsymbol{n}=\boldsymbol{u}_0\cdot\boldsymbol{n} \tag{17$'$}$$

$$\frac{d^2f}{dr^2}=-\frac{1}{2} \quad \text{for} \quad \boldsymbol{n}\times(\boldsymbol{v}_0\times\boldsymbol{n})=\boldsymbol{n}\times(\boldsymbol{u}_0\times\boldsymbol{n}) \tag{18$'$}$$

The deviatric stress components and the pressure are

$$\sigma'_{rr}=2\eta\frac{\partial v_r}{\partial r}=-4\eta\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right)u_0\cos\theta e^{-i\omega t} \tag{21$'$}$$

$$\sigma'_{r\theta}=\eta\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)=\eta\frac{d^3f}{dr^3}u_0\sin\theta e^{-i\omega t} \tag{22$'$}$$

$$p=p_0+\left[i\omega\rho\frac{df}{dr}u_0\cos\theta+\eta u_0\cos\theta\left(\frac{d^3f}{dr^3}+\frac{2}{r}\frac{d^2f}{dr^2}-\frac{2}{r^2}\frac{df}{dr}\right)\right]e^{-i\omega t} \tag{23$'$}$$

Finally the drag is expressed by the following formula:

$$F=\frac{-4\pi a^2}{3}\left[i\omega\rho u_0\left(\frac{df}{dr}\right)+3\eta\left(\frac{d^3f}{dr^3}+\frac{2}{r}\frac{d^2f}{dr^2}-\frac{2}{r^2}\frac{df}{dr}\right)u_0\right]_{r=a}e^{-i\omega t}$$

$$=i\omega u_0M'\left[\frac{1}{2}-\frac{9}{2}\frac{1}{Z^2}(1-iZ)\right]e^{-i\omega t} \tag{27$'$}$$

where $M'=(4/3)\pi\rho a^3$.

Since

$$q+iq'=\frac{1}{2}-\frac{9}{2}\frac{1}{Z^2}(1-iZ) \tag{33}'$$

$$\therefore\quad q=\frac{1}{2}+\frac{9}{4}\frac{\delta}{a} \tag{34}'$$

$$q'=\frac{9}{4}\left\{\frac{\delta}{a}+\left(\frac{\delta}{a}\right)^2\right\} \tag{35}'$$

Therefore we have

$$F=i\omega M'u_0\left\{\left(\frac{1}{2}+\frac{9}{4}\frac{\delta}{a}\right)+i\frac{9}{4}\left(\frac{\delta}{a}+\left(\frac{\delta}{a}\right)^2\right)\right\}e^{-i\omega t}$$

And we can easily show that this formula is equivalent to the Landau-Lifshitz's expression

$$F=6\pi\eta a\left(1+\frac{a}{\delta}\right)u+3\pi a^2\sqrt{\frac{2\eta\rho}{\omega}}\left(1+\frac{2}{9}\frac{a}{\delta}\right)\frac{du}{dt}$$

where $u=u_0e^{-i\omega t}$.

For $\omega=0$ this becomes Stokes' formula $F=6\pi\eta au_0$, while for large frequencies we have

$$F=\frac{2}{3}\pi\rho a^3\frac{du}{dt}+3\pi a^2\sqrt{2\eta\rho\omega}\,u.$$

The first term in this expression corresponds to the inertial force in potential flow past a sphere, while the second gives limit of the dissipative force.

Acknowledgements

The author would like to thank Dr. ANDO, T. and Mr. HASEGAWA, E. for their advice.

REFERENCES

HILDEBRAND, F. B. (1962): Advanced Calculus for Application, Prentice-Hall, p. 143 and 147.

LANDAU, L. D. and LIFSHITZ, E. M. (1959): Fluid Mechanics, Pergamon Press, p. 51.

MARTIN, H. (1925): Über Tonhöhe und Dämpfung der Schwingungen von Saiten in verschieden Flüssikeiten, Ann. Phys., Lpz. (4), **77**, 627.

OSEEN, C.W. (1910): Über die Stokessche Formel und über eine verwante Aufgabe in der Hydrodynamik, Ark. Mat. Astr. Fys. **6**, 29.

ROSENHEAD, L. (1963): Laminar Boundary Layers, Oxford University Press.
STOKES, G. G. (1851): On the effect of the internal friction of fluids on the motion of pendulums. Trans. Camb. Phil. Soc. **9**, Pt. II, 8–106.
STUART, J. T. and WOODGATE, L. (1955): Experimental determination of aerodynamic damping on a vibrating circular cylinder, Phil. Mag. (7), **46**, 40–46.
WATSON, G. N. (1958): Theory of Bessel Function, Cambridge University Press, p. 198.