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PART II—VISCOUS LIQUIDS

BY

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FLUID PRESSURE TRANSIENTS IN A TAPERED TRANSMISSION LINE

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ABSTRACT

In this paper is shown the extension of the theory for inviscid liquid (Part I) to viscous liquid. The same method is valid for the latter except a part of treatments. As examples, the time histories of the output pulses of pressure or volume flux are calculated in a constant, linear or exponential line on some boundary conditions.

Nomenclature

$A(x)$, A_0 = cross-sectional area of tube
a = wave velocity
$\boldsymbol{B}(x, s)$ = impedance-admittance matrix
$\boldsymbol{C}(x, s)$ = integrated impedance-admittance matrix
$\mathrm{erf}(x)$ = error function
$\mathrm{erfc}(x)$ = complementary error function
$F_1(t)$ = function defined by Eq. (53)
$F_2(t)$ = function defined by Eq. (66)
$\boldsymbol{G}(x, x_1, s)$ = transfer-function matrix
$I_p(x)$ = modified Bessel function of the first kind, of order p
$i = \sqrt{-1}$
$i^n \mathrm{erfc}\,(x)$ = the n-th repeated integral of complementary error function
$J_p(x)$ = Bessel function of the first kind, of order p

K=bulk compression modulus of liquid
$\boldsymbol{k}$=constant vector
L=total length of tube
$p, p_0, \tilde{p}, \hat{p}$=pressure
$q, q_0, \tilde{q}, \hat{q}$=volume fiux
$R(x), R_0$=radius of tube
$\boldsymbol{R}(x, s)$=matrix defined by Eq. (36)
r=radial coordinate
s=Laplace variable
$\boldsymbol{T}(x, s)$=matrix defined by Eq. (34)′
t=time
$u, u_0, \hat{u}$=axial component of velocity
$\boldsymbol{u}=\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix}$
$\boldsymbol{u}(x, 0)=\begin{pmatrix} p(x, 0) \\ q(x, 0) \end{pmatrix}$=initial vector
v=radial component of velocity
$\boldsymbol{w}$=velocity vector
x=axial coordinate
$Y(x, s)$=characteristic admittance
$Y_p(x)$=Bessel function of the second kind, of order p
$Z(x, s)$=characteristic impedance
α, β=magnification constant of tube
ρ, ρ_0=density of liquid
μ=absolute viscosity
ν=kinematic viscosity
λ=eigenvalue defined by Eq. (32)
$\boldsymbol{\Lambda}(x, s)$=diagonal matrix
τ=duration of time
ξ=dummy variable

Subscript

0=value at $t=0$ or $x=0$
-1=inverse matrix
$\wedge$=Laplace-transformed value
$\sim$=input pulse
h=homogeneous solution
p=particular solution or order
t=transposed matrix

Introduction

A pulse input from the entrance into a small-diameter tapered transmission line filled with viscous liquid appear with the distortion of shape and different height at the exit after a short time. If the variation of cross-sectional area is not large $\frac{\tau}{T}\frac{L}{A}\cdot\frac{dA}{dx}\ll 1$, as shown in Part I, distortion is not present even in a tapered line. In spite of this fact we will find in this Part II that the distortion is caused even in a constant line. This means that the shear stress induced by viscosity brings about the distortion in pulse and the variation of cross-sectional area mainly gives the difference of height between input and output pulses. This output pulse of pressure is inversely proportional to the square of the cross-sectional area and the output pulse of volume flux is directly proportional to it. Giving the wide scope of the above-mentioned treatments for a viscous liquid is the purpose of Part II.

Basic Equations

The flow of a slightly compressible viscous liquid is completely described by the following equations:

The Navier-Stokes equation

$$\frac{\partial \boldsymbol{w}}{\partial t}+(\boldsymbol{w}\cdot\text{grad})\boldsymbol{w}=-\frac{1}{\rho}\text{grad}\,p+\nu\nabla^2\boldsymbol{w} \tag{1}$$

the continuity equation

$$\text{div}\,(\rho\boldsymbol{w})+\frac{\partial\rho}{\partial t}=0 \tag{2}$$

the equation of state for a liquid

$$\frac{dp}{d\rho}=\frac{K}{\rho} \tag{3}$$

where $\boldsymbol{w}$ is the fluid velocity,

ρ is density,

t is time,

p is static pressure,

and K is the isothermal bulk compression modulus.

The following assumptions will be made:

(i) Variations in density are small.

(ii) Two-dimensional flow, viz., velocity profile is parabolic.

(iii) Pressure is uniform over a cross-sectional area of a conduit.

(iv) Convective acceleration is negligible compared to the local acceleration.

(v) The changes of a cross-sectional area in the longitudinal dimension are small.

We shall deal only with tubes that are circular in cross section so that the flow will be axisymmetric. With these assumptions the axial component of the Navier-Stokes equations (1) becomes

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\nu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right\} \tag{4}$$

where x is the axial coordinate, ρ_0 is the mean density, and u is the axial component of velocity. From the assumption (iii) p is a function of x and t and u is a function of x, r and t. The two-dimensional continuity equation may be written from Eq. (2) as follows,

$$\frac{\partial \rho}{\partial t}+\rho_0\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}\right\}=0 \tag{5}$$

where v is the radial component of velocity. From Eq. (3)

$$\frac{\partial p}{\partial t}=\frac{K}{\rho_0}\frac{\partial \rho}{\partial t} \tag{6}$$

Substituting Eq. (6) into Eq. (5)

$$\frac{1}{K}\frac{\partial p}{\partial t}+\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}\right\}=0 \tag{7}$$

We define $\bar{p}$ as the average pressure and q as the mean discharge in the x-direction across the cross-sectional area of a conduit; i.e.,

$$\bar{p}(x,t)=\frac{1}{A(x)}\int_0^{R(x)}2\pi r p(x,t)dr=p(x,t) \tag{8}$$

$$q(x,t)=\int_0^{R(x)}2\pi r u(x,r,t)dr \tag{9}$$

where $R(x)$ and $A(x)$ are radius and cross-sectional area, respectively. While

$$\int_0^{R(x)}2\pi r\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)dr=2\pi\int_0^{R(x)}\frac{\partial}{\partial r}(rv)dr=0$$

Hence averagining Eq. (7) over the cross-sectional area, we obtain

$$\frac{\partial p}{\partial t}+\frac{K}{A(x)}\frac{\partial q}{\partial x}=0 \tag{10}$$

Laplace-transformation of Eq. (10) gives

$$\frac{d\hat{q}(x,s)}{ds}=-\frac{A(x)s}{K}\hat{p}(x,s)+\frac{A(x)}{K}p(x,0) \tag{11}$$

where $\hat{q}$ and $\hat{p}$ are the Laplace-transforms of $q(x,t)$ and $p(x,t)$, respectively. $p(x,0)$ is determined by the initial condition.

And also Laplace-transformation of Eq. (4) gives

$$\frac{\partial^2\hat{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\hat{u}}{\partial r}-\frac{s}{\nu}\hat{u}=\frac{1}{\nu\rho_0}\frac{d\hat{p}}{dx}-\frac{1}{\nu}u(x,r,0) \tag{12}$$

where $\hat{u}=\hat{u}(x, r, s)$ and is the Laplace transforms of $u(x, r, t)$, and $u(x, r, 0)$ is also determined by the initial condition.

For a liquid at steady state in a tapered line, the initial conditions may be approximately stated as

$$u(x, r, 0)=2u_0\left\{1-\frac{r^2}{R^2(x)}\right\}\frac{A_0}{A(x)} \tag{13}$$

$$P(x, 0)=P_0-\frac{8\mu u_0 x A_0}{R^2(x)A(x)} \tag{14}$$

where u_0 and p_0 are the mean velocity and pressure $x=0$ and at $t=0$, respectively. A_0 is the cross sectional area on $x=0$ (see Fig. 1). Here let us solve Eq. (12). The corresponding homogeneous equation is expressed as follows, setting $\hat{u}(x, r)=\alpha(x)\beta(r)$ to separate the independent variables and $z=i\sqrt{s/\nu}\,r$ for β:

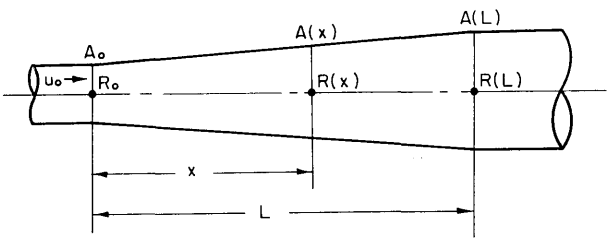

Fig. 1 Tapered transmission line

$$z^2\frac{d^2\beta}{dz^2}+z\frac{d\beta}{dz}+(z^2-0)\beta=0 \tag{15}$$

The solutions of this differential equation are known as Bessel function of order zero and expressed in the following form:

$$\beta=C_1J_0(z)+C_2Y_0(z) \tag{16}$$

where $J_0(z)$ and $Y_0(z)$ are Bessel functions of the first and second kinds, respectively. Since β is finite at $r=0$, C_2 must be zero. Therefore we have as a homogeneous solution

$$\hat{u}_h(x, r)=C_1\alpha(x)J_0\left(i\sqrt{\frac{s}{\nu}}r\right) \tag{17}$$

The particular solution $\hat{u}_p(x, r)$ is obtained by the method of undetermined coefficients, viz.,

$$\hat{u}_p(x, r)=B\frac{d\hat{p}}{dx}+Cr^2+Dr+E \tag{18}$$

Substituting Eq. (18) into Eq. (12) and using Eq. (13) gives

$$\hat{u}_p(x, r) = -\frac{1}{s\rho_0}\frac{d\hat{p}}{dx} + \frac{u(x, r, 0)}{s} - \frac{8\nu u_0 A_0}{s^2 R^2 A(x)} \tag{19}$$

Hence the general solution is expressed in the form; $\hat{u}=\hat{u}_h+\hat{u}_p$,

$$\hat{u}(x, r) = C_1\alpha(x) J_0\left(i\sqrt{\frac{s}{\nu}}\,r\right) - \frac{1}{s\rho_0}\frac{d\hat{p}}{dx} + \frac{u(x, r, 0)}{s} - \frac{8\nu u_0 A_0}{s^2 R^2 A(x)} \tag{20}$$

Here $C_1\alpha(x)$ is determined by the no-slip condition of $\hat{u}(x, R)=0$ at the wall.

$$C_1\alpha(x) = \frac{1}{J_0\left(i\sqrt{\frac{s}{\nu}}\,R\right)}\left[\frac{1}{s\rho_0}\frac{d\hat{p}}{dx} + \frac{8\nu u_0 A_0}{s^2 R^2 A(x)}\right] \tag{21}$$

Then

$$\hat{u}(x, r) = -\frac{1}{s\rho_0}\left[\frac{d\hat{p}}{dx} + \frac{8\nu\rho_0 u_0 A_0}{sR^2 A(x)}\right]\left[1 - \frac{J_0\left(j\sqrt{\frac{s}{\nu}}\,r\right)}{J_0\left(i\sqrt{\frac{s}{\nu}}\,R\right)}\right] + \frac{u(x, r, 0)}{s} \tag{22}$$

The volume flux $\hat{q}(x, s)$ is given by

$$\hat{q}(x, s) = \int_0^R 2\pi r\hat{u}(x, r)dr \tag{23}$$

By making use of the formula

$$\int_0^x x^n J_{n-1}(x)dx = x^n J_n(x) \tag{24}$$

Multiplying both sides of Eq. (22) by $2\pi r dr$ and integrating over the cross section of radius R yields

$$\hat{q}(x, s) = -\frac{A(x)}{s\rho_0}\left[\frac{d\hat{p}}{dx} + \frac{8\nu\rho_0 u_0 A_0}{sR^2 A(x)}\right]\left[1 - \frac{2J_1\left(i\sqrt{\frac{s}{\nu}}\,R\right)}{i\sqrt{\frac{s}{\nu}}\,R J_0\left(i\sqrt{\frac{s}{\nu}}\,R\right)}\right] + \frac{q_0}{s} \tag{25}$$

where

$$q_0 = \int_0^R 2\pi r u(x, r, 0)dr = u_0 A_0 \tag{26}$$

Equations (11) and (25), by defining

$$Y(x, s) = \frac{A(x)s}{K} \tag{27}$$

$$Z(x,s)=\frac{s\rho_0}{A(x)}\left[1-\frac{2J_1\left(i\sqrt{\frac{s}{\nu}}R\right)}{i\sqrt{\frac{s}{\nu}}RJ_0\left(i\sqrt{\frac{s}{\nu}}R\right)}\right]^{-1} \tag{28}$$

take the general forms

$$\frac{d\hat{q}(x,s)}{dx}+Y(x,s)\left\{\hat{p}(x,s)-\frac{p(x,0)}{s}\right\}=0 \tag{29}$$

$$\frac{d\hat{p}(x,s)}{dx}+Z(x,s)\left\{\hat{q}(x,s)-\frac{q_0}{s}\right\}=-\frac{8\nu\rho_0u_0A_0}{sR^2A(x)} \tag{30}$$

Equations (29) and (30) can be written in the form

$$\begin{pmatrix}\frac{d\hat{p}}{dx}\\ \frac{d\hat{q}}{dx}\end{pmatrix}=-\begin{pmatrix}0 & Z(x,s)\\ Y(x,s) & 0\end{pmatrix}\begin{pmatrix}\hat{p}\\ \hat{q}\end{pmatrix}+\begin{pmatrix}0 & \frac{1}{s}\left\{Z(x,s)-\frac{8\nu\rho_0}{R^2A(x)}\right\}\\ \frac{Y(x,s)}{s} & 0\end{pmatrix}\begin{pmatrix}p(x,0)\\ q(x,0)\end{pmatrix}$$

and rewritten in the vector-matrix form

$$\frac{d\hat{\boldsymbol{u}}}{dx}=-\boldsymbol{B}(x,s)\hat{\boldsymbol{u}}+\boldsymbol{C}(x,s)\boldsymbol{u}(x,0) \tag{31}$$

where

$$\hat{\boldsymbol{u}}=\begin{pmatrix}\hat{p}\\ \hat{q}\end{pmatrix},\quad \frac{d\hat{\boldsymbol{u}}}{dx}=\begin{pmatrix}\frac{d\hat{p}}{dx}\\ \frac{d\hat{q}}{dx}\end{pmatrix},\quad \boldsymbol{B}(x,s)=\begin{pmatrix}0 & Z\\ Y & 0\end{pmatrix}$$

$$\boldsymbol{C}(x,s)=\begin{pmatrix}0 & \frac{1}{s}\left\{Z-\frac{8\nu\rho_0}{R^2A(x)}\right\}\\ \frac{Y}{s} & 0\end{pmatrix},\quad \boldsymbol{u}(x,0)=\begin{pmatrix}p(x,0)\\ q(x,0)\end{pmatrix}$$

For fluid initially at rest, Eq. (31) becomes the same discussions in Part I

$$\frac{d\hat{\boldsymbol{u}}}{dx}=-\boldsymbol{B}(x,s)\hat{\boldsymbol{u}} \tag{31$'$}$$

where $\hat{p}(x,s)$ is regarded as $\hat{p}(x,s)-p_0/s$.

Derivation of solution in transformed space

With use of the same notations in Part I we can obtain the same type solution of Eq. (31). We will discuss the difference between Part I and Part II.

First, the eigenvalue λ of the impedance-admittance matrix $\boldsymbol{B}$ is not constant. From Eqs. (27) and (28)

$$\lambda(x, s)=\sqrt{ZY}=\frac{\dfrac{s}{a}}{\sqrt{1-\dfrac{2J_1\left(i\sqrt{\dfrac{s}{\nu}}R\right)}{i\sqrt{\dfrac{s}{\nu}}RJ_0\left(i\sqrt{\dfrac{s}{\nu}}\right)}}} \tag{32}$$

Secondly,

$$\boldsymbol{T}^{-1}\frac{d\boldsymbol{T}}{dx}\approx\frac{1}{2A(x)}\frac{dA(x)}{dx}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{33}$$

because R is a function of x and is included in the Bessel functions, but $\boldsymbol{T}^{-1}\dfrac{d\boldsymbol{T}}{dx}$ can be neglected for the same reason as in Part I. Then the solution of Eq. (31) is expressed by

$$\hat{\boldsymbol{u}}=\boldsymbol{T}(\boldsymbol{R}^{-1})^t\left\{\int^x \boldsymbol{R}^t\boldsymbol{T}^{-1}\boldsymbol{C}\boldsymbol{u}(\xi, 0)d\xi+\boldsymbol{k}\right\} \tag{34}$$

corresponding to Eq. (29) in Part I, where

$$\boldsymbol{T}=\begin{pmatrix}\sqrt{Z} & -\sqrt{Z}\\ \sqrt{Y} & \sqrt{Y}\end{pmatrix} \tag{34$'$}$$

and $\boldsymbol{k}$ is an arbitrary integration constant vector determined by the boundary conditions. The solution of the fluid intially at rest is given by

$$\hat{\boldsymbol{u}}=\boldsymbol{T}(\boldsymbol{R}^{-1})^t\boldsymbol{k} \tag{35}$$

where, if $\lambda(x, s)=\lambda(x)$ in brief,

$$\boldsymbol{R}(x, s)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\int_0^x\begin{pmatrix}\lambda(\xi_1) & 0\\ 0 & -\lambda(\xi_1)\end{pmatrix}d\xi_1+\int_0^x\int_0^{\xi_1}\begin{pmatrix}\lambda(\xi_1) & 0\\ 0 & -\lambda(\xi^1)\end{pmatrix}\begin{pmatrix}\lambda(\xi_2) & 0\\ 0 & -\lambda(\xi_2)\end{pmatrix}d\xi_1 d\xi_2+\cdots$$

$$=\begin{pmatrix}e^{\int_0^x\lambda(\xi)d\xi} & 0\\ 0 & e^{-\int_0^x\lambda(\xi)d\xi}\end{pmatrix}=e^{\begin{pmatrix}\int_0^x\lambda(\xi)d\xi & 0\\ 0 & -\int_0^x\lambda(\xi)d\xi\end{pmatrix}}=e^{\int_0^x\begin{pmatrix}\lambda(\xi) & 0\\ 0 & -\lambda(\xi)\end{pmatrix}d\xi}$$

$$=e^{\int_0^x\Lambda(\xi, s)d\xi}$$

Equation (35) is expressed with the following components,

$$\hat{p}(x, s)-\frac{p_0}{s}=\sqrt{Z}\left\{k_1e^{-\int_0^x\lambda(\xi)d\xi}-k_2e^{\int_0^x\lambda(\xi)d\xi}\right\} \tag{37}$$

$$\hat{q}(x, s)=\sqrt{Y}\left\{k_1e^{-\int_0^x\lambda(\xi)d\xi}+k_2e^{\int_0^x\lambda(\xi)d\xi}\right\} \tag{38}$$

where the asymptotic expansions of $\lambda(x)$, $Z(x)$, $\sqrt{Z(x)}$, $1/\sqrt{Z(x)}$ with respect to s are respectively given by

$$\lambda(x)=\frac{s}{a}\left\{1+\left(\frac{\nu}{sR^2}\right)^{\frac{1}{2}}+\left(\frac{\nu}{sR^2}\right)+\frac{7}{8}\left(\frac{\nu}{sR^2}\right)^{\frac{3}{2}}+\frac{1}{2}\left(\frac{\nu}{sR^2}\right)^2+\cdots\right\} \tag{39}$$

$$Z(x)=\frac{s\rho_0}{A(x)}\left\{1+2\left(\frac{\nu}{sR^2}\right)^{\frac{1}{2}}+3\left(\frac{\nu}{sR^2}\right)+\frac{15}{4}\left(\frac{\nu}{sR^2}\right)^{\frac{3}{2}}+\frac{15}{4}\left(\frac{\nu}{sR^2}\right)^2+\cdots\right\} \tag{40}$$

$$\sqrt{Z(x)}=\sqrt{\frac{s\rho_0}{A(x)}}\left\{1+\left(\frac{\nu}{sR^2}\right)^{\frac{1}{2}}+\left(\frac{\nu}{sR^2}\right)+\frac{7}{8}\left(\frac{\nu}{sR^2}\right)^{\frac{3}{2}}+\frac{1}{2}\left(\frac{\nu}{sR^2}\right)^2+\cdots\right\} \tag{41}$$

$$\frac{1}{\sqrt{Z(x)}}=\sqrt{\frac{A(x)}{s\rho_0}}\left\{1-\left(\frac{\nu}{sR^2}\right)^{\frac{1}{2}}+\frac{1}{8}\left(\frac{\nu}{sR^2}\right)^{\frac{3}{2}}+\frac{1}{4}\left(\frac{\nu}{sR^2}\right)^2+\cdots\right\} \tag{42}$$

with the help of the following equation;

$$\frac{2J_1\left[i\left(\frac{\nu}{s}\right)^{\frac{1}{2}}R\right]}{i\left(\frac{\nu}{s}\right)^{\frac{1}{2}}RJ_0\left[i\left(\frac{\nu}{s}\right)^{\frac{1}{2}}R\right]}=\frac{2I_1\left[\left(\frac{\nu}{s}\right)^{\frac{1}{2}}R\right]}{\left(\frac{\nu}{s}\right)^{\frac{1}{2}}RI_0\left[\left(\frac{\nu}{s}\right)^{\frac{1}{2}}R\right]}$$
$$=2\left(\frac{\nu}{sR^2}\right)^{\frac{1}{2}}-\left(\frac{\nu}{sR^2}\right)-\frac{1}{4}\left(\frac{\nu}{sR^2}\right)^{\frac{3}{2}}-\frac{1}{4}\left(\frac{\nu}{sR^2}\right)^2+\cdots \tag{43}$$

here $I_p(x)=i^{-p}J_p(ix)$ and this is called the modified Bessel function of the first kind, of order p.

Transfer-function matrix

In the same way we derived Eq. (36) in Part I, we obtain the following transfer-function matrix:

$$\boldsymbol{G}(x, x_1, s)=\boldsymbol{T}(x, s)\boldsymbol{R}^{-1}(x, s)\boldsymbol{R}(x_1, s)\boldsymbol{T}^{-1}(x_1, s)$$
$$=\frac{1}{2}\begin{pmatrix}\sqrt{Z} & -\sqrt{Z}\\ \sqrt{Y} & \sqrt{Y}\end{pmatrix}\begin{pmatrix}e^{-\int_0^x\lambda(\xi)d\xi} & 0\\ 0 & e^{\int_0^x\lambda(\xi)d\xi}\end{pmatrix}\begin{pmatrix}e^{\int_0^{x_1}\lambda(\xi)d\xi} & 0\\ 0 & e^{-\int_0^{x_1}\lambda(\xi)d\xi}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{Z_1}} & \frac{1}{\sqrt{Y_1}}\\ -\frac{1}{\sqrt{Z_1}} & \frac{1}{\sqrt{Y_1}}\end{pmatrix}$$
$$=\begin{pmatrix}\sqrt{\frac{Z}{Z_1}}\cosh\left(\int_{x_1}^x\lambda(\xi)d\xi\right) & -\sqrt{\frac{Z}{Y_1}}\sinh\left(\int_{x_1}^x\lambda(\xi)d\xi\right)\\ -\sqrt{\frac{Y}{Z_1}}\sinh\left(\int_{x_1}^x\lambda(\xi)d\xi\right) & \sqrt{\frac{Y}{Y_1}}\cosh\left(\int_{x_1}^x\lambda(\xi)d\xi\right)\end{pmatrix} \tag{44}$$

where $\lambda(x)$=Eq. (39), $Z(x)$=Eq. (40) and $Y(x)$=Eq. (27).

Inverse Laplace-transformation

The fluid is initially at rest all over the cases in this Part II. Hence we try to find the inverse Laplace-transformation of Eqs. (37) and (38).

A. Dynamic response of pressure pulse

EXAMPLE 1. Travelling pulse for pressure pulse input

The following input pulse is put in a semi-infinite tapered line as a boundary condition:

$$p(0,t)-p_0=\begin{cases}0 & t<0\\ \tilde{p} & 0<t<\tau\\ 0 & \tau<t\end{cases} \qquad \therefore \hat{p}(0,s)-\frac{p_0}{s}=\frac{\tilde{p}}{s}(1-e^{-\tau s}) \tag{45}$$

There is no reflected wave in this case, so $k_2=0$ in Eq. (37). Eliminating k_1 from Eq. (37) with the aid of Eq. (45) yields

$$\hat{p}(x,s)-\frac{p_0}{s}=\sqrt{\frac{Z(x)}{Z(0)}}\frac{\tilde{p}}{s}\{1-e^{-s\tau}\}e^{-\int_0^x \lambda(\xi)d\xi} \tag{46}$$

Substituting Eqs. (41), (42) and (39) in the places of $\sqrt{Z(x)}$, $\frac{1}{\sqrt{Z(0)}}$ and $\lambda(\xi)$ of Eq. (46) respectively gives

$$\hat{p}(x,s)-\frac{p_0}{s}\doteqdot\sqrt{\frac{A_0}{A(x)}}\left[1+\left(\frac{1}{R}-\frac{1}{R_0}\right)\left(\frac{\nu}{s}\right)^{\frac{1}{2}}+\frac{1}{R}\left(\frac{1}{R}-\frac{1}{R_0}\right)\left(\frac{\nu}{s}\right)\right.$$
$$\left.+\left(\frac{\frac{7}{8}}{R^3}-\frac{1}{R^2R_0}+\frac{\frac{1}{8}}{R_0^3}\right)\left(\frac{\nu}{s}\right)^{\frac{3}{2}}+\left(\frac{\frac{1}{2}}{R^4}-\frac{\frac{7}{8}}{R^3R_0}+\frac{\frac{1}{8}}{RR_0^3}+\frac{\frac{1}{4}}{R_0^4}\right)\left(\frac{\nu}{s}\right)^2\right]\frac{\tilde{p}}{s}\{1-e^{-s\tau}\}$$
$$\exp\left[-\frac{s}{a}\left\{x+\int_0^x\frac{1}{R}dx\left(\frac{\nu}{s}\right)^{\frac{1}{2}}+\int_0^x\frac{1}{R^2}dx\left(\frac{\nu}{s}\right)+\frac{7}{8}\int_0^x\frac{1}{R^3}dx\left(\frac{\nu}{s}\right)^{\frac{3}{2}}\right.\right.$$
$$\left.\left.+\frac{1}{2}\int\frac{1}{R^4}dx\left(\frac{\nu}{s}\right)^2\right\}\right] \tag{47}$$

Inverse Laplace-transforming Eq. (47) with the aid of the relations from Eq. (48) through Eq. (50) gives the following solution (52).

RELATION 1.

$$\mathcal{L}^{-1}[e^{-\tau s}\hat{f}(s)]=\begin{cases}0 & 0<t<\tau\\ f(t-\tau) & \tau<t\end{cases} \tag{48}$$

RELATION 2.

$$\mathcal{L}^{-1}\left[\frac{1}{s}\hat{f}(s)\right]=\int_0^t f(t)dt \tag{49}$$

RELATION 3.

$$\mathcal{L}\left[(2\sqrt{t})^n i^n \operatorname{erfc}\left(\frac{\mu}{2\sqrt{t}}\right)\right]=\frac{1}{s}\left(\frac{1}{\sqrt{s}}\right)^n e^{-\mu\sqrt{s}}$$
$$n=0,1,2,\cdots \tag{50}$$

RELATION 4.

$$\operatorname{erfc}(t)=1-\operatorname{erf}(t)=\frac{2}{\sqrt{\pi}}\int_t^\infty e^{-y^2}dy \tag{51}$$

where $i^n \operatorname{erfc}(x)$ is the n-th repeated integral of the complementary error function.

$$\frac{p(x,t)-p_0}{\tilde{p}} \doteqdot \begin{cases} 0 & 0<t<\dfrac{x}{a} \\ \sqrt{\dfrac{A_0}{A(x)}}\, e^{-\frac{\nu}{a}\int_0^x \frac{1}{R^2}dx} F_1\left(t-\dfrac{x}{a}\right) & \dfrac{x}{a}<t<\dfrac{x}{a}+\tau \\ \sqrt{\dfrac{A_0}{A(x)}}\, e^{-\frac{\nu}{a}\int_0^x \frac{1}{R^2}dx}\left[F_1\left(t-\dfrac{x}{a}\right)-F_1\left(t-\dfrac{x}{a}-\tau\right)\right] & \dfrac{x}{a}+\tau<t \end{cases} \tag{52}$$

where

$$\begin{aligned} F_1(t) \doteqdot & \operatorname{erfc}\left[\frac{1}{2a}\sqrt{\frac{\nu}{t}}\int_0^x \frac{1}{R}\,dx\right]+2\sqrt{\nu t}\left\{\left(\frac{1}{R}-\frac{1}{R_0}\right)-\frac{7}{8}\left(\frac{\nu}{a}\right)\int_0^x \frac{1}{R^3}\,dx\right\} \\ & \operatorname{ierfc}\left[\frac{1}{2a}\sqrt{\frac{\nu}{t}}\int_0^x \frac{1}{R}\,dx\right]+4\nu t\left[\left(\frac{1}{R}-\frac{1}{R_0}\right)\left\{\frac{1}{R}-\frac{7}{8}\left(\frac{\nu}{a}\right)\int_0^x \frac{1}{R^3}\,dx\right\}\right. \\ & \left.-\frac{\nu}{2a}\left\{\int_0^x \frac{1}{R^4}\,dx-\frac{\nu}{a}\left(\frac{7}{8}\int_0^x \frac{1}{R^3}\,dx\right)^2\right\}\right] i^2\operatorname{erfc}\left[\frac{1}{2a}\sqrt{\frac{\nu}{t}}\int_0^x \frac{1}{R}\,dx\right] \end{aligned} \tag{53}$$

Some calculated results are shown in Fig. 2(a) and (b) for the values; $a=4.8\times10^3$ ft/sec, $\nu=1\times10^{-5}$ ft^2/sec, $L=10$ ft, $\tau=0.1\times L/a=2.08\times10^{-4}$sec, $R_0=0.3$ in, and $R(L)=2/3R_0$. Equation (52) shows that the output pulse for a viscous liquid is different

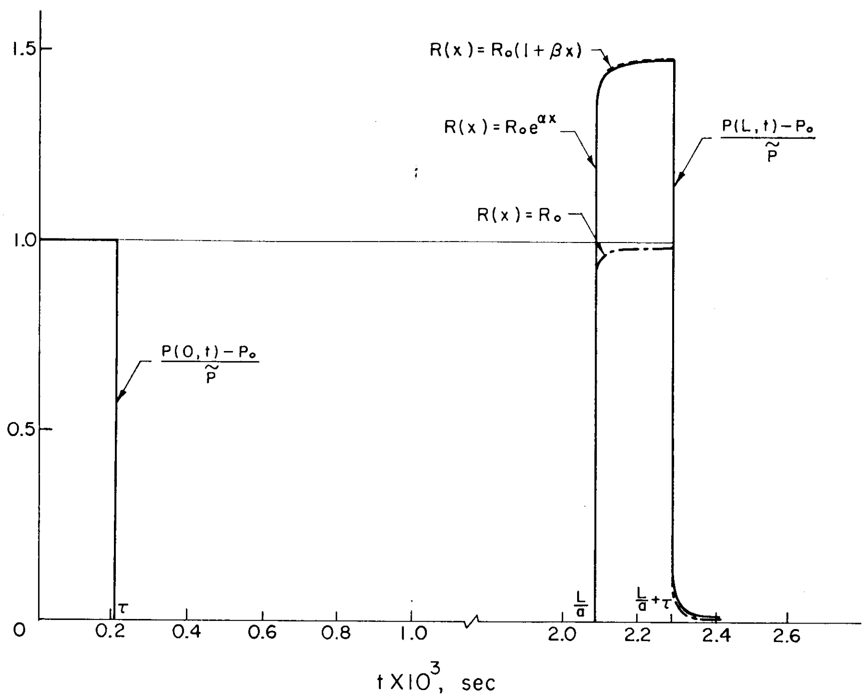

Fig. 2(a) Travelling pulses in a semi-infinite water-filled line

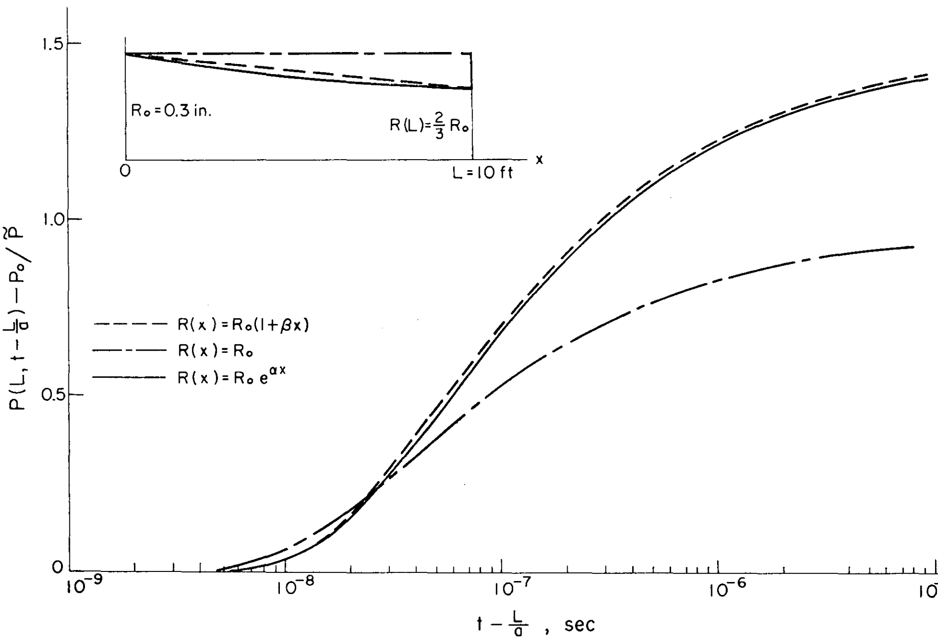

Fig. 2(b) Leading edges of output pressure pulses in water-filled line of Example 1

from one for an inviscid liquid in two ways. First, the term of $e^{-\frac{\nu}{a}\int_0^x \frac{1}{R^2}dx}$ makes the height of output pulse decrease slightly. Secondly, the function $F_1(t)$ makes the leading edge round, so $F_1\left(t-\frac{L}{a}\right)-F_1\left(t-\frac{L}{a}-\tau\right)$ does not become zero at $t=L/a+\tau$. Figure 2(b) shows the extended time scale in order to make standing up clear.

EXAMPLE 2. Travelling pulse and reflected pulse at the dead end

The boundary conditions for an input pulse with a dead end are in Laplace-transformed way

$$\hat{p}(0, s)-\frac{p_0}{s}=\frac{\tilde{p}}{s}(1-e^{-s\tau}) \tag{54}$$

$$\hat{q}(L, s)=0 \tag{55}$$

Substituting Eqs. (54) and (55) in Eqs. (37) and (38) respectively and specializing the unknown coefficients k_1 and k_2 give

$$\hat{p}(x, s)-\frac{p_0}{s}=\sqrt{\frac{Z(x)}{Z(0)}}\frac{\tilde{p}}{s}\{1-e^{-s\tau}\}\frac{\cosh\left(\int_x^L \lambda(\xi)d\xi\right)}{\cosh\left(\int_0^L \lambda(\xi)d\xi\right)} \tag{56}$$

The approximation of Eq. (56) for the first period of time is given by

$$\hat{p}(x,s)-\frac{p_0}{s}\doteqdot\sqrt{\frac{Z(x)}{Z(0)}}\frac{\tilde{p}}{s}\{1-e^{-s\tau}\}\left\{e^{-\int_0^x\lambda(\xi)d\xi}+e^{-2\int_0^L\lambda(\xi)d\xi+\int_0^x\lambda(\xi)d\xi}\right\}$$
$$=\sqrt{\frac{Z(x)}{Z(0)}}\frac{\tilde{p}}{s}\{1-e^{-s\tau}\}e^{-\int_0^x\lambda(\xi)d\xi}$$
$$+\sqrt{\frac{Z(x)}{Z(0)}}\frac{\tilde{p}}{s}\{1-e^{-s\tau}\}e^{-2\int_0^L\lambda(\xi)d\xi+\int_0^x\lambda(\xi)d\xi} \tag{57}$$

The first term of Eq. (57) is equal to Eq. (46), therefore all we have to do is to find the inverse Laplace-transformation of the second term of Eq. (57). Also the second term is equal to the first term for $x=L$.

$$\hat{p}(L,s)-\frac{p_0}{s}\doteqdot2\sqrt{\frac{Z(L)}{Z(0)}}\frac{\tilde{p}}{s}\{1-e^{-s\tau}\}e^{-\int_0^L\lambda(\xi)d\xi} \tag{58}$$

The inverse Laplace-transformation of Eq. (58) is easily found from Eq. (47). In this case the period is given by

$$T=4\left\{\frac{L}{a}+\left(\frac{\nu}{s}\right)^{\frac{1}{2}}\int_0^L\frac{dx}{R}+\left(\frac{\nu}{s}\right)\int_0^L\frac{dx}{R^2}+\frac{7}{8}\left(\frac{\nu}{s}\right)^{\frac{3}{2}}\int_0^L\frac{dx}{R^3}+\frac{1}{2}\left(\frac{\nu}{s}\right)^2\int_0^L\frac{dx}{R^4}+\cdots\right\} \tag{59}$$

This means the fact that the period is a function of viscosity, variation of radius and time. Obviously $4L/a$ is the well-known period for $\nu=0$.

EXAMPLE 3. Travelling pulse and reflected pulse at the open end.

The boundary conditions for an input pulse with an open end are in the Laplace-transformed way

$$\hat{p}(0,s)-\frac{p_0}{s}=\frac{\tilde{p}}{s}(1-e^{-s\tau}) \tag{60}$$

$$\hat{p}(L,s)-\frac{p_0}{s}=0 \tag{61}$$

Using the same way as in Example 2,

$$\hat{p}(x,s)-\frac{p_0}{s}=\sqrt{\frac{Z(x)}{Z(0)}}\frac{\tilde{p}}{s}\{1-e^{-s\tau}\}\frac{\sinh\left(\int_x^L\lambda(\xi)d\xi\right)}{\sinh\left(\int_0^L\lambda(\xi)d\xi\right)} \tag{62}$$

The approximation of Eq. (62) for the first period of time is given by

$$\hat{p}(x,s)-\frac{p_0}{s}\doteqdot\sqrt{\frac{Z(x)}{Z(0)}}\frac{\tilde{p}}{s}\{1-e^{-s\tau}\}\left\{e^{-\int_0^x\lambda(\xi)d\xi}-e^{-2\int_0^L\lambda(\xi)d\xi+\int_0^x\lambda(\xi)d\xi}\right\} \tag{63}$$

If we change the positive sign of the second term of Eq. (57) into a negative sign, Eq. (63) becomes equal to Eq. (57). This means that the reflected wave has a different sign between open and dead ends. Next let us consider the inverse Laplace-transformation of the second term of Eq. (57). Substituting Eqs. (39), (41) and (42) in this term yields

$$\sqrt{\frac{Z(x)}{Z(0)}}\frac{\tilde{p}}{s}\{1-e^{-s\tau}\}e^{-2\int_0^L\lambda(\xi)d\xi+\int_0^x\lambda(\xi)d\xi}$$

$$\doteqdot\sqrt{\frac{A_0}{A(x)}}\left[1+\left(\frac{1}{R}-\frac{1}{R_0}\right)\left(\frac{\nu}{s}\right)^{\frac{1}{2}}+\frac{1}{R}\left(\frac{1}{R}-\frac{1}{R_0}\right)\left(\frac{\nu}{s}\right)+\left(\frac{\frac{7}{8}}{R^3}-\frac{1}{R^2R_0}+\frac{\frac{1}{8}}{R_0^3}\right)\left(\frac{\nu}{8}\right)^{\frac{3}{2}}\right.$$

$$\left.+\left(\frac{\frac{1}{2}}{R^4}-\frac{\frac{7}{8}}{R^3R_0}+\frac{\frac{1}{8}}{RR_0^3}+\frac{\frac{1}{4}}{R_0^4}\right)\left(\frac{\nu}{s}\right)^2\right]\frac{\tilde{p}}{s}\{1-e^{-s\tau}\}\exp\left[-\frac{s}{a}\left\{(2L-x)\right.\right.$$

$$+\left(\frac{\nu}{s}\right)^{\frac{1}{2}}\left(2\int_0^L\frac{1}{R}dx-\int_0^x\frac{1}{R}dx\right)+\left(\frac{\nu}{s}\right)\left(2\int_0^L\frac{1}{R^2}dx-\int_0^x\frac{1}{R^2}dx\right)$$

$$\left.\left.+\frac{7}{8}\left(\frac{\nu}{s}\right)^{\frac{3}{2}}\left(2\int_0^L\frac{1}{R^3}dx-\int_0^x\frac{1}{R^3}dx\right)+\frac{1}{2}\left(\frac{\nu}{s}\right)^2\left(2\int_0^L\frac{1}{R^3}dx-\int_0^x\frac{1}{R^3}dx\right)\right\}\right] \tag{64}$$

Inverse Laplace-transforming Eq. (64) with the aid of the relations from Eq. (48) through Eq. (50) gives

$$\begin{cases} 0 & 0<t<\frac{2L-x}{a} \\ \sqrt{\frac{A_0}{A(x)}}e^{-\frac{\nu}{a}\left(2\int_0^L\frac{1}{R^2}dx-\int_0^x\frac{1}{R^2}dx\right)}F_2\left(t-\frac{2L-x}{a}\right) & \frac{2L-x}{a}<t<\frac{2L-x}{a}+\tau \\ \sqrt{\frac{A_0}{A(x)}}e^{-\frac{\nu}{a}\left(2\int_0^L\frac{1}{R^2}dx-\int_0^x\frac{1}{R^2}dx\right)}\left[F_2\left(t-\frac{2L-x}{a}\right)-F_2\left(t-\frac{2L-x}{a}-\tau\right)\right] & \\ & \frac{2L-x}{a}+\tau<t<\frac{2L+x}{a} \end{cases} \tag{65}$$

where

$$F_2(t)\doteqdot\operatorname{erfc}\left[\frac{1}{2a}\sqrt{\frac{\nu}{t}}\left(2\int_0^L\frac{1}{R}dx-\int_0^x\frac{1}{R}dx\right)\right]+2\sqrt{\nu t}\left\{\left(\frac{1}{R}-\frac{1}{R_0}\right)\right.$$

$$-\frac{7}{8}\left(\frac{\nu}{a}\right)\left(2\int_0^L\frac{1}{R^3}dx-\int_0^x\frac{1}{R^3}dx\right)\operatorname{ierfc}\left[\frac{1}{2a}\sqrt{\frac{\nu}{t}}\left(2\int_0^L\frac{1}{R}dx-\int_0^x\frac{1}{R}dx\right)\right]$$

$$+4\nu t\left[\left(\frac{1}{R}-\frac{1}{R_0}\right)\left\{\frac{1}{R}-\frac{7}{8}\left(\frac{\nu}{a}\right)\left(2\int_0^L\frac{1}{R^3}dx-\int^x\frac{1}{R^3}dx\right)\right\}\right.$$

$$\left.-\frac{\nu}{2a}\left\{\left(2\int_0^L\frac{1}{R^4}dx-\int_0^x\frac{1}{R^4}dx\right)-\frac{\nu}{a}\left(\frac{7}{8}\left(2\int_0^L\frac{1}{R^3}dx-\int_0^x\frac{1}{R^3}dx\right)\right)^2\right\}\right]$$

$$i^2\operatorname{erfc}\left[\frac{1}{2a}\sqrt{\frac{\nu}{t}}\left(2\int_0^L\frac{1}{R}dx-\int_0^x\frac{1}{R}dx\right)\right] \tag{66}$$

Hence the inverse Laplace-transformation of Eq. (63) is given by

$$\frac{p(x,t)-p_0}{\tilde{p}} \doteqdot \begin{cases} 0 & 0<t<\dfrac{x}{a} \\ \sqrt{\dfrac{A_0}{A(x)}}\, e^{-\frac{\nu}{a}\int_0^x \frac{1}{R^2}dx} F_1\!\left(t-\dfrac{x}{a}\right) & \dfrac{x}{a}<t<\dfrac{x}{a}+\tau \\ \sqrt{\dfrac{A_0}{A(x)}}\, e^{-\frac{\nu}{a}\int_0^x \frac{1}{R^2}dx}\left[F_1\!\left(t-\dfrac{x}{a}\right)-F_1\!\left(t-\dfrac{x}{a}-\tau\right)\right] & \dfrac{x}{a}+\tau<t<\dfrac{2L-x}{a} \\ \sqrt{\dfrac{A_0}{A(x)}}\left\{ e^{-\frac{\nu}{a}\int_0^x \frac{1}{R^2}dx}\left[F_1\!\left(t-\dfrac{x}{a}\right)-F_1\!\left(t-\dfrac{x}{a}-\tau\right)\right] \right. & \\ \qquad \left. -e^{-\frac{\nu}{a}\left(2\int_0^L \frac{1}{R^2}dx-\int_0^x \frac{1}{R^2}dx\right)} F_2\!\left(t-\dfrac{2L-x}{a}\right)\right\} & \dfrac{2L-x}{a}<t<\dfrac{2L-x}{a}+\tau \\ \sqrt{\dfrac{A_0}{A(x)}}\left\{ e^{-\frac{\nu}{a}\int_0^x \frac{1}{R^2}dx}\left[F_1\!\left(t-\dfrac{x}{a}\right)-F_1\!\left(t-\dfrac{x}{a}-\tau\right)\right] \right. & \\ \qquad \left. -e^{-\frac{\nu}{a}\left(2\int_0^L \frac{1}{R^2}dx-\int_0^x \frac{1}{R^2}dx\right)}\left[F_2\!\left(t-\dfrac{2L-x}{a}\right)-F_2\!\left(t-\dfrac{2L-x}{a}-\tau\right)\right]\right\} & \\ & \dfrac{2L-x}{a}+\tau<t<\dfrac{2L+x}{a} \end{cases} \tag{67}$$

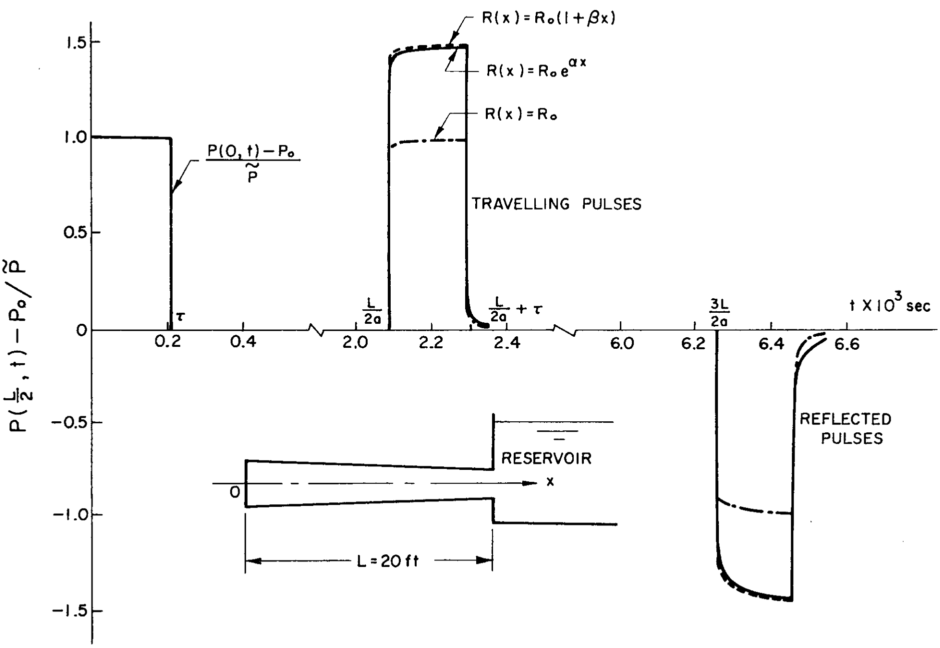

Fig. 3 Input and output pulses of pressure in water-filled line of Example 3 as determined from Eq. (67)

In Fig. 3 are shown some calculated results of Eq. (67) for the values; $\alpha=-0.04055$ 1/ft, $\beta=-1/30$ 1/ft, $L=20$ ft, $\tau=0.05\,L/a=2.08\times10^{-4}$ sec, $\nu=1\times10^{-5}$ ft^2/sec, $a=4.8\times10^3$ ft/sec, and $R_0=0.3$ in. The total length of the tube in Example 3 is twice as long as in Example 1 in order to make reflected pulses clear.

B. Dynamic response of volume flux

EXAMPLE 4. Travelling pulse for pulse input of volume flux

As the last example, let us consider the dynamic response of the volume flux in a semi-infinite tapered line. The boundary condition for a pulse of volume flux is in the Laplace-transformed way

$$\hat{q}(0, s)=\frac{\tilde{q}}{s}\{1-e^{-s\tau}\} \tag{68}$$

There is no reflected wave in this case, so $k_2=0$ in Eq. (38). Eliminating k_1 from Eq. (38) with the help of Eq. (68) yields

$$\hat{q}(x, s)=\sqrt{\frac{Y(x)}{Y(0)}}\frac{\tilde{q}}{s}\{1-e^{-s\tau}\}e^{-\int_0^x \lambda(\xi)d\xi} \tag{69}$$

Substituting Eqs. (27) and (39) in Eq. (46) gives

$$\hat{q}(x, s)\fallingdotseq\sqrt{\frac{A(x)}{A_0}}\frac{\tilde{q}}{s}\{1-e^{-s\tau}\}\exp\left[-\frac{s}{a}\left\{x+\left(\frac{\nu}{s}\right)^{\frac{1}{2}}\int_0^x\frac{1}{R}dx+\left(\frac{\nu}{s}\right)\int_0^x\frac{1}{R^2}dx+\frac{7}{8}\left(\frac{\nu}{s}\right)^{\frac{3}{2}}\int_0^x\frac{1}{R^3}dx+\frac{1}{2}\left(\frac{\nu}{s}\right)^2\int_0^x\frac{1}{R^4}dx\right\}\right] \tag{70}$$

Inverse Laplace-transforming Eq. (70) yields

$$\frac{q(x,t)}{\tilde{q}}\fallingdotseq\begin{cases}0 & 0<t<\dfrac{x}{a}\\ \sqrt{\dfrac{A(x)}{A_0}}\,e^{-\frac{\nu}{a}\int_0^x\frac{1}{R^2}dx}\,G_1\!\left(t-\dfrac{x}{a}\right) & \dfrac{x}{a}<t<\dfrac{x}{a}+\tau\\ \sqrt{\dfrac{A(x)}{A_0}}\,e^{-\frac{\nu}{a}\int_0^x\frac{1}{R^2}dx}\left[G_1\!\left(t-\dfrac{x}{a}\right)-G_1\!\left(t-\dfrac{x}{a}-\tau\right)\right] & \dfrac{x}{a}+\tau<t\end{cases} \tag{71}$$

where

$$G_1(t)\fallingdotseq\operatorname{erfc}\left[\frac{1}{2a}\sqrt{\frac{\nu}{t}}\int_0^x\frac{1}{R}dx\right]-\frac{7}{4}\frac{\nu^{\frac{3}{2}}\sqrt{t}}{a}\int_0^x\frac{1}{R^3}dx\,\operatorname{ierfc}\left[\frac{1}{2a}\sqrt{\frac{\nu}{t}}\int_0^x\frac{1}{R}dx\right]-\frac{2\nu^2t}{a}\left\{\int_0^x\frac{1}{R^4}dx-\frac{\nu}{a}\left(\frac{7}{8}\int_0^x\frac{1}{R^3}dx\right)^2\right\}\mathrm{i}^2\operatorname{erfc}\left[\frac{1}{2a}\sqrt{\frac{\nu}{t}}\int_0^x\frac{1}{R}dx\right] \tag{72}$$

In Fig. 4(a) and (b) are shown some calculated results for the values; $a=4.8\times10^3$ ft/sec, $\nu=1\times10^{-5}$ ft^2/sec, $L=10$ ft, $\tau=0.1\times L/2=2.08\times10^{-4}$ sec, $R_0=0.3$ in, and $\alpha=-0.04055$ 1/ft.

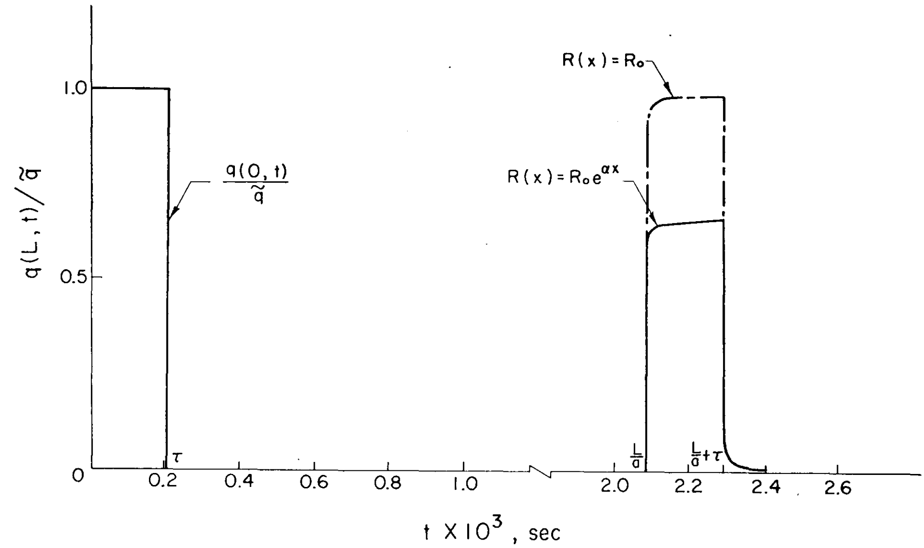

Fig. 4(a) Input and output pulses of volume flux in water-filled line of Example 4

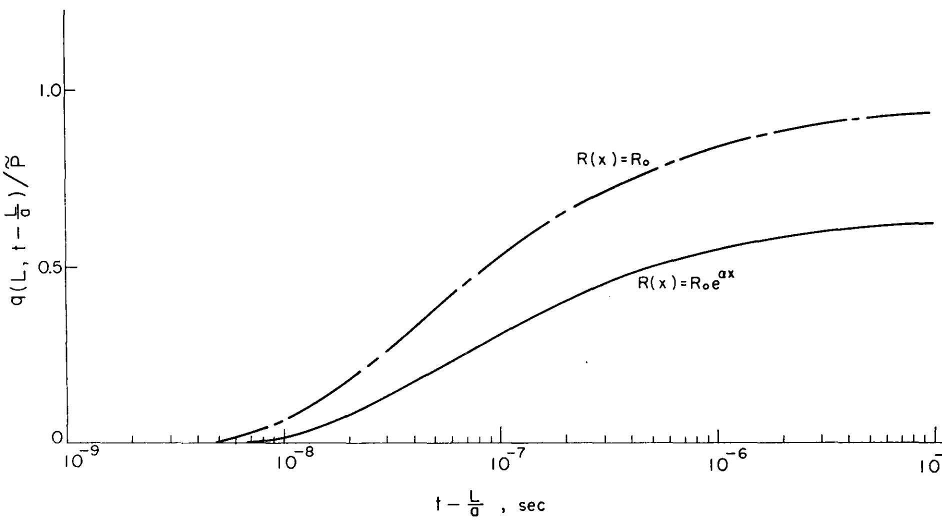

Fig. 4(b) Leading deges of output pulses of volume flux of Example 4

Conclusions

(1) Equation (34) is a fundamental equation and finding the inverse Laplace-transform of Eqs. (37) and (38) is sufficient for a viscous liquid initially at rest.

This new simple method may be applied to the wide range of the transmission line with a circular cross-sectional area as far as the assumptions are valid.

(2) Although distortion is not caused, as shown in Part I, in the tapered transmission line with the small variation of the cross-sectional area $\left(\frac{\tau}{T}\frac{L}{A}\frac{dA}{dx}\ll 1\right)$ filled with inviscid liquid, viscosity has a function of distorting the leading edge of the pulse in the line filled with viscous liquid as shown in Part II even in the constant radius pipe. As a result of this, a part of the pulse is left even after $t=x/a+\tau$ (see Fig. 2(a)).

(3) From Eqs. (52) and (71) we can deduce that

$$\frac{p(x,t)-p_0}{\tilde{p}}\sim\sqrt{\frac{A_0}{A(x)}}e^{-\frac{\nu}{a}\int_0^x\frac{1}{R^2}dx}$$

$$\frac{q(x,t)}{\tilde{q}}\sim\sqrt{\frac{A(x)}{A_0}}e^{-\frac{\nu}{a}\int_0^x\frac{1}{R^2}dx}$$

where

$$e^{-\frac{\nu}{a}\int_0^x\frac{1}{R^2}dx}\doteqdot 1 \quad \text{in general.}$$

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