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## PART I—INVISCID LIQUIDS

BY

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# FLUID PRESSURE TRANSIENTS IN A TAPERED TRANSMISSION LINE

## PART I INVISCID LIQUIDS

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**ABSTRACT**

In this paper are shown a general solution and a generalized transfer-function matrix to analyze dynamic responses of the pulse of pressure or volume flux in a small-diameter tapered transmission line and hydraulic transients for inviscid liquid initially not at rest.

## Nomenclature

$A(x)$, $A_0$ = cross-sectional area of tube
$a$ = wave velocity
$\boldsymbol{B}(x, s)$ = impedance-admittance matrix
$\boldsymbol{C}(x, s)$ = integrated impedance-admittance matrix
$\boldsymbol{E}$ = unit matrix
$\boldsymbol{F}(x, s)$ = semi-transfer-function matrix
$\boldsymbol{G}(x, x_1, s)$ = transfer-function matrix
$K$ = bulk compression modulus of liquid
$\boldsymbol{k}$ = constant vector
$L$ = total length of conduit
$p, p_0, \tilde{p}, \hat{p}$ = pressure
$q, q_0, \tilde{q}, \hat{q}$ = volume flux
$R(x)$, $R_0$ = radius of tube
$\boldsymbol{R}(x)$ = matrix defined by equation (25)
$r$ = radial coordinate
$s$ = Laplace variable

$\boldsymbol{T}(x, s)$ = matrix defined by equation (17)
$t$ = time
$u, u_0, \hat{u}$ = velocity
$\hat{\boldsymbol{u}} = \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix}$
$\boldsymbol{u}(x, 0) = \begin{pmatrix} p(x, 0) \\ q(x, 0) \end{pmatrix}$ = initial vector
$\hat{\boldsymbol{v}} = \boldsymbol{T}^{-1}\hat{\boldsymbol{u}}$
$\boldsymbol{w}$ = velocity vector
$x$ = axial coordinate
$Y(x, s)$, $Y_c$ = characteristic admittance
$Z(x, s)$, $Z_c$ = characteristic impedance
$\rho, \rho_0$ = density of liquid
$\lambda$ = eigenvalue
$\boldsymbol{\Lambda}(s)$ = diagonal matrix
$\tau$ = duration of pulse

## Subscripts

$0$ = value at $t=0$ or $x=0$
$-1$ = inverse matrix
$t$ = transposed matrix
$\wedge$ = Laplace-transformed value
$c$ = characteristic
$\sim$ = input pulse
$'$ = derivative with respect to $x$

## Introduction

The purpose of early investigation on unsteady fluid flow was to examine the surging phenomena so called water-hammer in pumping plants employing large-diameter conduits and to determine the velocity of wave propagation in the fluid.[9),10),11)] But more recently, unsteady flow in small-diameter conduits plays a major role in liquid-propellant rocket systems, hydraulic and pneumatic control systems, the circulation system of the blood and elsewhere. If not properly accounted for, the fluid transients of small-diameter transmission lines may cause combustion instability in rockets or unsatisfactory performance elsewhere. The first analytically detailed paper concerned with these problems was issued by D'SOUZA, A. F. and OLDENBURGER, R.[3)] who showed the formula for Dynamic Response of fluid lines valid for an infinite or long line. Next, BROWN, F. T. and NELSON, S. E.[1)] calculated pressure (flow) responses to step of flow (pressure) for semi-infinite liquid lines with frequency-dependent effects of viscosity respectively.

While these methods are applicable to uniform pipes, ROULEAU, W.T. and YOUNG, F. J.[5),6)] applied similar techniques to non-uniform pipes, in which pressure pulses become distorted, dependent upon the geometry of the pipe and the viscous shear forces. TARANTINE, F. J. and ROULEAU, W. T.[7),8)] presented and illustrated a method for the determination of the output for inviscid-liquid-filled tapered tubes terminated by a very long uniform line. However, this method gave no information concerning the distortion following the arrival of the wave front.

Experiments, which showed the strong distortion of waves traveling through uniform fluid line, were made by HOLMBOE, F. L. and ROULEAU, W. T.[4)].

The above-mentioned methods for the linear transfer techniques had never taken into account so-called convective terms, which are non-linearities in the basic differential equations. Recently, ZIELKE, W.[13)] has shown that the method of characteristics can be adapted to handle frequency-dependent wall shear, taking convective terms into consideration. And this quasi method of characteristics with application to fluid line with frequency-dependent wall shear has been developed by BROWN, F. T.[2)]. In most of these recent papers, the initial condition has been carried as fluid is at rest. Because that, when initial fluid moves in steady state, initial discharge in tapered pipes and pressure in viscous liquid even in constant pipes are not constant or a function of location. This has made the problems difficult. In this paper is shown one generalization of analytical solutions for tapered transmission systems filled with liquid initially not at rest.

This new method significantly extends the scope of the analysis of the tapered transmission lines.

## Basic Equations

The unsteady flow of a slightly compressible inviscid liquid is expressed by the following equations:

Momentum equations

$$\rho\frac{\partial \boldsymbol{w}}{\partial t}+\rho(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}=-\nabla p \tag{1}$$

Continuity equation

$$\nabla\cdot(\rho\boldsymbol{w})+\frac{\partial\rho}{\partial t}=0 \tag{2}$$

State equation

$$\frac{dp}{d\rho}=\frac{K}{\rho} \tag{3}$$

where $\boldsymbol{w}$ is the fluid velocity, $\rho$ is density, $t$ is time, $p$ is pressure, and $K$ is the isothermal bulk compression modulus. The following assumptions will be made:

(i) Inviscid fluid.
(ii) Variations in density are small.
(iii) One-dimensional (plane) flow.
(iv) Convective acceleration negligible compared to the local acceleration.

We shall deal only with tubes that are circular in cross section so that the flow will be axisymmetric. With these assumptions the axial component of the momentum equations (1) becomes

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{4}$$

where $x$ is the axial coordinate, $\rho_0$ is the mean density, and $u$ is the fluid mean velocity. Since one-dimensional (plane) flow has been assumed, $p$ and $u$ are functions only of $x$ and $t$.

The one-dimensional continuity equation may be written from Eq. (2) as follows,

$$\frac{\partial \rho}{\partial t}+\rho_0\frac{\partial u}{\partial x}=0 \tag{5}$$

For a liquid at steady state in a tapered line, the initial conditions may be stated as

$$u(x,0)=\frac{A(0)}{A(x)}u(0,0)=\frac{A_0}{A(x)}u_0 \tag{6}$$

$$p(x,0)=p_0+\frac{1}{2}\rho_0u_0^2\left\{1-\left(\frac{A_0}{A(x)}\right)^2\right\} \tag{7}$$

where $A_0$, $u_0$, and $p_0$ are now defined as the area, mean velocity, and pressure at $x=0$ and $t=0$ respectively.

Laplace-transforming Eq. (4) and using the initial conditions (6) gives

$$s\hat{u}-u(x,0)=-\frac{1}{\rho_0}\frac{d\hat{p}}{dx} \tag{8}$$

$$\therefore\quad \hat{u}=\frac{1}{s}\left\{\frac{A_0}{A(x)}u_0-\frac{1}{\rho_0}\frac{d\hat{p}}{dx}\right\} \tag{9}$$

where $\hat{u}$ and $\hat{p}$ are the Laplace transforms of $u(x,t)$ and $p(x,t)$, respectively.

The volume flux $q$ is given by

$$q(x,t)=\int_0^{R(x)}2\pi ru(x,t)dr=A(x)u(x,t) \tag{10}$$

$$\therefore\quad \hat{q}(x,s)=A(x)\hat{u}(x,s),\qquad q_0=A_0u_0 \tag{11}$$

Eliminating $\hat{u}$ from Eq. (9) gives

$$\hat{q}=\frac{1}{s}\left\{q_0-\frac{A(x)}{\rho_0}\frac{d\hat{p}}{dx}\right\} \tag{12}$$

Equation (3) may be used to eliminate density variations in Eq. (5), which is then averaged across the pipe to give

$$\frac{1}{\rho_0}\frac{\partial\rho}{\partial t}=\frac{1}{K}\frac{\partial p}{\partial t}=-\frac{\partial u}{\partial x}$$

so that

$$\frac{\partial p}{\partial t}=-\frac{K}{A(x)}\frac{\partial q}{\partial x} \tag{13}$$

Laplace transform of Eq. (13), using the initial condition (7), gives

$$s\hat{p}-p(x,0)=-\frac{K}{A(x)}\frac{d\hat{q}}{dx} \tag{14}$$

or

$$\hat{p}=\frac{1}{s}\left\{p_0+\frac{1}{2}\rho_0 u_0^2\left[1-\left(\frac{A_0}{A(x)}\right)^2\right]-\frac{K}{A(x)}\frac{d\hat{q}}{dx}\right\} \tag{15}$$

Equations (12) and (14) can be written in the form

$$\begin{pmatrix}\dfrac{d\hat{p}}{dx}\\[2ex] \dfrac{d\hat{q}}{dx}\end{pmatrix}=-\begin{pmatrix}0 & Z\\ Y & 0\end{pmatrix}\begin{pmatrix}\hat{p}\\ \hat{q}\end{pmatrix}+\begin{pmatrix}0 & \dfrac{Z}{s}\\[2ex] \dfrac{Y}{s} & 0\end{pmatrix}\begin{pmatrix}p(x,0)\\ q(x,0)\end{pmatrix}$$

and rewritten in the vector-matrix form

$$\frac{d\hat{\boldsymbol{u}}(x,s)}{dx}=-\boldsymbol{B}(x,s)\hat{\boldsymbol{u}}(x,s)+\boldsymbol{C}(x,s)\boldsymbol{u}(x,0) \tag{16}$$

where

$$\hat{\boldsymbol{u}}=\begin{pmatrix}\hat{p}\\ \hat{q}\end{pmatrix},\quad \frac{d\boldsymbol{u}}{dx}=\begin{pmatrix}\dfrac{d\hat{p}}{dx}\\[2ex] \dfrac{d\hat{q}}{dx}\end{pmatrix},\quad \boldsymbol{B}(x,s)=\begin{pmatrix}0 & Z\\ Y & 0\end{pmatrix},$$

$$\boldsymbol{C}(x,s)=\begin{pmatrix}0 & \dfrac{Z}{s}\\[2ex] \dfrac{Y}{s} & 0\end{pmatrix},\quad \boldsymbol{u}(x,0)=\begin{pmatrix}p(x,0)\\ q(x,0)\end{pmatrix},\quad Z=\frac{s\rho_0}{A(x)},\quad Y=\frac{A(x)s}{K}$$

here $Z$ and $Y$ are called the characteristic impedance and the characteristic admittance. For fluid initially at rest, it is merely necessary to let $u_0=0$, i.e., $q(x,0)=0$ and $p(x,0)=p_0=$constant. So that, if $\hat{p}(x,s)$ is regarded as $\hat{p}(x,s)-\dfrac{p_0}{s}$ we have $\boldsymbol{u}(x,0)=\boldsymbol{0}$. Hence the simple form of Eq. (16) is obtained as

$$\frac{d\hat{\boldsymbol{u}}(x,s)}{dx}=-\boldsymbol{B}(x,s)\hat{\boldsymbol{u}}(x,s) \tag{16$'$}$$

## Derivation of Solution in Transformed Space

The general matrix is diagonalizable by operating some non-singular matrix. The impedance-admittance matrix $\boldsymbol{B}$, together with the relations, $a=\sqrt{\dfrac{K}{\rho_0}}$, $\lambda=\sqrt{ZY}=\dfrac{s}{a}=\text{constant}$

$$\boldsymbol{T}=\begin{pmatrix}\sqrt{Z} & -\sqrt{Z}\\ \sqrt{Y} & \sqrt{Y}\end{pmatrix}\qquad \boldsymbol{T}^{-1}=\frac{1}{2}\begin{pmatrix}\dfrac{1}{\sqrt{Z}} & \dfrac{1}{\sqrt{Y}}\\ -\dfrac{1}{\sqrt{Z}} & \dfrac{1}{\sqrt{Y}}\end{pmatrix}\qquad \boldsymbol{\Lambda}=\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}$$

can be placed in the form

$$\boldsymbol{T}^{-1}\boldsymbol{B}\boldsymbol{T}=\boldsymbol{\Lambda} \tag{17}$$

where $\lambda$ is the eigenvalue of the impedance-admittance matrix $\boldsymbol{B}$ and a is the velocity of wave propagation.

If the vector $\hat{\boldsymbol{v}}$ is transformed by the non-singular matrix $\boldsymbol{T}$ and becomes the vector $\hat{\boldsymbol{u}}$, i.e., $\hat{\boldsymbol{u}}=\boldsymbol{T}\hat{\boldsymbol{v}}$, Eq. (16), using $\dfrac{d\hat{\boldsymbol{u}}}{dx}=\boldsymbol{T}\dfrac{d\hat{\boldsymbol{v}}}{dx}+\dfrac{d\boldsymbol{T}}{dx}\hat{\boldsymbol{v}}$, can be rewritten in the following form;

$$\frac{d\hat{\boldsymbol{v}}}{dx}=-\boldsymbol{\Lambda}\hat{\boldsymbol{v}}+\boldsymbol{T}^{-1}\boldsymbol{C}\boldsymbol{u}(x,0)-\boldsymbol{T}^{-1}\frac{d\boldsymbol{T}}{dx}\hat{\boldsymbol{v}} \tag{18}$$

Then we examine the third term on the right-hand side of Eq. (18). This term can be written in the form

$$\begin{aligned}\boldsymbol{T}^{-1}\frac{d\boldsymbol{T}}{dx}&=\frac{1}{2}\begin{pmatrix}\dfrac{1}{\sqrt{Z}} & \dfrac{1}{\sqrt{Y}}\\ -\dfrac{1}{\sqrt{Z}} & \dfrac{1}{\sqrt{Y}}\end{pmatrix}\begin{pmatrix}\dfrac{1}{2\sqrt{Z}}\dfrac{dZ}{dx} & -\dfrac{1}{2\sqrt{Z}}\dfrac{dZ}{dx}\\ \dfrac{1}{2\sqrt{Y}}\dfrac{dY}{dx} & \dfrac{1}{2\sqrt{Y}}\dfrac{dY}{dx}\end{pmatrix}\\ &=\frac{1}{4}\begin{pmatrix}\dfrac{Y'}{Y}+\dfrac{Z'}{Z} & \dfrac{Y'}{Y}-\dfrac{Z'}{Z}\\ \dfrac{Y'}{Y}-\dfrac{Z'}{Z} & \dfrac{Y'}{Y}+\dfrac{Z'}{Z}\end{pmatrix}\end{aligned} \tag{19}$$

where prime means derivative with respect to $x$. Substituting the definitions of $Y$ and $Z$ into Eq. (19), we obtain

$$\boldsymbol{T}^{-1}\frac{d\boldsymbol{T}}{dx}=\frac{1}{2A(x)}\frac{dA(x)}{dx}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{20}$$

One-dimensional flow of the assumption (iii)[12] restricts the analysis to tapered lines in which the changes in the lateral dimension are small compared to the corresponding changes in length. This implies that

$$\frac{\left\| \boldsymbol{T}^{-1}\dfrac{d\boldsymbol{T}}{dx} \right\|}{||\boldsymbol{\Lambda}||}=\frac{\tau}{T}\frac{L}{A(x)}\frac{dA(x)}{dx}\ll 1 \tag{21}$$

Hence neglecting this term in Eq. (18), we have

$$\frac{d\hat{\boldsymbol{v}}}{dx}=-\boldsymbol{\Lambda}\hat{\boldsymbol{v}}+\boldsymbol{T}^{-1}\boldsymbol{C}\boldsymbol{u}(x,0) \tag{22}$$

The adjoint differential equation of Eq. (22) is

$$\frac{d\hat{\boldsymbol{r}}}{dx}={}^{t}\boldsymbol{\Lambda}\hat{\boldsymbol{r}}=\boldsymbol{\Lambda}\hat{\boldsymbol{r}} \tag{23}$$

where $t$ means the transposed matrix and in this case ${}^{t}\boldsymbol{\Lambda}=\boldsymbol{\Lambda}$ since $\boldsymbol{\Lambda}$ is diagonal. This equation is equivalent to the following matrix equation:

$$\frac{d\boldsymbol{R}}{dx}=\boldsymbol{\Lambda}^{t}\boldsymbol{R},\qquad \boldsymbol{R}(0)=\boldsymbol{E} \tag{24}$$

where $\boldsymbol{E}$ is the unit matrix of the second order and $\boldsymbol{R}$ is the characteristic matrix, i.e.,

$$\boldsymbol{R}(x)=\boldsymbol{E}+\int_0^x\boldsymbol{\Lambda}(\xi_1)d\xi_1+\int_0^x\int_0^{\xi_1}\boldsymbol{\Lambda}(\xi_1)\boldsymbol{\Lambda}(\xi_2)d\xi_1d\xi_2+\cdots \tag{25}$$

Since in general

$$\frac{d}{dx}(\boldsymbol{R}^{t}\hat{\boldsymbol{v}})=\frac{d\boldsymbol{R}^{t}}{dx}\hat{\boldsymbol{v}}+\boldsymbol{R}^{t}\frac{d\hat{\boldsymbol{v}}}{dx} \tag{26}$$

substituting Eqs. (24) and (22) into Eq. (26) gives

$$\frac{d}{dx}(\boldsymbol{R}^{t}\hat{\boldsymbol{v}})=(\boldsymbol{\Lambda}^{t}\boldsymbol{R})^{t}\hat{\boldsymbol{v}}+\boldsymbol{R}^{t}\{-\boldsymbol{\Lambda}\hat{\boldsymbol{v}}+\boldsymbol{T}^{-1}\boldsymbol{C}\boldsymbol{u}(x,0)\}=\boldsymbol{R}^{t}\boldsymbol{T}^{-1}\boldsymbol{C}\boldsymbol{u}(x,0) \tag{27}$$

Integrating Eq. (27), we have

$$\hat{\boldsymbol{v}}=\boldsymbol{R}^{-t}\left\{\int^x\boldsymbol{R}^{t}\boldsymbol{T}^{-1}\boldsymbol{C}\boldsymbol{u}(\xi,0)d\xi+\boldsymbol{k}\right\} \tag{28}$$

where $\boldsymbol{k}$ is an arbitrary integration constant vector and determined by the boundary conditions. Therefore we obtain as a general solution

$$\hat{\boldsymbol{u}}=\boldsymbol{T}\hat{\boldsymbol{v}}=\boldsymbol{T}\boldsymbol{R}^{-t}\left\{\int^x\boldsymbol{R}^{t}\boldsymbol{T}^{-1}\boldsymbol{C}\boldsymbol{u}(\xi,0)d\xi+\boldsymbol{k}\right\} \tag{29}$$

where $\boldsymbol{R}$ is calculated from Eq. (25), namely

$$\boldsymbol{R}=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\int_0^x\begin{pmatrix}\lambda&0\\0&-\lambda\end{pmatrix}d\xi_1+\int_0^x\int_0^{\xi_1}\begin{pmatrix}\lambda&0\\0&-\lambda\end{pmatrix}\begin{pmatrix}\lambda&0\\0&-\lambda\end{pmatrix}d\xi_1d\xi_2+\cdots$$

$$=\begin{pmatrix}1+\lambda x+\dfrac{\lambda^2}{2!}x^2+\cdots & 0\\ 0 & 1-\lambda x+\dfrac{\lambda^2}{2!}x^2-\cdots\end{pmatrix}=\begin{pmatrix}e^{\lambda x}&0\\0&e^{-\lambda x}\end{pmatrix}=e^{\int_0^x\Lambda d\xi}$$

so we have

$$\boldsymbol{R}^{-t}=\begin{pmatrix} e^{-\lambda x} & 0 \\ 0 & e^{\lambda x} \end{pmatrix}=e^{-\int_0^x \Lambda d\xi}$$

And also the solution of the fluid initially at rest is given by

$$\begin{aligned}\hat{\boldsymbol{u}}&=\boldsymbol{T}\boldsymbol{R}^{-t}\boldsymbol{k}\\&=\boldsymbol{T}\boldsymbol{R}^{-1}\boldsymbol{k}\\&=\boldsymbol{F}^{-1}\boldsymbol{k}\end{aligned}\tag{30}$$

where we term $\boldsymbol{F}=\boldsymbol{R}\boldsymbol{T}^{-1}$ the semi-transfer-function matrix,

$$\begin{aligned}\boldsymbol{F}(x, s)&=\frac{1}{2}\begin{pmatrix} e^{\lambda x} & 0 \\ 0 & e^{-\lambda x} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{Z}} & \frac{1}{\sqrt{Y}} \\ -\frac{1}{\sqrt{Z}} & \frac{1}{\sqrt{Y}} \end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix} \frac{e^{\lambda x}}{\sqrt{Z}} & \frac{e^{\lambda x}}{\sqrt{Y}} \\ -\frac{e^{-\lambda x}}{\sqrt{Z}} & \frac{e^{-\lambda x}}{\sqrt{Y}} \end{pmatrix}\end{aligned}$$

**Transfer-function Matrix**

From Eq. (22), we have for static fluid

$$\frac{d\hat{\boldsymbol{v}}}{dx}=-\boldsymbol{\Lambda}\hat{\boldsymbol{v}}\tag{31}$$

Integrating Eq. (31) with respect to $x$ from $x_1$ on the boundary condition $\hat{\boldsymbol{v}}(x_1, s)$ gives

$$\hat{\boldsymbol{v}}(x, s)=e^{-\int_{x_1}^x \Lambda d\xi}\,\hat{\boldsymbol{v}}(x_1, s)\tag{32}$$

Multiplying Eq. (32) by $\boldsymbol{T}$ yields

$$\boldsymbol{T}(x, s)\hat{\boldsymbol{v}}(x, s)=\boldsymbol{T}(x, s)e^{-\int_{x_1}^x \Lambda d\xi}\,\hat{\boldsymbol{v}}(x_1, s)\tag{33}$$

If noting $\hat{\boldsymbol{u}}(x, s)=\boldsymbol{T}(x, s)\hat{\boldsymbol{v}}(x, s)$ and $\boldsymbol{R}(x, s)=e^{\int_0^x \Lambda d\xi}$, we have, since $\boldsymbol{\Lambda}$ is diagonal,

$$\begin{aligned}\hat{\boldsymbol{u}}(x, s)&=\boldsymbol{T}(x, s)\boldsymbol{R}^{-1}(x, s)\boldsymbol{R}(x_1, s)\boldsymbol{T}^{-1}(x_1, s)\hat{\boldsymbol{u}}(x_1, s)\\&=\boldsymbol{F}^{-1}(x, s)\boldsymbol{F}(x_1, s)\hat{\boldsymbol{u}}(x_1, s)\end{aligned}$$

$$\therefore\quad \boldsymbol{F}(x, s)\hat{\boldsymbol{u}}(x, s)=\boldsymbol{F}(x_1, s)\hat{\boldsymbol{u}}(x_1, s)$$
$$=\boldsymbol{F}(0, s)\hat{\boldsymbol{u}}(0, s)$$

As a result,

$$\hat{\boldsymbol{u}}(x, s)=\boldsymbol{G}(x, x_1, s)\hat{\boldsymbol{u}}(x_1, s) \tag{34}$$

where $\boldsymbol{G}(x, x_1, s)=\boldsymbol{F}^{-1}(x, s)\boldsymbol{F}(x_1, s)$ is usually called the transfer-function matrix.

$$\boldsymbol{G}(x, x_1, s)=\boldsymbol{T}(x, s)\boldsymbol{R}^{-1}(x, s)\boldsymbol{R}(x_1, s)\boldsymbol{T}^{-1}(x_1, s) \tag{35}$$

$$=\frac{1}{2}\begin{pmatrix}\sqrt{Z} & -\sqrt{Z}\\ \sqrt{Y} & \sqrt{Y}\end{pmatrix}\begin{pmatrix}e^{-\int_0^x \lambda dx} & 0\\ 0 & e^{\int_0^x \lambda dx}\end{pmatrix}\begin{pmatrix}e^{\int_0^{x_1}\lambda dx} & 0\\ 0 & e^{-\int_0^{x_1}\lambda dx}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{Z_1}} & \frac{1}{\sqrt{Y_1}}\\ -\frac{1}{\sqrt{Z_1}} & \frac{1}{\sqrt{Y_1}}\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt{\frac{Z}{Z_1}}\cosh\left(\int_{x_1}^x \lambda d\xi\right) & -\sqrt{\frac{Z}{Y_1}}\sinh\left(\int_{x_1}^x \lambda d\xi\right)\\ -\sqrt{\frac{Y}{Z_1}}\sinh\left(\int_{x_1}^x \lambda d\xi\right) & \sqrt{\frac{Y}{Y_1}}\cosh\left(\int_{x_1}^x \lambda d\xi\right)\end{pmatrix} \tag{36}$$

where $Z_1=Z(x_1, s)$ and $Y_1=Y(x_1, s)$.

From Eq. (35) we have the following formula;

$$\boldsymbol{G}^{-1}(x, x_1, s)=\boldsymbol{G}(x_1, x, s) \tag{37}$$

As examples, for inviscid liquid, $\lambda=\frac{s}{a}$,

$$\therefore\quad \boldsymbol{G}(x, x_1, s)=\begin{pmatrix}\sqrt{\frac{Z}{Z_1}}\cosh\frac{s}{a}(x-x_1) & -\sqrt{\frac{Z}{Y_1}}\sinh\frac{s}{a}(x-x_1)\\ -\sqrt{\frac{Y}{Z_1}}\sinh\frac{s}{a}(x-x_1) & \sqrt{\frac{Y}{Y_1}}\cosh\frac{s}{a}(x-x_1)\end{pmatrix} \tag{38}$$

and for inviscid liquid and the constant radius pipe, $\lambda=\frac{s}{a}$,

$$\left.\begin{aligned}Z(x, s)=Z(x_1, s)=Z(0, s)=Z_c(s)\\ Y(x, s)=Y(x_1, s)=Y(0, s)=Y_c(s)\end{aligned}\right\} \tag{39}$$

we obtain

$$\boldsymbol{G}(x, x_1, s)=\begin{pmatrix}\cosh\frac{s}{a}(x-x_1) & -\sqrt{\frac{Z_c}{Y_c}}\sinh\frac{s}{a}(x-x_1)\\ -\sqrt{\frac{Y_c}{Z_c}}\sinh\frac{s}{a}(x-x_1) & \cosh\frac{s}{a}(x-x_1)\end{pmatrix} \tag{40}$$

and also, we have for $x_1=0$ at the inlet

$$G(x, 0, s)=\begin{pmatrix} \cosh\frac{s}{a}x & -\sqrt{\frac{Z_c}{Y_c}}\sinh\frac{s}{a}x \\ -\sqrt{\frac{Y_c}{Z_c}}\sinh\frac{s}{a}x & \cosh\frac{s}{a}x \end{pmatrix} \tag{41}$$

In this case the inverse matrix will become

$$\begin{aligned} G^{-1}(x, 0, s) &= G(0, x, s) \\ &= \begin{pmatrix} \cosh\frac{s}{a}x & \sqrt{\frac{Z_c}{Y_c}}\sinh\frac{s}{a}x \\ \sqrt{\frac{Y_c}{Z_c}}\sinh\frac{s}{a}x & \cosh\frac{s}{a}x \end{pmatrix} \end{aligned} \tag{42}$$

Eq. (42) is equal to the transfer matrix which was given by ICHIKAWA, T. and YAMAGUCHI, U..

## Inverse Laplace-Transformation

We now consider dynamic response for static fluid and water-hammer initially not at rest.

### A. Dynamic response of pressure pulse

EXAMPLE 1: Travelling wave for pressure pulse input.

In a semi-infinite tube exists only travelling wave. From Eq. (30),

$$\begin{aligned} \hat{u} &= F^{-1}k = TR^{-1}k \\ &= \begin{pmatrix} \sqrt{Z} & -\sqrt{Z} \\ \sqrt{Y} & \sqrt{Y} \end{pmatrix}\begin{pmatrix} e^{-\lambda x} & 0 \\ 0 & e^{\lambda x} \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\ &= \begin{pmatrix} k_1\sqrt{Z}\,e^{-\lambda x}-k_2\sqrt{Z}\,e^{\lambda x} \\ k_1\sqrt{Y}\,e^{-\lambda x}+k_2\sqrt{Y}\,e^{\lambda x} \end{pmatrix} \end{aligned} \tag{43}$$

Here $k_2=0$ because of no existence of reflected wave. Therefore, we obtain the following equations:

$$\left.\begin{aligned} \hat{p}(x, s)-\frac{p_0}{s} &= k_1\sqrt{\frac{s\rho_0}{A(x)}}\,e^{-\frac{s}{a}x} \\ \hat{q}(x, s) &= k_1\sqrt{\frac{A(x)s}{K}}\,e^{-\frac{s}{a}x} \end{aligned}\right\} \tag{44}$$

The boundary condition for pressure pulse input is

$$p(0,t)-p_0=\begin{cases}0 & t<0\\ \tilde{p} & 0<t<\tau\\ 0 & \tau<t\end{cases} \tag{45}$$

where $\tilde{p}$ is the magnitude of pressure pulse. Laplace-transforming Eq. (45) and using Eq. (44) gives

$$\hat{p}(0,s)-\frac{p_0}{s}=(1-e^{-s\tau})\frac{\tilde{p}}{s}=k_1\sqrt{\frac{s\rho_0}{A_0}} \tag{46}$$

$$\therefore\quad k_1=\sqrt{\frac{A_0}{s\rho_0}}(1-e^{-s\tau})\frac{\tilde{p}}{s} \tag{47}$$

Eliminating $k_1$ from Eq. (44) and dividing it by $\tilde{p}$,

$$\frac{\hat{p}(x,s)-\dfrac{p_0}{s}}{\tilde{p}}=\frac{1}{s}\sqrt{\frac{A_0}{A(x)}}\left\{e^{-\frac{x}{a}s}-e^{-\left(\frac{x}{a}+\tau\right)s}\right\} \tag{48}$$

Inverse Laplace-transforming Eq. (48) gives

$$\frac{p(x,t)-p_0}{\tilde{p}}=\begin{cases}0 & 0<t<\dfrac{x}{a}\\ \sqrt{\dfrac{A_0}{A(x)}} & \dfrac{x}{a}<t<\dfrac{x}{a}<\tau\\ 0 & \dfrac{x}{a}+\tau<t\end{cases} \tag{49}$$

This is illustrated in Fig. 2.

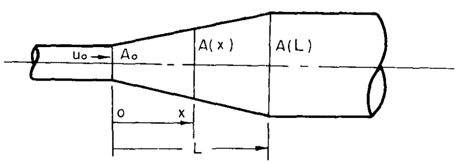

Fig. 1 Tapered transmission line

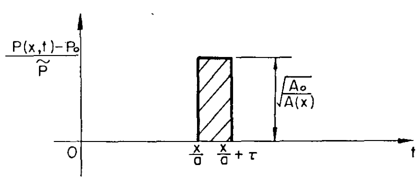

Fig. 2 Travelling pulse in a semi-infinite tube

Figure 2 shows that the height of output pulse is inversely proportional to the square root of the ratio of $A(x)$ to $A_0$ in a tapered transmission line. Here

a. $\sqrt{\dfrac{A_0}{A(x)}}=1$ for $R(x)=R_0$ constant radius

b. $\sqrt{\dfrac{A_0}{A(x)}}=e^{-\alpha x}$ $R(x)=R_0 e^{\alpha x}$ exponential radius

c. $\sqrt{\dfrac{A_0}{A(x)}}=(1+\beta x)^{-1}$ $R(x)=R_0(1+\beta x)$ linear radius

EXAMPLE 2: Travelling wave and reflected wave at the closed end.

From Eq. (43),

$$\hat{p}(x, s)-\frac{p_0}{s}=k_1\sqrt{\frac{s\rho_0}{A(x)}}e^{-\frac{s}{a}x}-k_2\sqrt{\frac{s\rho_0}{A(x)}}e^{\frac{s}{a}x} \tag{50}$$

$$\hat{q}(x, s)=k_1\sqrt{\frac{A(x)s}{K}}e^{-\frac{s}{a}x}+k_2\sqrt{\frac{A(x)s}{K}}e^{\frac{s}{a}x} \tag{51}$$

The boundary conditions with a dead end (Fig. 3) are

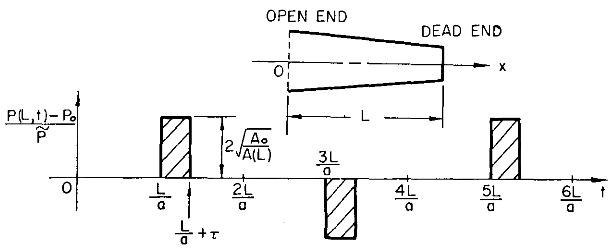


Fig. 3 Travelling and reflected pulse at the dead end

$$p(0, t)-p_0=\begin{cases} 0 & t<0 \\ \tilde{p} & 0<t<\tau \quad \left(\tau \ll \dfrac{L}{a}\right) \\ 0 & \tau<t \end{cases} \tag{52}$$

$$q(L, t)=0 \tag{53}$$

Laplace-transforming Eqs. (52) and (53) gives

$$\hat{p}(0, s)-\frac{p_0}{s}=(1-e^{-s\tau})\frac{\tilde{p}}{s} \tag{54}$$

$$\hat{q}(L, s)=0 \tag{55}$$

Using these boundary conditions, we uniquely specify two arbitrary constants of Eqs. (50) and (51):

$$k_1=\sqrt{\frac{A_0}{s\rho_0}}\frac{(1-e^{-s\tau})}{(1+e^{-2\frac{L}{a}s})}\frac{\tilde{p}}{s} \tag{56}$$

$$k_2=-\sqrt{\frac{A_0}{s\rho_0}}\frac{(1-e^{-s\tau})}{(1+e^{-2\frac{L}{a}s})}e^{-2\frac{L}{a}s}\frac{\tilde{p}}{s} \tag{57}$$

Eliminating $k_1$ and $k_2$ from Eq. (50), we have

$$\hat{p}(x,s)-\frac{p_0}{s}=\sqrt{\frac{A_0}{A(x)}}\frac{\cosh\left(\frac{L}{a}-\frac{x}{a}\right)s}{\cosh\left(\frac{L}{a}s\right)}(1-e^{-s\tau})\frac{\tilde{p}}{s} \tag{58}$$

for $x=L$,

$$\hat{p}(L,s)-\frac{p_0}{s}=\sqrt{\frac{A_0}{A(L)}}\frac{(1-e^{-s\tau})}{\cosh\left(\frac{L}{a}s\right)}\frac{\tilde{p}}{s} \tag{59}$$

Inverse Laplace-transforming Eq. (59) gives

$$\frac{p(L,t)-p_0}{\tilde{p}}=\begin{cases} 0 & 0<t<\frac{L}{a} \\ 2\sqrt{\frac{A_0}{A(L)}} & \frac{L}{a}<t<\frac{L}{a}+\tau \\ 0 & \frac{L}{a}+\tau<t<\frac{3L}{a} \\ -2\sqrt{\frac{A_0}{A(L)}} & \frac{3L}{a}<t<\frac{3L}{a}+\tau \end{cases} \tag{60}$$

This is illustrated in Fig. 3.

Obviously, this Fig. 3 means that Eq. (59) is a periodic function with a period $\frac{4L}{a}$ and the magnitude of measured pulse is proportional to $2\sqrt{\frac{A_0}{A(L)}}$.

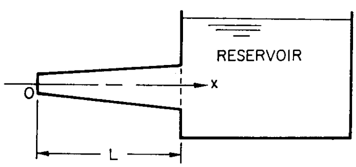


Fig. 4 Reservoir

EXAMPLE 3: Travelling wave and reflected wave at the open end.

Let us take a reservoir as an open end.
The boundary conditions with a reservoir (Fig. 4) are

$$p(0,t)-p_0=\begin{cases}0 & t<0\\ \tilde{p} & 0<t<\tau \qquad \left(\tau\ll\dfrac{L}{a}\right)\\ 0 & \tau<t\end{cases} \tag{61}$$

$$p(L,t)=p_0 \tag{62}$$

In the same way, we have as a solution

$$\hat{p}(x,s)-\frac{p_0}{s}=\sqrt{\frac{A_0}{A(x)}}\frac{\sinh\left(\dfrac{L}{a}-\dfrac{x}{a}\right)s}{\sinh\left(\dfrac{L}{a}s\right)}(1-e^{-s\tau})\frac{\tilde{p}}{s} \tag{63}$$

## B. Dynamic response of volume flux

EXAMPLE 4: Travelling wave for pulse input of volume flux.

We consider only travelling wave in a semi-infinite tube. From Eq. (44)

$$\hat{q}(x,s)=k_1\sqrt{\frac{A(x)s}{K}}e^{-\frac{s}{a}x} \tag{64}$$

The boundary condition for volume flux pulse is

$$q(0,t)=\begin{cases}0 & t<0\\ \tilde{q} & 0<t<\tau \qquad \left(\tau\ll\dfrac{L}{a}\right)\\ 0 & \tau<t\end{cases} \tag{65}$$

where $\tilde{q}$ is the magnitude of volume flux pulse. Laplace-transforming Eq. (65) and using Eq. (64) gives

$$k_1=\sqrt{\frac{K}{A_0 s}}(1-e^{-s\tau})\frac{\tilde{q}}{s} \tag{66}$$

Eliminating $k_1$ from Eq. (64) and dividing by $\tilde{q}$,

$$\frac{\hat{q}(x,s)}{\tilde{q}}=\frac{1}{s}\sqrt{\frac{A(x)}{A(0)}}\left\{e^{-\frac{x}{a}s}-e^{-\left(\frac{x}{a}+\tau\right)s}\right\} \tag{67}$$

Equation (67) is different from Eq. (48) for pressure pulse in the coefficient, viz, the height of output pulse for volume flux is proportional to the square root of the ratio of $A(x)$ to $A_0$ in a tapered transmission line.

### C. Water-hammer

Next we consider hydraulic transients accompanied by a rapid value closure for fluid initially not at rest. Complexity in this case exists in $p(x,0)$ generally depending on $x$, i.e.,

$$\boldsymbol{u}(x,0)=\begin{pmatrix} p(x,0) \\ q(x,0) \end{pmatrix} \neq \text{constant vector}$$

EXAMPLE 5: $R(x)=R_0$, constant radius

From Eq. (29),

$$\hat{\boldsymbol{u}}(x,s)=\begin{pmatrix} \sqrt{\frac{s\rho_0}{A(x)}} & -\sqrt{\frac{s\rho_0}{A(x)}} \\ \sqrt{\frac{A(x)s}{K}} & \sqrt{\frac{A(x)s}{K}} \end{pmatrix} \begin{pmatrix} e^{-\frac{s}{a}x} & 0 \\ 0 & e^{\frac{s}{a}x} \end{pmatrix} \left\{ \int^x \frac{1}{2} \begin{pmatrix} e^{\frac{s}{a}x} & 0 \\ 0 & e^{-\frac{s}{a}x} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{A(x)}{s\rho_0}} & \sqrt{\frac{K}{A(x)s}} \\ -\sqrt{\frac{A(x)}{s\rho_0}} & \sqrt{\frac{K}{A(x)s}} \end{pmatrix} \right.$$

$$\left. \begin{pmatrix} 0 & \frac{\rho_0}{A(x)} \\ \frac{A(x)}{K} & 0 \end{pmatrix} \begin{pmatrix} p(x,0) \\ q(x,0) \end{pmatrix} dx + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right\} \tag{68}$$

therefore

$$\hat{\boldsymbol{u}}(x,s)=\begin{pmatrix} \sqrt{\frac{s\rho_0}{A(x)}}e^{-\frac{s}{a}x} & -\sqrt{\frac{s\rho_0}{A(x)}}e^{\frac{s}{a}x} \\ \sqrt{\frac{A(x)s}{K}}e^{-\frac{s}{a}x} & \sqrt{\frac{A(x)s}{K}}e^{\frac{s}{a}x} \end{pmatrix}$$

$$\left\{ \int^x \frac{1}{2s} \begin{pmatrix} \sqrt{\frac{s\rho_0}{A(x)}} q(x,0) e^{\frac{s}{a}x} + \sqrt{\frac{A(x)s}{K}} p(x,0) e^{\frac{s}{a}x} \\ -\sqrt{\frac{s\rho_0}{A(x)}} q(x,0) e^{-\frac{s}{a}x} + \sqrt{\frac{A(x)s}{K}} p(x,0) e^{-\frac{s}{a}x} \end{pmatrix} dx + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right\} \tag{69}$$

Integrating Eq. (69) with $A(x)=A_0$, $p(x,0)=p_0$ and $q(x,0)=q_0$ for $R(x)=R_0$,

$$\hat{p}(x,s)=\frac{p_0}{s}+\sqrt{\frac{s\rho_0}{A_0}}\left(k_1 e^{-\frac{s}{a}x}-k_2 e^{\frac{s}{a}x}\right) \tag{70}$$

$$\hat{q}(x,s)=\frac{q_0}{s}+\sqrt{\frac{A_0 s}{K}}\left(k_1 e^{-\frac{s}{a}x}+k_2 e^{\frac{s}{a}x}\right) \tag{71}$$

The boundary conditions (Fig. 5(a)) are

$$q(L,t)=\begin{cases} q_0 & t<0 \\ 0 & 0<t \end{cases} \tag{72}$$

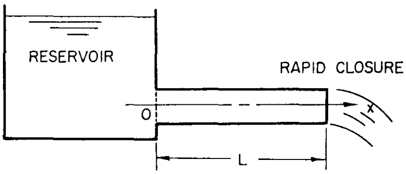


Fig. 5(a) $R(x)=R_0$, constant radius

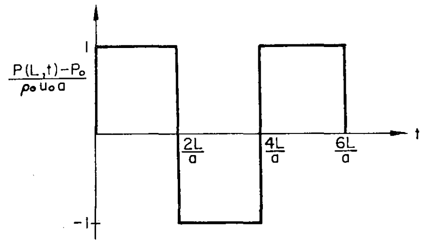


Fig. 5(b) Water-hammer with a rapid closure

$$p(0, t)=p_0 \qquad t>0 \tag{73}$$

Substituting Eqs. (72) and (73) in Eqs. (71) and (70) respectively,

$$k_1=k_2=-\frac{q_0}{2s}\sqrt{\frac{K}{A_0 s}}\frac{1}{\cosh\left(\frac{L}{a}s\right)} \tag{74}$$

$$\hat{p}(x, s)-\frac{p_0}{s}=\frac{\rho_0 u_0 a}{s}\frac{\sinh\left(\frac{x}{a}s\right)}{\cosh\left(\frac{L}{a}s\right)} \tag{75}$$

for $x=L$

$$\hat{p}(L, s)-\frac{p_0}{s}=\frac{\rho_0 u_0 a}{s}\tanh\left(\frac{L}{a}s\right) \tag{76}$$

Inverse Laplace-transforming Eq. (76) gives (see Fig. 5(b))

$$\frac{p(L, t)-p_0}{\rho_0 u_0 a}\begin{cases} 1 & 0<t<\frac{2L}{a} \\ -1 & \frac{2L}{a}<t<\frac{4L}{a} \\ 1 & \frac{4L}{a}<t<\frac{6L}{a} \\ \vdots & \vdots \end{cases} \tag{77}$$

where $\rho_0 u_0 a$ means the maximum pressure rise in rigid water column theory.

EXAMPLE 6: $R(x)=R_0 e^{\alpha x}$, exponential radius

For $R(x)=R_0 e^{\alpha x}$,

$$A(x)=A_0 e^{2\alpha x} \tag{78}$$

$$p(x, 0)=p_0+\frac{1}{2}\rho_0 u_0^2(1-e^{-2\alpha x}) \tag{79}$$

$$q(x, 0)=q_0 \tag{80}$$

Substitution of these equations in Eq. (69) gives

$$\hat{p}(x, s)=\frac{1}{s}\left(p_0+\frac{1}{2}\rho_0 u_0^2\right)\frac{\lambda^2}{\lambda^2-\alpha^2}+\frac{\rho_0 u_0 a}{s}\frac{\alpha\lambda e^{-2\alpha x}}{\lambda^2-\alpha^2}-\frac{\frac{1}{2}\rho_0 u_0^2}{s}\frac{\lambda^2 e^{-4\alpha x}}{\lambda^2-9\alpha^2}$$
$$+\sqrt{\frac{s\rho_0}{A_0}}e^{-\alpha x}\{k_1 e^{-\lambda x}-k_2 e^{\lambda x}\} \tag{81}$$

$$\hat{q}(x, s)=\frac{q_0}{s}\frac{\lambda^2}{\lambda^2-\alpha^2}-\frac{A_0}{K}\left(p_0+\frac{1}{2}\rho_0 u_0^2\right)\frac{\alpha e^{2\alpha x}}{\lambda^2-\alpha^2}-\frac{A_0}{K}\left(\frac{1}{2}\rho_0 u_0^2\right)\frac{3\alpha e^{-2\alpha x}}{\lambda^2-9\alpha^2}$$
$$+\sqrt{\frac{A_0 s}{K}}e^{\alpha x}\{k_1 e^{-\lambda x}+k_2 e^{\lambda x}\} \tag{82}$$

where $\lambda=\frac{s}{a}$. Equations (81) and (82) for $\alpha=0$ are equal to Eqs. (70) and (71) respectively.

The unknown coefficients $k_1$ and $k_2$ are specified by the boundary conditions (72) and (73);

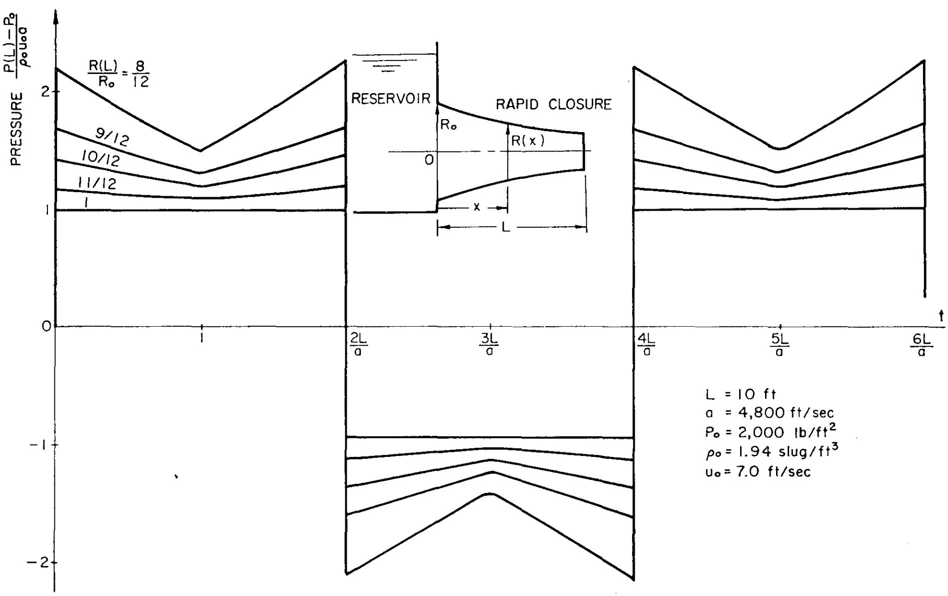


Fig. 6 Time history history in an exponential line, $R(x)=R_0 e^{\alpha x}$

$$k_1 = k_2 - A \tag{83}$$

$$k_2 = \frac{B + Ae^{-\lambda L}}{2\cosh(\lambda L)} \tag{84}$$

Here

$$A = \sqrt{\frac{A_0}{s\rho_0}}\left[\frac{p_0}{s}\frac{\alpha^2}{\lambda^2-\alpha^2} + \frac{\rho_0 u_0^2}{2s}\left\{\frac{\lambda^2}{\lambda^2-\alpha^2} - \frac{\lambda^2}{\lambda^2-9\alpha^2}\right\} + \frac{\rho_0 u_0 a}{s}\frac{\alpha\lambda}{\lambda^2-\alpha^2}\right] \tag{85}$$

$$B = e^{-\alpha L}\sqrt{\frac{K}{A_0 s}}\left[-\frac{q_0}{s}\frac{\lambda^2}{\lambda^2-\alpha^2} + \frac{A_0}{K}\left(p_0 + \frac{1}{2}\rho_0 u_0^2\right)\frac{\alpha e^{2\alpha L}}{\lambda^2-\alpha^2} + \frac{A_0}{K}\left(\frac{1}{2}\rho_0 u_0^2\right)\frac{3\alpha e^{-2\alpha L}}{\lambda^2-9\alpha^2}\right. \tag{86}$$

Substituting Eqs. (83) and (84) in Eq. (81), after some calculations, we get the following inverse Laplace-transformed equations at $x=L$ (see Fig. 6)

i. For $0 < t < \frac{L}{a}$

$$p(L,t) = \left(p_0 + \frac{1}{2}\rho_0 u_0^2\right)e^{-a\alpha t} - \frac{1}{2}\rho_0 u_0^2 e^{a\alpha\left(3t-4\frac{L}{a}\right)}$$
$$+ \rho_0 u_0 a e^{a\alpha\left(t-2\frac{L}{a}\right)} \tag{87}$$

ii. For $\frac{L}{a} < t < \frac{2L}{a}$

$$p(L,t) = \frac{1}{2}\rho_0 u_0^2 e^{-a\alpha\left(3t-2\frac{L}{a}\right)} - \left(p_0 + \frac{1}{2}\rho_0 u_0^2\right)e^{a\alpha\left(t-2\frac{L}{a}\right)}$$
$$+ \rho_0 u_0 a e^{-a\alpha t} + 2p_0 e^{-\alpha L} \tag{88}$$

iii. For $\frac{2L}{a} < t < \frac{3L}{a}$

$$p(L,t) = \frac{1}{2}\rho_0 u_0^2 e^{a\alpha\left(3t-10\frac{L}{a}\right)} - \left(p_0 + \frac{1}{2}\rho_0 u_0^2\right)e^{-a\alpha\left(t-2\frac{L}{a}\right)}$$
$$+ 2p_0 e^{-\alpha L} - \rho_0 u_0 a e^{a\alpha\left(t-4\frac{L}{a}\right)} \tag{89}$$

iv. For $\frac{3L}{a} < t < \frac{4L}{a}$

$$p(L,t) = \left(p_0 + \frac{1}{2}\rho_0 u_0^2\right)e^{a\alpha\left(t-4\frac{L}{a}\right)} - \frac{1}{2}\rho_0 u_0^2 e^{-a\alpha\left(3t-8\frac{L}{a}\right)}$$
$$- \rho_0 u_0 a e^{-a\alpha\left(t-2\frac{L}{a}\right)} \tag{90}$$

v. For $\frac{4L}{a}<t<\frac{5L}{a}$

$$p(L,t)=\left(p_0+\frac{1}{2}\rho_0u_0^2\right)e^{-a\alpha\left(t-4\frac{L}{a}\right)}-\frac{1}{2}\rho_0u_0^2e^{a\alpha\left(3t-16\frac{L}{a}\right)}$$

$$+\rho_0u_0ae^{a\alpha\left(t-6\frac{L}{a}\right)} \tag{91}$$

vi. For $\frac{5L}{a}<t<\frac{6L}{a}$

$$p(L,t)=\frac{1}{2}\rho_0u_0^2e^{-a\alpha\left(3t-14\frac{L}{a}\right)}-\left(p_0+\frac{1}{2}\rho_0u_0^2\right)e^{a\alpha\left(t-6\frac{L}{a}\right)}$$

$$+\rho_0u_0ae^{-a\alpha\left(t-4\frac{L}{a}\right)}+2p_0e^{-\alpha L} \tag{92}$$

EXAMPLE 7: $R(x)=R_0(1+\beta x)$, linear radius

For $R(x)=R_0(1+\beta x)$,

$$A(x)=A_0(1+\beta x)^2 \tag{93}$$

$$p(x,0)=p_0+\frac{1}{2}\rho_0u_0^2\{1-(1+\beta x)^{-4}\} \tag{94}$$

$$q(x,0)=q_0 \tag{95}$$

Substitution of these equations yields as an approximation

$$\hat{p}(x,s)=\frac{1}{s}\left(p_0+\frac{1}{2}\rho_0u_0^2\right)+\frac{\rho_0u_0a}{s}\frac{\lambda\beta(1+\beta x)^{-2}}{\lambda^2-\beta^2}-\frac{\rho_0u_0^2}{2s}\frac{\lambda^2(1+3\beta x)^{-1}(1+\beta x)^{-1}}{\lambda^2-\beta^2}$$

$$+\sqrt{\frac{s\rho_0}{A_0}}(1+\beta x)^{-1}\{k_1e^{-\lambda x}-k_2e^{\lambda x}\} \tag{96}$$

$$\hat{q}(x,s)=\frac{q_0}{s}\frac{\lambda^2}{\lambda^2-\beta^2}-\frac{A_0}{K}\left(p_0+\frac{1}{2}\rho_0u_0^2\right)\frac{\beta(1+\beta x)}{\lambda^2}-\frac{A_0}{K}\left(\frac{1}{2}\rho_0u_0^2\right)\frac{\beta(1+3\beta x)^{-1}(1+\beta x)}{\lambda^2-\beta^2}$$

$$+\sqrt{\frac{A_0s}{K}}(1+\beta x)\{k_1e^{-\lambda x}+k_2e^{\lambda x}\} \tag{97}$$

The unknown coefficients $k_1$ and $k_2$ are specified by the boundary conditions (72) and (73):

$$k_1=k_2-A \tag{98}$$

$$k_2=\frac{B+Ae^{-\lambda L}}{2\cosh(\lambda L)} \tag{99}$$

Here

$$A=\sqrt{\frac{A_0}{s\rho_0}}\left[-\frac{\rho_0 u_0^2}{2s}\frac{\beta^2}{\lambda^2-\beta^2}+\frac{\rho_0 u_0 a}{s}\frac{\beta\lambda}{\lambda^2-\beta^2}\right] \tag{100}$$

$$B=\sqrt{\frac{K}{A_0 s}}\left[-\frac{q_0}{s}\frac{\lambda^2(1+\beta L)^{-1}}{\lambda^2-\beta^2}+\frac{A_0}{K}\left(p_0+\frac{1}{2}\rho_0 u_0^2\right)\frac{\beta}{\lambda^2}+\frac{A_0}{K}\left(\frac{1}{2}\rho_0 u_0^2\right)\frac{\beta(1+3\beta L)^{-1}}{\lambda^2-\beta^2}\right] \tag{101}$$

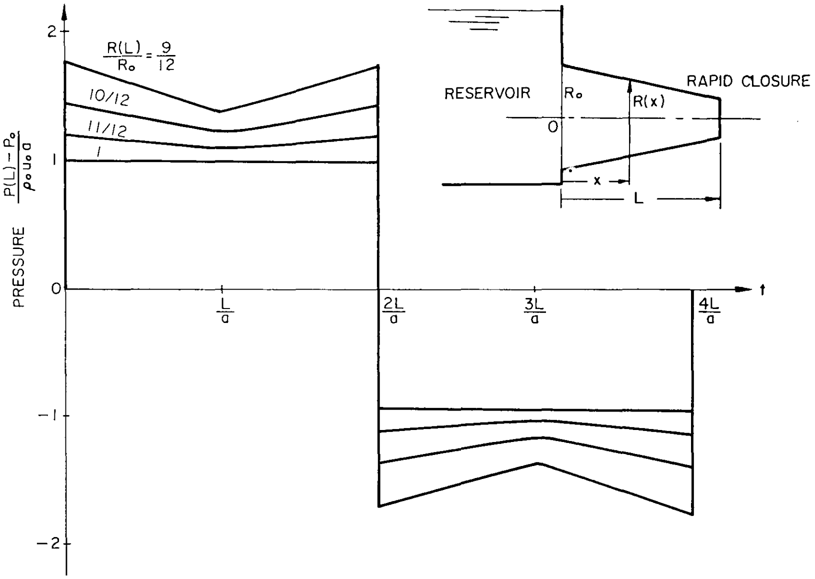


Fig. 7 Time history in a linear line, $R(x)=R_0(1+\beta x)$

Substituting Eqs. (98) and (99) in Eq. (96), after some calculations, we get the following inverse Laplace-transformed equations at $x=L$ (see Fig. 7)

i. For $0<t<\dfrac{L}{a}$

$$p(L,t)=p_0+\frac{1}{2}\rho_0 u_0^2+\rho_0 u_0 a(1+\beta L)^{-2}e^{a\beta t}-\frac{1}{2}\rho_0 u_0^2(1+\beta L)^{-1}(1+3\beta L)^{-1}e^{a\beta t}$$

$$-a\beta\left(p_0+\frac{1}{2}\rho_0 u_0^2\right)(1+\beta L)^{-1}t \tag{102}$$

ii. For $\dfrac{L}{a}<t<\dfrac{2L}{a}$

$$p(L,t)=p_0+\frac{1}{2}\rho_0 u_0^2+\rho_0 u_0 a(1+\beta L)^{-2}e^{a\beta t}-2\rho_0 u_0 a(1+\beta L)^{-1}\sinh a\beta\left(t-\frac{L}{a}\right)$$

$$-\frac{1}{2}\rho_0 u_0^2(1+\beta L)^{-1}(1+3\beta L)^{-1}e^{a\beta t}+\rho_0 u_0^2(1+\beta L)^{-1}\cosh a\beta\left(t-\frac{L}{a}\right)$$

$$-\rho_0 u_0^2(1+\beta L)^{-1}-a\beta\left(p_0+\frac{1}{2}\rho_0 u_0^2\right)(1+\beta L)^{-1}t \tag{103}$$

iii. For $\frac{2L}{a}<t<\frac{3L}{a}$

$$p(L,t)=p_0+\frac{1}{2}\rho_0 u_0^2+\rho_0 u_0 a(1+\beta L)^{-2}e^{a\beta t}-2\rho_0 u_0 a(1+\beta L)^{-1}\sinh a\beta\left(t-\frac{L}{a}\right)$$

$$-2\rho_0 u_0 a(1+\beta L)^{-2}\cosh a\beta\left(t-2\frac{L}{a}\right)-\frac{1}{2}\rho_0 u_0^2(1+\beta L)^{-1}(1+3\beta L)^{-1}e^{a\beta t}$$

$$+\rho_0 u_0^2(1+\beta L)^{-1}\cosh a\beta\left(t-\frac{L}{a}\right)+\rho_0 u_0^2(1+\beta L)^{-1}(1+3\beta L)^{-1}\sinh a\beta\left(t-2\frac{L}{a}\right)$$

$$-\rho_0 u_0^2(1+\beta L)^{-1}+a\beta\left(p_0+\frac{1}{2}\rho_0 u_0^2\right)(1+\beta L)^{-1}\left(t-4\frac{L}{a}\right) \tag{104}$$

iv. For $\frac{3L}{a}<t<\frac{4L}{a}$

$$p(L,t)=p_0+\frac{1}{2}\rho_0 u_0^2+\rho_0 u_0 a(1+\beta L)^{-2}e^{a\beta t}-2\rho_0 u_0 a(1+\beta L)^{-1}\sinh a\beta\left(t-\frac{L}{a}\right)$$

$$-2\rho_0 u_0 a(1+\beta L)^{-2}\cosh a\beta\left(t-2\frac{L}{a}\right)+2\rho_0 u_0 a(1+\beta L)^{-1}\sinh a\beta\left(t-3\frac{L}{a}\right)$$

$$-\frac{1}{2}\rho_0 u_0^2(1+\beta L)^{-1}(1+3\beta L)^{-1}e^{a\beta t}+\rho_0 u_0^2(1+\beta L)^{-1}\cosh a\beta\left(t-\frac{L}{a}\right)$$

$$+\rho_0 u_0^2(1+\beta L)^{-1}(1+3\beta L)^{-1}\sinh a\beta\left(t-2\frac{L}{a}\right)-\rho_0 u_0^2(1+\beta L)^{-1}\cosh a\beta\left(t-3\frac{L}{a}\right)$$

$$+a\beta\left(p_0+\frac{1}{2}\rho_0 u_0^2\right)(1+\beta L)^{-1}\left(t-4\frac{L}{a}\right) \tag{105}$$

v. For $\frac{4L}{a}<t<\frac{5L}{a}$

$$p(L,t)=p_0+\frac{1}{2}\rho_0 u_0^2+\rho_0 u_0 a(1+\beta L)^{-2}e^{a\beta t}-2\rho_0 u_0 a(1+\beta L)^{-1}\sinh a\beta\left(t-\frac{L}{a}\right)$$

$$-2\rho_0 u_0 a(1+\beta L)^{-2}\cosh a\beta\left(t-2\frac{L}{a}\right)+2\rho_0 u_0 a(1+\beta L)^{-1}\sinh a\beta\left(t-3\frac{L}{a}\right)$$

$$+2\rho_0 u_0 a(1+\beta L)^{-2}\cosh a\beta\left(t-4\frac{L}{a}\right)-\frac{1}{2}\rho_0 u_0^2(1+\beta L)^{-1}(1+3\beta L)^{-1}e^{a\beta t}$$

$$+\rho_0 u_0^2(1+\beta L)^{-1}\cosh a\beta\left(t-\frac{L}{a}\right)+\rho_0 u_0^2(1+\beta L)^{-1}(1+3\beta L)^{-1}\sinh a\beta\left(t-2\frac{L}{a}\right)$$

$$-\rho_0 u_0^2(1+\beta L)^{-1}\cosh a\beta\left(t-3\frac{L}{a}\right)-\rho_0 u_0^2(1+\beta L)^{-1}(1+3\beta L)^{-1}\sinh a\beta\left(t-4\frac{L}{a}\right)$$

$$-a\beta\left(p_0+\frac{1}{2}\rho_0 u_0^2\right)(1+\beta L)^{-1}\left(t-4\frac{L}{a}\right) \tag{106}$$

vi. For $\frac{5L}{a}<t<\frac{6L}{a}$

$$p(L,t)=p_0+\frac{1}{2}\rho_0 u_0^2+\rho_0 u_0 a(1+\beta L)^{-2}e^{a\beta t}-2\rho_0 u_0 a(1+\beta L)^{-1}\sinh a\beta\left(t-\frac{L}{a}\right)$$

$$-2\rho_0 u_0 a(1+\beta L)^{-2}\cosh a\beta\left(t-2\frac{L}{a}\right)+2\rho_0 u_0 a(1+\beta L)^{-1}\sinh a\beta\left(t-3\frac{L}{a}\right)$$

$$+2\rho_0 u_0 a(1+\beta L)^{-2}\cosh a\beta\left(t-4\frac{L}{a}\right)-2\rho_0 u_0 a(1+\beta L)^{-1}\sinh a\beta\left(t-5\frac{L}{a}\right)$$

$$-\frac{1}{2}\rho_0 u_0^2(1+\beta L)^{-1}(1+3\beta L)^{-1}e^{a\beta t}+\rho_0 u_0^2(1+\beta L)^{-1}\cosh a\beta\left(t-\frac{L}{a}\right)$$

$$+\rho_0 u_0^2(1+\beta L)^{-1}(1+3\beta L)^{-1}\sinh a\beta\left(t-2\frac{L}{a}\right)-\rho_0 u_0^2(1+\beta L)^{-1}\cosh a\beta\left(t-3\frac{L}{a}\right)$$

$$-\rho_0 u_0^2(1+\beta L)^{-1}(1+3\beta L)^{-1}\sinh a\beta\left(t-4\frac{L}{a}\right)+\rho_0 u_0^2(1+\beta L)^{-1}\cosh a\beta\left(t-5\frac{L}{a}\right)$$

$$-\rho_0 u_0^2(1+\beta L)^{-1}-a\beta\left(p_0+\frac{1}{2}\rho_0 u_0^2\right)(1+\beta L)^{-1}\left(t-4\frac{L}{a}\right) \tag{107}$$

## Conclusions

1. All we have to do is to find the inverse Laplace-transformation of the Eq. (29) for many problems different from boundary and initial conditions.

2. The transfer-function matrix expressed in the general form of the Eq. (36) will play an important role in the analysis of the frequency response in a small-diameter tapered transmission line.

3. The distortion is not caused in the output pulse for inviscid fluid if $\frac{\tau}{T}\frac{L}{A(x)}\frac{dA(x)}{dx}\ll 1$. In large tapered lines where this assumption is not valid, we must treat the motion of fluid as two or three-dimensional flow.

4. It is confirmed in another way that

$$\frac{p(L,t)-p_0}{\tilde{p}}\sim\sqrt{\frac{A_0}{A(L)}} \quad \text{and} \quad \frac{q(L,t)-q_0}{\tilde{q}}\sim\sqrt{\frac{A(L)}{A_0}}$$

5. A concave on the pressure head of water-hammer is found to be formed in a tapered tube.

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