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BY

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A CENTRAL LIMIT THEOREM FOR THE RANDOM NOISE PROCESS

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ABSTRACT

In this paper a central limit theorem for a random noise process, which was introduced by RICE (1944), is given.

Introduction

Suppose that certain events occur in accordance with a Poisson process with stationary independent increments at the rate $\lambda>0$, and each event has a certain intensity U and has an after-effect $U\Phi(t)$ after t time units. Let the intensities $U_1, U_2, \cdots$ at occurences of events be mutually independent. The sum of after-effects at time t can be represented as

$$X_1(t)=\int_{-\infty}^{+\infty}\Phi(t-s)dy(s), \tag{1. 1}$$

where $y(t)$, $-\infty<t<+\infty$, is a stochastic process whose sample functions are constant between events and increase by the corresponding intensity U_i at each event. This is called the *random noise process*, since it is a model of the shot effect in a thermionic vacuum tube. This kind of process was considered by RICE (1944) and was formulated by DOOB (1953). KAWATA (1955) proved some results, in connection with Rice theory, with $X_1(t)$ defined by (1. 1) in a rigorous way from mathematical view points. In this paper we shall prove a central limit theorem for the random noise process.

The random noise process

We give the mathematical definition of the random noise process following DOOB (1953). Also see KAWATA (1955).

We assume throughout this paper that $\Phi(t)$ is a real-valued function defined on $(-\infty, +\infty)$ such that

$$\Phi(t) \geqslant 0 \tag{2.1}$$

$$\int_{-\infty}^{\infty} \Phi(t)dt = a, \tag{2.2}$$

$$\int_{-\infty}^{\infty} \Phi^2(t)dt = b, \tag{2.3}$$

$(0<b<+\infty)$. $\Phi(t)$ is not necessarily zero on $(-\infty, 0)$. Let each intensity U_i have a common distribution function $F(x)$ and $E\{U_i\}=\alpha$, $E\{U_i^2\}=\beta$, $(\beta<\infty)$. It is easily shown that the process $y(t)$, $-\infty<t<+\infty$, defined above has stationary independent increments whose distribution function is given by,

$$\begin{aligned} F_s(x) &= Pr\{y(t+s)-y(t) \leqslant x\} \\ &= 0, \quad x<0, \\ &= e^{-\lambda s}, \quad x=0, \\ &= e^{-\lambda s} + \sum_{k=1}^{\infty} e^{-\lambda s} \frac{(\lambda s)^k}{k!} F^{k*}(x), \qquad x>0, \end{aligned} \tag{2.4}$$

where $s>0$ and $F^k(x)=\overbrace{F*\cdots*F}^{k}(x)$.

Put

$$Y(t) = y(t)-y(0)-m(t), \qquad -\infty<t<\infty, \tag{2.5}$$

where $m(t)=E\{y(t)-y(0)\}=\lambda\alpha t$. Then the process $Y(t)$, $-\infty<t<\infty$, also has stationary independent increments, which satisfy that

$$\begin{aligned} &E\{Y(t+s)-Y(t)\}=0, \\ &E\{|Y(t+s)-Y(t)|^2\}=\lambda\beta s. \end{aligned} \tag{2.6}$$

Thus $Y(t)$, $-\infty<t<\infty$, has necessarily orthogonal increments. Let

$$\begin{aligned} &V(t+s)-V(t)=E\{|Y(t+s)-Y(t)|^2\} \\ &Y((t,\ t+s])=Y(t+s)-Y(t) \\ &Y((t,\ t+s])=E\{|Y((t,\ t+s])|^2\}, \qquad (s>0). \end{aligned} \tag{2.7}$$

The random measure $Y(\cdot)$ and the set function $V(\cdot)$ can be extended to be defined over any Borel set (c.f. ROZANOV (1963)).

The stochastic integral

$$X(t)=\int_{-\infty}^{\infty}\Phi(t-s)Y(ds) \tag{2. 8}$$

is well-defined in mean square sense, since $\Phi(t)\in L_2(-\infty, \infty)$. As usual, we define the stochastic integral (1. 1) as follows;

$$\begin{aligned} X_1(t)&=\int_{-\infty}^{\infty}\Phi(t-s)dy(s)\\ &=\int_{-\infty}^{\infty}\Phi(t-s)d[y(s)-m(s)]+\int_{-\infty}^{\infty}\Phi(t-s)dm(s)\\ &=\int_{-\infty}^{\infty}\Phi(t-s)Y(ds)+\lambda\alpha\int_{-\infty}^{\infty}\Phi(s)ds\\ &=X(t)+\lambda\alpha a. \end{aligned} \tag{2. 9}$$

Characteristic functions of the random noise process

When the process $Y(t)$, $-\infty<t<\infty$, has stationary independent increments, LUGANNANI and THOMAS (1967) have obtained the form of the characteristic function of a random measure generated by $Y(t)$. For a special case of random noise, the formula takes the form

$$\begin{aligned} E\{\exp[iuY(B)]\}&=\exp\left[\lambda\int_B\int_{-\infty}^{\infty}\{\exp[iux]-iux-1\}dF(x)ds\right]\\ &=\exp[\lambda\mu(B)\{g(u)-iu-1\}]. \end{aligned} \tag{3. 1}$$

where B is any Borel set and $g(u)$ is the characteristic function of U_i and $\mu(\cdot)$ is the Lebesgue measure. The characteristic function of the random noise process $X(t)$ is given by

$$E\{\exp[iuX(t)]\}=\exp\left[\lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\exp[iu\Phi(t-s)x]-iu\Phi(t-s)x-1\}dF(x)ds\right].$$

This formula was directly derived by KAWATA (1955). Actually this is a special special case of Lemma 2 given later. It follows that the process $X(t)$, $-\infty<t<\infty$, is strictly stationary as well as weakly stationary. $X(t)$ itself and its covariance function have the following representations (see KAWATA (1955));

$$X(t)=\int_{-\infty}^{\infty}e^{i\omega t}d\zeta(\omega), \tag{3. 2}$$

$$\rho(\tau)=EX(t+\tau)X(t)=\lambda\beta\int_{-\infty}^{\infty} e^{i\tau\omega}|\hat{\Phi}(\omega)|^2 d\omega, \tag{3.3}$$

where $\zeta(t)$ is a process of orthogonal increments defined by (3. 5) below and $\hat{\Phi}(\omega)$ is the Fourier transform of $\Phi(t)$ in L_2-sense. Define the process $Y^*(x)$, $-\infty<x<\infty$, by the stochastic integral

$$Y^*(\mu)-Y^*(\nu)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-is\mu}-e^{-is\nu}}{is}Y(ds). \tag{3.4}$$

It is easily seen that $Y^*(x)$, $-\infty<x<\infty$, has orthogonal increments and

$$E[|Y^*(x+y)-Y^*(x)|^2]=\lambda\beta y.$$

We define the process $\zeta(\omega)$, $-\infty<\omega<\infty$, by

$$\zeta(\omega)=\int_{-\infty}^{\omega}\hat{\Phi}(x)dY^*(x), \tag{3.5}$$

which is shown to be $\zeta(\omega)$ in (3. 3). We easily see that

$$E[|\zeta(\omega_2)-\zeta(\omega_1)|^2]=\lambda\beta\int_{\omega_1}^{\omega_2}|\hat{\Phi}(x)|^2 dx. \tag{3.6}$$

Consider the stochastic integral

$$\int_{-T}^{T}X(t)dt$$

which is shown to exist in mean square sense, since the integral

$$\int_{-T}^{T}\Phi(t-s)dt,$$

as a function of s, belongs to $L_2(-\infty, \infty)$. Put

$$Z(T)=C(T)\int_{-T}^{T}X(t)dt, \tag{3.7}$$

where

$$C(T)=\left[E\left|\int_{-T}^{T}X(t)dt\right|^2\right]^{-\frac{1}{2}}.$$

We give some lemmas.

LEMMA 1. *For sufficiently large* $T>0$,

$$C(T)=O(1/\sqrt{T}). \tag{3.8}$$

Proof.

$$\frac{1}{T[C(T)]^2}=\frac{1}{T}E\left|\int_{-T}^{T}X(t)dt\right|^2$$

$$=\frac{1}{T}\int_{-T}^{T}\int_{-T}^{T}\rho(t-t')dtdt'$$

$$=\frac{\lambda\beta}{T}\int_{-T}^{T}\int_{-T}^{T}dt\,dt'\int_{-\infty}^{\infty}e^{i(t-t')x}|\hat{\Phi}(x)|^2dx$$

$$=4\lambda\beta\int_{-\infty}^{\infty}\frac{\sin^2 Tx}{Tx^2}|\hat{\Phi}(x)|^2\,dx. \tag{3. 9}$$

which, by the property of Fejér integral, converges to

$$4\pi\lambda\beta|\hat{\Phi}(0)|^2=4\pi\lambda\beta\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(t)dt\right]^2$$

$$=2a^2\lambda\beta,$$

as $T\to\infty$.

LEMMA 2. *Let* $\varphi(s)\in L_2(-\infty,\infty)$ *and define*

$$Z=\int_{-\infty}^{\infty}\varphi(s)Y(ds). \tag{3. 10}$$

Then the characteristic function of Z *is given by*

$$E\{\exp[iuZ]\}=\exp\left[\lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\exp[iu\varphi(s)x]-iu\varphi(s)x-1\}dF(x)ds\right]. \tag{3. 11}$$

For the proof, see LUGANNANI *et al.* (1967).

A central limit theorem

We now give the following central limit theorem.

THEOREM. *Let* $Y(\cdot)$ *be a random measure defined in* 2 *and* $X(t)$ *be the random noise process defined by*

$$X(t)=\int_{-\infty}^{\infty}\Phi(t-s)Y(ds),$$

where $\Phi(t)$ *satisfies* (2. 1)–(2. 3). *Let* $Z(T)$ *be defined by* (3. 7). *Then for every real* x, *the distribution function* $F_T(x)$ *of* $Z(T)$ *converges to the normal distribution*

$$\mathfrak{N}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt, \qquad as\ T\to\infty. \tag{4. 1}$$

Proof. Since

$$Z(T)=C(T)\int_{-\infty}^{\infty}\left[\int_{-T}^{T}\Phi(t-s)dt\right]Y(ds), \tag{4.2}$$

and

$$\int_{-T}^{T}\Phi(t-s)dt\in L_2(-\infty,\ \infty),$$

applying Lemma 2, we have

$$E\{\exp[iuZ(T)]\}=\exp\left[\lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\exp\left[iuxC(T)\int_{-T}^{T}\Phi(t-s)dt\right]\right.\right.$$
$$\left.\left.-iuxC(T)\int_{-T}^{T}\Phi(t-s)dt-1\right\}dF(x)ds\right] \tag{4.3}$$

Write

$$E|uZ(T)|^2=u^2C^2(T)E\left|\int_{-T}^{T}X(t)dt\right|^2$$
$$=u^2. \tag{4.4}$$

This is also described by

$$E|uZ(T)|^2=E\left|uC(T)\int_{-\infty}^{\infty}\left[\int_{-T}^{T}\Phi(t-s)dt\right]Y(ds)\right|^2$$
$$=\lambda\beta\int_{-\infty}^{\infty}\left[uC(T)\int_{-T}^{T}\Phi(t-s)dt\right]^2ds,$$

which is, using the fact $\beta=E\{U_i^2\}=\int_{-\infty}^{\infty}x^2dF(x)$,

$$=\lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[uxC(T)\int_{-T}^{T}\Phi(t-s)dt\right]^2dF(x)ds. \tag{4.5}$$

Since the convergence of $F_T(x)$ to $\mathfrak{N}(x)$ implies and implied by

$$E\{\exp[iuZ(T)]\}\longrightarrow-\frac{1}{2}u^2,\ \text{as}\ T\longrightarrow\infty,$$

it is sufficient to prove, putting (4. 3), (4. 4) and (4. 5) together, that

$$J(T)=\lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\exp\left[iuxC(T)\int_{-T}^{T}\Phi(t-s)dt\right]\right.$$
$$\left.-iuxC(T)\int_{-T}^{T}\Phi(t-s)dt-1\right\}dF(x)ds+\frac{1}{2}u^2$$
$$=\lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\exp\left[iuxC(T)\int_{-T}^{T}\Phi(t-s)dt\right]\right.$$

$$-\frac{1}{2}\left[iuxC(T)\int_{-T}^{T}\Phi(t-s)dt\right]^2 - iuxC(T)\int_{-T}^{T}\Phi(t-s)dt$$

$$-1\Bigg\}dF(x)ds \longrightarrow 0, \text{ as } T \longrightarrow \infty. \tag{4. 6}$$

From the obvious inequality,

$$\left|\exp(ix) - \frac{1}{2}(ix)^2 - (ix) - 1\right| \leqslant K\frac{x^3}{1+|x|}, \qquad (-\infty < x < \infty),$$

K being a positive constant, we have for every $\varepsilon > 0$,

$$|J(T)| \leqslant Ku^3\lambda \iint_{|xC(T)\Psi_T(s)|<\varepsilon} |xC(T)\Psi_T(s)|^2 dF(x)ds$$

$$+Ku^2\lambda \iint_{|xC(T)\Psi_T(s)|\geqslant\varepsilon} |xC(T)\Psi_T(s)|^2 dF(x)ds$$

$$= Ku^3\varepsilon J_1(T) + Ku^2\lambda J_2(T), \text{ say,} \tag{4. 7}$$

where $\Psi_T(s) = \int_{-T}^{T}\Phi(t-s)dt$.

Let us evaluate $J_1(T)$ and $J_2(T)$ as follows.

$$J_1(T) \leqslant \lambda \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |xC(T)\Psi_T(s)|^2 dF(x)ds$$

$$= E|Z(T)|^2$$

$$= 1. \tag{4. 8}$$

Since $C(T) = O(1/\sqrt{T})$, by Lemma 1, we have $\{x: |xC(T)\Psi_T(s)| \geqslant \varepsilon\} \subset \{x: |x| \geqslant \varepsilon M\sqrt{T}\}$ for some positive constant M. It follows that

$$J_2(T) \leqslant C^2(T)\int_{-\infty}^{\infty}[\Psi_T(s)]^2 ds \int_{|x|>\varepsilon M\sqrt{T}} x^2 dF(x).$$

Noting that

$$C^2(T)\int_{-\infty}^{\infty}[\Psi_T(s)]^2 ds = \frac{1}{\lambda\beta},$$

we have

$$J_2(T) \leqslant \frac{1}{\lambda\beta}\int_{|x|>\varepsilon M\sqrt{T}} x^2 dF(x). \tag{4. 9}$$

Thus we have by (4. 7), (4. 8) and (4. 9), for any $\varepsilon > 0$,

$$\limsup_{T\to\infty} |J(T)| \leqslant Ku^2\varepsilon. \tag{4. 10}$$

This completes the proof.

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