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ABSTRACT

Necessary conditions for optimal solutions of games are derived when pure strategies exist both for a minimizer and for a maximizer. The Kuhn-Tucker condition for a static game (continuous game) is obtained by use of a pair of Lagrangian functions. Next, Euler, Clebsch & Weierstrass conditions for a dynamic game (differential game) are also derived by use of a pair of Lagrangian functions, based on calculus of variations. Finally, the relationship between the Kuhn-Tucker condition and the Euler, Clebsch & Weierstrass conditions is discussed.

§ 1. Introduction

The purpose of this paper is to derive necessary conditions that optional solutions of games must satisfy. We study only the case when pure strategies exist both for a minimizer $\underset{\sim}{u}$ and for a maximizer $\underset{\sim}{v}$. By a static game is meant a so-called continuous game (BLACKWELL & GIRSHICK 1954) and by a dynamic game an ordinary differential game (BERKOWITZ 1961, 1964).

As to the static game necessary conditions when a pay-off function $F(\underset{\sim}{u}, \underset{\sim}{v})$ and an inequality constraint $\underset{\sim}{g}(\underset{\sim}{u}, \underset{\sim}{v}) \leq \underset{\sim}{0}$ are defined as functions of only $\underset{\sim}{u}$ and $\underset{\sim}{v}$ have been obtained already (SHIMIZU 1969). The Kuhn-Tucker condition for such game was derived from consideration of a pair of Lagrangian functions. This paper extends that theorem to the case when it contains a state vector $\underset{\sim}{x}$ which is defined by a process equation $\underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{v}) = \underset{\sim}{0}$.

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In his pioneering studies on differential games, Berkowitz derived necessary conditions based on calculus of variations. He considered the case when optimal strategies for the minimizer $\underset{\sim}{u}$ and the maximizer $\underset{\sim}{v}$ are of feed back regulation of $\underset{\sim}{x}$ and inequality constrains are given by $\underset{\sim}{h}(\underset{\sim}{x}(t), \underset{\sim}{u}(t))\leq\underset{\sim}{0}$ and $\underset{\sim}{h}'(\underset{\sim}{x}(t), \underset{\sim}{v}(t))\leq\underset{\sim}{0}$.

In this paper we consider a differencial game for which there is a constraint of type $\underset{\sim}{g}(\underset{\sim}{u}, \underset{\sim}{v})\leq\underset{\sim}{0}$. Similarly to against the static game we consider a pair of Lagrangian functions and obtain the Euler equation and Clebsh condition corresponding to $\underset{\sim}{g}(\underset{\sim}{u}, \underset{\sim}{v})\leq\underset{\sim}{0}$, etc. A pursuer-evader game (Ho 1965) is regarded as a special case of the general differential game.

In appendix the relationship between static and dynamic problems is discussed showing that from the Euler, Clebsch & Weierstrass conditions (BERKOWITZ 1961, BLISS 1946) the Kuhn-Tucker condition (KUHN & TUCKER 1951) is derived.

§ 2. Necessary Conditions for Pure Strategies of Static Game

We will discuss a static game consisting of two players A (a minimizer) and B (a maximizer). The player A wishes to minimize a pay-off function selecting a strategic variable $\underset{\sim}{u}$, and the player B wishes to maximize the same function choosing $\underset{\sim}{v}$ under given constraints.

The game is governed by a process equation

$$\underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{v})=\underset{\sim}{0} \tag{1}$$

and an inequality constraint

$$\underset{\sim}{g}(\underset{\sim}{u}, \underset{\sim}{v})\leqslant\underset{\sim}{0} \tag{2}$$

where $\underset{\sim}{x}$ is an n-dimensional state vector, $\underset{\sim}{u}$ an v-dimensional minimizing vector and $\underset{\sim}{v}$ an q-dimensional maximizing vector. The state of game is determined by Eq. (1) and choices of strategic variables (the minimizer $\underset{\sim}{u}$ and the maximizer $\underset{\sim}{v}$) are constrained by Eq. (2).

The pay-off function is expressed as

$$P=F(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{v}) \tag{3}$$

We assume that $\underset{\sim}{f}, \underset{\sim}{g}, F$ are continuous differentiable functions with respect to $\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{v}$ and $\partial\underset{\sim}{f}/\partial\underset{\sim}{x}$ is a non-singuler matrix. In addition, dimension of $\underset{\sim}{g}$ is less than dimension of $\begin{pmatrix}\underset{\sim}{u}\\ \underset{\sim}{v}\end{pmatrix}$ and $\partial\underset{\sim}{g}/\partial\underline{w}$, $\underline{w}=\begin{pmatrix}\underset{\sim}{u}\\ \underset{\sim}{v}\end{pmatrix}$ has a maximum rank.

It is assumed that there exist pure strategies both for the minimizer and for the maximizer. Then our problem is to find a saddle point solution (the optimal minimizer $\underset{\sim}{u}^0$ and the optimal maximizer $\underset{\sim}{v}^0$) satisfying the constraints (1) (2). That is, for any $\underset{\sim}{u}$ satisfying $\underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{v}^0)=\underset{\sim}{0}$ and $\underset{\sim}{g}(\underset{\sim}{u}, \underset{\sim}{v}^0)\leqslant\underset{\sim}{0}$ and for any $\underset{\sim}{v}$ satisfying $\underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u}^0, \underset{\sim}{v})=\underset{\sim}{0}$ and $\underset{\sim}{g}(\underset{\sim}{u}^0, \underset{\sim}{v})\leqslant 0$, the following inequality holds.

$$F(\underset{\sim}{x}(\underset{\sim}{u}^0, \underset{\sim}{v}), \underset{\sim}{u}^0, \underset{\sim}{v})\leqslant F(\underset{\sim}{x}^0(\underset{\sim}{u}^0, \underset{\sim}{v}^0), \underset{\sim}{u}^0, \underset{\sim}{v}^0)\leqslant F(\underset{\sim}{x}(\underset{\sim}{u}, \underset{\sim}{v}^0), \underset{\sim}{u}, \underset{\sim}{v}^0) \tag{4}$$

Constraint Qualification: Let $\{\underline{u}^0, \underline{v}^0\}$ belong to the boundary of the constraint set. Let the inequality $\underline{g}(\underline{u}, \underline{v}) \leqslant \underline{0}$ be separated into two groups

$$\underline{g}^1(\underline{u}^0, \underline{v}^0) = \underline{0} \qquad \text{and} \qquad \underline{g}^2(\underline{u}^0, \underline{v}^0) < \underline{0}$$

Then it is assumed that for each $\{\underline{u}^0, \underline{v}^0\}$ of the boundary of the constraint set any vector differential $d\underline{u}$, $d\underline{v}$ satisfying homogeneous linear inequalities

$$\frac{\partial \underline{g}^1}{\partial \underline{u}}(\underline{u}^0, \underline{v}^0) d\underline{u} \leqslant \underline{0} \tag{5}$$

$$\frac{\partial \underline{g}^1}{\partial \underline{v}}(\underline{u}^0, \underline{v}^0) d\underline{v} \leqslant \underline{0} \tag{6}$$

is tangent to an arc contained in the constraint set. Furthermore, there exists some vector differential $d\underline{u}$, $d\underline{v}$, $d\underline{x}$ satisfying

$$\frac{\partial \underline{f}(\underline{x}^0, \underline{u}^0, \underline{v}^0)}{\partial \underline{u}} d\underline{u} = \underline{0}, \qquad \frac{\partial \underline{f}(\underline{x}^0, \underline{u}^0, \underline{v}^0)}{\partial \underline{v}} d\underline{v} = \underline{0}, \qquad \frac{\partial \underline{f}(\underline{x}^0, \underline{u}^0, \underline{v}^0)}{\partial \underline{x}} d\underline{x} = \underline{0}.$$

Now let us define a pair of Lagrangian functions

$$\Phi_1(\underline{x}, \underline{u}, \underline{v}, \underline{\psi}, \underline{\lambda}) = F(\underline{x}, \underline{u}, \underline{v}) + \underline{\psi}^{\mathsf{T}} \underline{f}(\underline{x}, \underline{u}, \underline{v}) + \underline{\lambda}^{\mathsf{T}} \underline{g}(\underline{u}, \underline{v}) \tag{7}$$

$$\Phi_2(\underline{x}, \underline{u}, \underline{v}, \underline{\psi}, \underline{\lambda}) = F(\underline{x}, \underline{u}, \underline{v}) + \underline{\psi}^{\mathsf{T}} \underline{f}(\underline{x}, \underline{u}, \underline{v}) + \underline{\lambda}^{\mathsf{T}} \underline{g}(\underline{u}, \underline{v}) \tag{8}$$

THEOREM 1. In order that $\underline{u}^0$, $\underline{v}^0$ be the pure strategies of the static game for the optimal minimizer and the maximizer, respectively it is necessary that $\underline{u}^0$, $\underline{v}^0$ and some $\underline{\psi}^0$, $\underline{\lambda}^0$ satisfy the following conditions.

$$\frac{\partial \Phi_1}{\partial \underline{x}}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \underline{\psi}^0, \underline{\lambda}^0) = \underline{0} \tag{9}$$

$$\frac{\partial \Phi_1}{\partial \underline{u}}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \underline{\psi}^0, \underline{\lambda}^0) = \underline{0} \tag{10}$$

$$\frac{\partial \Phi_2}{\partial \underline{v}}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \underline{\psi}^0, \underline{\lambda}^0) = \underline{0} \tag{11}$$

$$\frac{\partial \Phi_1}{\partial \underline{\psi}}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \underline{\psi}^0, \underline{\lambda}^0) = \underline{0} \qquad (\underline{\psi} = \text{unrestricted}) \tag{12}$$

$$\frac{\partial \Phi_1}{\partial \underline{\lambda}}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \underline{\psi}^0, \underline{\lambda}^0) \leqslant \underline{0}, \quad \frac{\partial \Phi_1}{\partial \underline{\lambda}}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \underline{\psi}^0, \underline{\lambda}^0)\underline{\lambda}^0 = 0, \quad \underline{\lambda}^0 \geqslant \underline{0} \tag{13}$$

Proof. This theorem can be proved by use of Farkas' lemma* in the case of

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* *Farkas' Lemma in the case of both Inequality and Equality:*

An inequality $\underline{b}^{\mathsf{T}}\underline{x} \geqslant 0$ holds for all vectors $\underline{x}$ satisfying $A\underline{x} \geqslant \underline{0}$ and $B\underline{x} = \underline{0}$ where A, B are matrices if and only if $\underline{b} = A^{\mathsf{T}}\underline{t} + B^{\mathsf{T}}\underline{s}$ for some vectors $\underline{t} \geqslant \underline{0}$ and $\underline{s}$ = unrestricted.

containing equalities also and by a method analogous to standard derivation of the Kuhn-Tucker condition* for a problem subject to inequality and equality constraints.

We assume existence of a saddle point solution $\{\underset{\sim}{u}^0, \underset{\sim}{v}^0\}$. Let the vector $\underset{\sim}{u}^0$ minimize $F(\underset{\sim}{x}(\underset{\sim}{u}, \underset{\sim}{v}^0), \underset{\sim}{u}, \underset{\sim}{v}^0)$ under $\underset{\sim}{f}(\underset{\sim}{x}(\underset{\sim}{u}, \underset{\sim}{v}^0), \underset{\sim}{u}, \underset{\sim}{v}^0)=\underset{\sim}{0}$ and $\underset{\sim}{g}(\underset{\sim}{u}, \underset{\sim}{v}^0)\leqslant\underset{\sim}{0}$. Let $\underset{\sim}{v}^0$ maximize $F(\underset{\sim}{x}(\underset{\sim}{u}^0, \underset{\sim}{v}), \underset{\sim}{u}^0, \underset{\sim}{v})$ under $\underset{\sim}{f}(\underset{\sim}{x}(\underset{\sim}{u}^0, \underset{\sim}{v}), \underset{\sim}{u}^0, \underset{\sim}{v})=\underset{\sim}{0}$ and $\underset{\sim}{g}(\underset{\sim}{u}^0, \underset{\sim}{v})\leqslant\underset{\sim}{0}$ (The constraint qualification (5) (6) is assumed to be satisfied). Therefore, at the saddle point (stationary point) $\underset{\sim}{u}^0$ and $\underset{\sim}{v}^0$ constitute minimum and maximum points of F, respectively. Under this situation, however, the state vector $\underset{\sim}{x}$ must constitute a saddle point also. Because the vector $\{\underset{\sim}{u}^0, \underset{\sim}{v}^0\}$ cannot constitute a saddle point under $\underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{v})=\underset{\sim}{0}$ if a saddle point is not formed also with respect to components of $\underset{\sim}{x}$. Thus $\underset{\sim}{x}^0(\underset{\sim}{u}^0, \underset{\sim}{v}^0)$ may be separated into $\underset{\sim}{x}_1^0$, a set of components forming a minimum and $\underset{\sim}{x}_2^0$, a set of components forming a maximum. Here a composite vector of $\underset{\sim}{x}_1^0$ and $\underset{\sim}{x}_2^0$ becomes $\underset{\sim}{x}^0$. For simplicity of description a function f and derivative $\partial f/\partial \underset{\sim}{x}$ evaluated at the point $\{\underset{\sim}{x}^0, \underset{\sim}{u}^0, \underset{\sim}{v}^0\}$ are written as f^0 and $\partial f^0/\partial \underset{\sim}{x}$ below.

Then inequalities

$$\frac{\partial F^0}{\partial \underset{\sim}{u}} d\underset{\sim}{u} \geqslant 0 \tag{14. a}$$

$$\frac{\partial F^0}{\partial \underset{\sim}{x}} d\underset{\sim}{x}_1 \geqslant 0 \tag{14. b}$$

must hold for any vector differentials $\{d\underset{\sim}{u}, d\underset{\sim}{x}_1\}$ satisfying

$$\frac{\partial \underset{\sim}{g}^{10}}{\partial \underset{\sim}{u}} d\underset{\sim}{u} \leqslant \underset{\sim}{0} \tag{15. a}$$

$$\frac{\partial \underset{\sim}{g}^{10}}{\partial \underset{\sim}{x}} d\underset{\sim}{x}_1 \leqslant \underset{\sim}{0} \tag{15. b}$$

$$\frac{\partial \underset{\sim}{f}^0}{\partial \underset{\sim}{u}} d\underset{\sim}{u} = \underset{\sim}{0} \qquad \text{therefore} \qquad -\frac{\partial \underset{\sim}{f}^0}{\partial \underset{\sim}{u}} d\underset{\sim}{u} = \underset{\sim}{0} \tag{15. c}$$

$$\frac{\partial \underset{\sim}{f}^0}{\partial \underset{\sim}{x}_1} d\underset{\sim}{x}_1 = \underset{\sim}{0} \qquad \text{therefore} \qquad -\frac{\partial \underset{\sim}{f}^0}{\partial \underset{\sim}{x}_1} d\underset{\sim}{x}_1 = \underset{\sim}{0} \tag{15. d}$$

Furthermore, inequalities

$$\frac{\partial F^0}{\partial \underset{\sim}{v}} d\underset{\sim}{v} \leqslant 0 \tag{16. a}$$

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* $\min\limits_{\underset{\sim}{x}} F(\underset{\sim}{x})$

subj. to $\underset{\sim}{f}(\underset{\sim}{x})=\underset{\sim}{0}$

$\underset{\sim}{g}(\underset{\sim}{x})\leq\underset{\sim}{0}$

The constraint qualification is assumed. Then necessary Kuhn-Tucker condition is given by $L_{\underset{\sim}{x}}=\underset{\sim}{0}$, $L_{\underset{\sim}{\psi}}=\underset{\sim}{0}$, $L_{\underset{\sim}{\lambda}}\leqslant\underset{\sim}{0}$, $\underset{\sim}{\lambda}\geqslant\underset{\sim}{0}$, $\underset{\sim}{\psi}$=unrestricted where $L=F+\underset{\sim}{\psi}^{\mathrm{T}}\underset{\sim}{f}+\underset{\sim}{\lambda}^{\mathrm{T}}\underset{\sim}{g}$.

$$\frac{\partial F^0}{\partial \underline{x}_2} d\underline{x}_2 \leqslant 0 \tag{16. b}$$

must hold for any vector differentials $d\underline{v}$, $d\underline{x}$ satisfying

$$\frac{\partial g^{10}}{\partial \underline{v}} d\underline{v} \leqslant \underline{0} \tag{17. a}$$

$$\frac{\partial g^{10}}{\partial \underline{x}_2} d\underline{x}_2 \leqslant \underline{0} \tag{17. b}$$

$$\frac{\partial \underline{f}^0}{\partial \underline{v}} d\underline{v} = \underline{0} \tag{17. c}$$

$$\frac{\partial \underline{f}^0}{\partial \underline{x}_2} d\underline{x}_2 = \underline{0} \tag{17. d}$$

Eqs. (15. b) (17. b) are trivial since g does not include $\underline{x}$. Eqs. (15. c, d) (17. c, d) are conditions given against the equality constraint. Eqs. (14. a, b) (16. a, b) are obvious from geometric consideration like the Kuhn-Tucker's proof (KUHN & TUCKER 1951). Then applying the Farkas' lemma in the case of both inequality and equality to Eqs. (14) (15) and Eqs. (16) (17), we obtain

$$\frac{\partial F^{0\mathrm{T}}}{\partial \underline{u}} = -\frac{\partial \underline{f}^{0\mathrm{T}}}{\partial \underline{u}} \underline{\psi}^0 - \frac{\partial \underline{g}^{10\mathrm{T}}}{\partial \underline{u}} \underline{\lambda}^{10} = \underline{0} \tag{18. a}$$

(for some $\underline{\lambda}^{10} \geqslant \underline{0}$, $\underline{\psi}^0$=unrestricted)

$$\frac{\partial F^{0\mathrm{T}}}{\partial \underline{x}_1} = -\frac{\partial \underline{f}^{0\mathrm{T}}}{\partial \underline{x}_1} \underline{\psi}^0 - \underline{0} = \underline{0} \tag{18. b}$$

$$-\frac{\partial F^{0\mathrm{T}}}{\partial \underline{v}} = \frac{\partial \underline{f}^{0\mathrm{T}}}{\partial \underline{v}} \bar{\underline{\psi}}^0 - \frac{\partial \underline{g}^{10\mathrm{T}}}{\partial \underline{v}} \bar{\underline{\lambda}}^{10} = \underline{0} \tag{19. a}$$

(for some $\bar{\underline{\lambda}}^{10} \geqslant 0$, $\bar{\underline{\psi}}^0$=unrestricted)

$$-\frac{\partial F^{0\mathrm{T}}}{\partial \underline{x}_2} = \frac{\partial \underline{f}^{0\mathrm{T}}}{\partial \underline{x}_2} \bar{\underline{\psi}}^0 - \underline{0} = \underline{0} \tag{19. b}$$

Eqs. (18. a) (19. a) may be written as

$$\frac{\partial F^{0\mathrm{T}}}{\partial \underline{u}} = -\frac{\partial \underline{f}^{0\mathrm{T}}}{\partial \underline{u}} \underline{\psi}^0 - \frac{\partial \underline{g}^{0\mathrm{T}}}{\partial \underline{u}} \underline{\lambda}^0$$

$$-\frac{\partial F^{0\mathrm{T}}}{\partial \underline{v}} = \frac{\partial \underline{f}^{0\mathrm{T}}}{\partial \underline{v}} \bar{\underline{\psi}}^0 - \frac{\partial \underline{g}^{0\mathrm{T}}}{\partial \underline{v}} \bar{\underline{\lambda}}^0$$

by adding zeros as components to $\underline{\lambda}^{10}$ and $\bar{\underline{\lambda}}^{10}$ to form $\underline{\lambda}^0$ and $\bar{\underline{\lambda}}^0$. Consequently,

$$\frac{\partial \Phi_1}{\partial \underline{u}}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \underline{\psi}^0, \underline{\lambda}^0) = \frac{\partial F}{\partial \underline{u}}(\underline{x}^0, \underline{u}^0, \underline{v}^0) + \underline{\psi}^{0\mathrm{T}} \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{x}^0, \underline{u}^0, \underline{v}^0) + \underline{\lambda}^{0\mathrm{T}} \frac{\partial \underline{g}}{\partial \underline{u}}(\underline{u}^0, \underline{v}^0) = \underline{0} \tag{20}$$

$$\frac{\partial \Phi_1}{\partial \underline{x}_1}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \underline{\psi}^0, \underline{\lambda}^0) = \frac{\partial F}{\partial \underline{x}_1}(\underline{x}^0, \underline{u}^0, \underline{v}^0) + \underline{\psi}^{0\mathrm{T}} \frac{\partial \underline{f}}{\partial \underline{x}_1}(\underline{x}^0, \underline{u}^0, \underline{v}^0) = \underline{0} \tag{21}$$

$$\frac{\partial \Phi_2}{\partial \underline{v}}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \bar{\underline{\phi}}^0, \bar{\underline{\lambda}}^0) = \frac{\partial F}{\partial \underline{v}}(\underline{x}^0, \underline{u}^0, \underline{v}^0) + \bar{\underline{\phi}}^{0\mathrm{T}} \frac{\partial \underline{f}}{\partial \underline{v}}(\underline{x}^0, \underline{u}^0, \underline{v}^0) + \bar{\underline{\lambda}}^{0\mathrm{T}} \frac{\partial \underline{g}}{\partial \underline{v}}(\underline{u}^0, \underline{v}^0) = \underline{0} \tag{22}$$

$$\frac{\partial \Phi_2}{\partial \underline{x}_2}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \bar{\underline{\phi}}^0, \bar{\underline{\lambda}}^0) = \frac{\partial F}{\partial \underline{x}_2}(\underline{x}^0, \underline{u}^0, \underline{v}^0) + \bar{\underline{\phi}}^{0\mathrm{T}} \frac{\partial \underline{f}}{\partial \underline{x}_2}(\underline{x}^0, \underline{u}^0, \underline{v}^0) = \underline{0} \tag{23}$$

On the other hand from the original inequality

$$\frac{\partial \Phi_1}{\partial \underline{\lambda}}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \underline{\phi}^0, \underline{\lambda}^0)^{\mathrm{T}} = \underline{g}(\underline{u}^0, \underline{v}^0) \leqslant \underline{0} \tag{24}$$

$$-\frac{\partial \Phi_2}{\partial \bar{\underline{\lambda}}}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \bar{\underline{\phi}}^0, \bar{\underline{\lambda}}^0)^{\mathrm{T}} = \underline{g}(\underline{u}^0, \underline{v}^0) \leqslant \underline{0} \tag{25}$$

Furthermore

$$\frac{\partial \Phi_1}{\partial \underline{\lambda}}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \underline{\phi}^0, \underline{\lambda}^0)\underline{\lambda}^0 = \underline{\lambda}^{10\mathrm{T}} \underline{g}^1(\underline{u}^0, \underline{v}^0) + \underline{\lambda}^{20\mathrm{T}} \underline{g}^2(\underline{u}^0, \underline{v}^0) = \underline{0} \tag{26}$$

$$\frac{\partial \Phi_2}{\partial \bar{\underline{\lambda}}}(\underline{x}^0, \underline{u}^0, \underline{v}^0, \bar{\underline{\phi}}^0, \bar{\underline{\lambda}}^0)\bar{\underline{\lambda}}^0 = -\bar{\underline{\lambda}}^{10\mathrm{T}} \underline{g}^1(\underline{u}^0, \underline{v}^0) - \bar{\underline{\lambda}}^{20\mathrm{T}} \underline{g}^2(\underline{u}^0, \underline{v}^0) = \underline{0} \tag{27}$$

As necessary conditions of the game, Eqs. (20) to (27) must be satisfied simultaneously for $\underline{\lambda}^0 \geqslant \underline{0}$, $\bar{\underline{\lambda}}^0 \geqslant \underline{0}$, $\underline{\phi}^0$, $\bar{\underline{\phi}}^0$ = unrestricted. We can, however, set $\underline{\phi} \equiv \bar{\underline{\phi}}$ and $\underline{\lambda} \equiv \bar{\underline{\lambda}}$ in general. This can be verified as follows[1].

Representing Eqs. (14) (16) and Eqs. (15) (17) in vector form at the same time, we have

$$\left(\frac{\partial F^0}{\partial \underline{u}}, \ \frac{\partial F^0}{\partial \underline{x}_1}, \ -\frac{\partial F^0}{\partial \underline{v}}, \ -\frac{\partial F^0}{\partial \underline{x}_2}\right)\begin{pmatrix} d\underline{u} \\ d\underline{x}_1 \\ d\underline{v} \\ d\underline{x}_2 \end{pmatrix} \geqslant 0 \tag{28}$$

for any vector differentials $\{d\underline{u}, d\underline{x}_1, d\underline{v}, d\underline{x}_2\}$ satisfying

$$\left(-\frac{\partial \underline{g}^{10}}{\partial \underline{u}}, \ \underline{0}, \ \frac{\partial \underline{g}^{10}}{\partial \underline{v}}, \ \underline{0}\right)\begin{pmatrix} d\underline{u} \\ d\underline{x}_1 \\ d\underline{v} \\ d\underline{x}_2 \end{pmatrix} \geqslant \underline{0} \tag{29}$$

$$\left(-\frac{\partial \underline{f}^0}{\partial \underline{u}}, \ -\frac{\partial \underline{f}^0}{\partial \underline{x}_1}, \ \frac{\partial \underline{f}^0}{\partial \underline{v}}, \ \frac{\partial \underline{f}^0}{\partial \underline{x}_2}\right)\begin{pmatrix} d\underline{u} \\ d\underline{x}_1 \\ d\underline{v} \\ d\underline{x}_2 \end{pmatrix} = \underline{0} \tag{30}$$

Therefore by application of the Farkas' lemma to Eqs. (28) (29) (30) we have for some vector $\underline{\lambda}^{10} \geqslant \underline{0}$ and $\underline{\psi}$=unrestricted.

$$\left(\frac{\partial F^0}{\partial \underline{u}},\ \frac{\partial F^0}{\partial \underline{x}_1},\ -\frac{\partial F^0}{\partial \underline{v}},\ \frac{\partial F^0}{\partial \underline{x}_2}\right)^{\mathrm{T}}$$

$$=\left(-\frac{\partial g^{10}}{\partial \underline{u}},\ 0,\ -\frac{\partial g^{10}}{\partial \underline{v}},\ \underline{0}\right)^{\mathrm{T}} \underline{\lambda}^{10}+\left(-\frac{\partial f^0}{\partial \underline{u}},\ -\frac{\partial f^0}{\partial \underline{x}_1},\ \frac{\partial f^0}{\partial \underline{v}},\ \frac{\partial f^0}{\partial \underline{x}_2}\right)^{\mathrm{T}} \underline{\psi}^0 \tag{31}$$

By adding zeros as components to $\underline{\lambda}^{10}$ we have

$$\left(\frac{\partial F^0}{\partial \underline{u}},\ \frac{\partial F^0}{\partial \underline{x}_1},\ -\frac{\partial F^0}{\partial \underline{v}},\ -\frac{\partial F^0}{\partial \underline{x}_2}\right)^{\mathrm{T}}$$

$$=\left(-\frac{\partial \underline{g}^0}{\partial \underline{u}},\ \underline{0},\ -\frac{\partial \underline{g}^0}{\partial \underline{v}},\ \underline{0}\right)^{\mathrm{T}} \underline{\lambda}^0+\left(-\frac{\partial f^0}{\partial \underline{u}},\ -\frac{\partial f^0}{\partial \underline{x}_1},\ \frac{\partial f^0}{\partial \underline{v}},\ \frac{\partial f^0}{\partial \underline{x}_2}\right)^{\mathrm{T}} \underline{\psi}^0 \tag{32}$$

Dimension of $\underline{\lambda}$ is equal to the number of columns of the transposed matrix $\left(-\frac{\partial \underline{g}}{\partial \underline{u}},\ \underline{0},\ -\frac{\partial \underline{g}}{\partial \underline{v}},\ \underline{0}\right)$ which is equal to dimension of $\underline{g}$. Similary, dimension of $\underline{\psi}$ is equal to dimension of $\underline{f}$. Eq. (32) may be written separately as Eqs. (20) (21) and Eqs. (22) (23) in which $\bar{\underline{\psi}}$, $\bar{\underline{\lambda}}$ are replaced by $\underline{\psi}$, $\underline{\lambda}$, respectively. Furthermore, since Eq. (21) and Eq. (23) appear to be the same type of equation, they can be expressed as Eq. (9) together with. Other conditions may be derived in the analogous manner. Hence the necessary conditions (20) to (27) are stated briefly as Eq. (9) to (13).

Theorem 1 may be regarded as Kuhn-Tucker condition for the game.

§ 3. Necessary Conditions for Pure Strategies of Dynamic Game

We will consider a dynamic game in which a state is determined by a system of differential equations

$$\frac{d\underline{x}}{dt}=\underline{f}(\underline{x},\ \underline{u},\ \underline{v}),\quad \underline{x}(t_0)=\underline{x}_0 \tag{33}$$

where $\underline{x} \in X$ is an n-state vector, $\underline{u} \in U$ is an r-dimensional minimizer and $\underline{v} \in V$ is a q-dimensional maximizer. The vectors $\underline{u}$, $\underline{v}$ are constrained by an inequality constraint.

$$\underline{g}(\underline{u},\ \underline{v}) \leqslant \underline{0} \tag{34}$$

A pay-off function of the game is given by the integral form

$$P=x_0=\int_{t_0}^{t_1} f_0(\underline{x},\ \underline{u},\ \underline{v})dt \tag{35}$$

Then our problem is to determine the function $\underset{\sim}{u}(t)$ which minimizes P subject to the process equation (33) and the constraint (34) and the function $\underset{\sim}{v}(t)$ which minimizes P. The following assumptions are made:

(i) The functions f_0, f, g are of the class $C^{(2)}$ with respect to $\underset{\sim}{x}$, $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

(ii) Dimension of g is less than dimension of $\underset{\sim}{u}$ plus dimension of $\underset{\sim}{v}$, i.e. $s<r+q$.

(iii) A matrix $[\partial g_i/\partial w_j]$, $\underset{\sim}{w}=\{\underset{\sim}{u}, \underset{\sim}{v}\}$, has maximum rank at each point in $X\times U\times V$.

(iv) Let the minimizer $\underset{\sim}{u}$ and the maximizer $\underset{\sim}{v}$ be admissible, where the word "admissible" means belonging to the bounded and closed set and yielding the trajectory which is a piece-wise continuously differentiable function with finite derivative values and satisfies all constraints.

Let us define a Hamiltonian function

$$H(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{v}, \psi_0, \underset{\sim}{\psi})=\psi_0 f_0(x, u, v)+\underset{\sim}{\psi}^{\mathsf{T}} \underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{v}) \tag{36}$$

and a pair of Lagrangian functions

$$\Phi_1(\underset{\sim}{x}, \dot{\underset{\sim}{x}}, \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{\xi}, \psi_0, \underset{\sim}{\psi}, \underset{\sim}{\lambda})=H(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{v}, \psi_0, \underset{\sim}{\psi})-\underset{\sim}{\psi}^{\mathsf{T}}\dot{\underset{\sim}{x}}+\underset{\sim}{\lambda}^{\mathsf{T}}(\underset{\sim}{g}(\underset{\sim}{u}, \underset{\sim}{v})+\underset{\sim}{\Xi}) \tag{37}$$

$$\Phi_2(\underset{\sim}{x}, \dot{\underset{\sim}{x}}, \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{\xi}, \psi_0, \underset{\sim}{\psi}, \underset{\sim}{\lambda})=H(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{v}, \psi_0, \underset{\sim}{\psi})-\underset{\sim}{\psi}^{\mathsf{T}}\dot{\underset{\sim}{x}}-\underset{\sim}{\lambda}^{\mathsf{T}}(\underset{\sim}{g}(\underset{\sim}{u}, \underset{\sim}{v})+\underset{\sim}{\Xi}) \tag{38}$$

where $\underset{\sim}{\xi}=(\xi_1, \xi_2, \cdots, \xi_s)^{\mathsf{T}}$ and $\underset{\sim}{\Xi}$ is a vector whose components are $\xi_i^2\geqslant 0$ $(i=1, 2, \cdots, s)$. Then the following theorem holds and gives necessary conditions for the solution to the differential game.

THEOREM 2. It is assumed that there exist pure strategies for the minimizer $\underset{\sim}{u}$ and the maximizer $\underset{\sim}{v}$. Let $\underset{\sim}{u}^0(t)\in U$, $\underset{\sim}{v}^0(t)\in V$, $\underset{\sim}{x}^0(t)\in X$, $t_0\leqslant t\leqslant t_1$, be an optimal admissible minimizer, an optimal admissible maximizer and the corresponding trajectory. Then there exist a constant $\psi_0\geqslant 0$ and a non-zero vector $\{\psi_0, \underset{\sim}{\psi}(t), \underset{\sim}{\lambda}(t)\}$ such that $\underset{\sim}{\psi}(t)$ and $\underset{\sim}{\lambda}(t)$ are continuous on the interval $t_0\leqslant t\leqslant t_1$, except perhaps at values of t corresponding to corners of the solution curve, where they possess unique right and left limits and satisfy the following conditions. Moreover, we can set $\psi_0=1$ assuming normality of the trajectory.

(i) (Euler equations) Along the optimal trajectory

$$\dot{\underset{\sim}{x}}=\frac{\partial H^{\mathsf{T}}}{\partial \underset{\sim}{\psi}} \tag{39}$$

$$\dot{\underset{\sim}{\psi}}=-\frac{\partial H^{\mathsf{T}}}{\partial \underset{\sim}{x}} \tag{40}$$

$$\frac{\partial H}{\partial \underset{\sim}{u}}+\underset{\sim}{\lambda}^{\mathsf{T}}\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}=\underset{\sim}{0} \tag{41}$$

$$\frac{\partial H}{\partial \underset{\sim}{v}}-\underset{\sim}{\lambda}^{\mathsf{T}}\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{v}}=\underset{\sim}{0} \tag{42}$$

$$\underset{\sim}{\lambda}^{\mathsf{T}}\underset{\sim}{g}=0 \tag{43}$$

(ii) (Result obtained from Clebsch condition)

$$\underset{\sim}{\lambda} \geqslant \underset{\sim}{0} \tag{44}$$

(iii) (Weierstrass-Erdmann's corner condition) At the corner of the solution curve $\underset{\sim}{\psi}(t)$ and $\underset{\sim}{\lambda}(t)$ have well defined one-sided limits that are equal.

(iv) (Weierstrass condition) For any element $\{\underset{\sim}{x}^0, \underset{\sim}{u}^0, \underset{\sim}{v}^0, \psi_0, \underset{\sim}{\psi}\}$ of the solution curve and for any admissible minimizer $\underset{\sim}{u}$ and admissible maximizer $\underset{\sim}{v}$

$$H(\underset{\sim}{x}^0, \underset{\sim}{u}^0, \underset{\sim}{v}^0, \psi_0, \underset{\sim}{\psi}) \leqslant H(\underset{\sim}{x}^0, \underset{\sim}{u}, \underset{\sim}{v}^0, \psi_0, \underset{\sim}{\psi}) \tag{45}$$

$$H(\underset{\sim}{x}^0, \underset{\sim}{u}^0, \underset{\sim}{v}^0, \psi_0, \underset{\sim}{\psi}) \geqslant H(\underset{\sim}{x}^0, \underset{\sim}{u}^0, \underset{\sim}{v}, \psi_0, \underset{\sim}{\psi}) \tag{46}$$

(v) (Transversality condition) $\underset{\sim}{\psi}(t_1) = \underset{\sim}{0}$ (47)

Proof. We assume that there exist pure strategies $\{\underset{\sim}{u}^0, \underset{\sim}{v}^0\}$ for the minimizer and the maximizer. Necessary conditions for the pure strategies are obtained by combining necessary conditions for the optimal minimizer $\underset{\sim}{u}^0$ given $\underset{\sim}{v} = \underset{\sim}{v}^0$ and necessary conditions for the optimal maximizer $\underset{\sim}{v}^0$ given $\underset{\sim}{u} = \underset{\sim}{u}^0$ (BERKOWITZ 1964). The necessary conditions for the optimal control obtained by variational calculus are all seen in the famous Berkowitz's paper (1961). Thus we will apply only his results to derive the necessary conditions for the pure strategies of the game.

Let us note that $\underset{\sim}{g}$ must not contain $\underset{\sim}{x}$ so as to yield the same adjoint equation both for the minimization problem and for the maximization one.*

Suppose that $\underset{\sim}{v} = \underset{\sim}{v}^0$ be fixed. Then using Φ_1 defined by (37) and applying the necessary condition for the optimal control (minimization) constrained by the inequality (34), we obtain the following condition in the case when $\underset{\sim}{g}$ does not depend on $\underset{\sim}{x}$. That is, from Euler equation and Clebsch condition for the optimal minimizer we have

$$\dot{\underset{\sim}{x}} = \frac{\partial H^{\mathrm{T}}}{\partial \underset{\sim}{\psi}} \tag{48}$$

$$\dot{\underset{\sim}{\psi}} = -\frac{\partial H^{\mathrm{T}}}{\partial \underset{\sim}{x}} \tag{49}$$

$$\frac{\partial H}{\partial \underset{\sim}{u}} + \underset{\sim}{\lambda}^{\mathrm{T}} \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}} = \underset{\sim}{0} \tag{50}$$

$$\underset{\sim}{g} \leqslant \underset{\sim}{0}, \qquad \underset{\sim}{\lambda}^{\mathrm{T}} \underset{\sim}{g} = \underset{\sim}{0}, \qquad \underset{\sim}{\lambda} \geqslant \underset{\sim}{0} \tag{51}$$

along the optimal trajectory. In above non-negativity of $\underset{\sim}{\lambda}$, being defferent from the Berkowitz's paper, is caused by direction of the inequality.

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* When $\underset{\sim}{g}$ contains $\underset{\sim}{x}$ as its argument, it becomes that $\Phi_{1\underset{\sim}{x}} \neq \Phi_{2\underset{\sim}{x}}$ and no matter how signs of $\psi_0, \underset{\sim}{\psi}, \underset{\sim}{\lambda}$ in Φ_2 are adjusted, we cannot set right description of stationarity conditions in which signs of $\underset{\sim}{\lambda}$ for minimization and maximization problems are consistent to each other.

Next suppose that $\underset{\sim}{u}=\underset{\sim}{u}^0$ be fixed. Define the following Lagrangian and Hamiltonian functions

$$\bar{\Phi}_2=\bar{H}-\bar{\underset{\sim}{\phi}}^{\mathrm{T}}\dot{\underset{\sim}{x}}-\bar{\underset{\sim}{\lambda}}^{\mathrm{T}}(\underset{\sim}{g}+\underset{\sim}{\xi})$$
$$\bar{H}=f_0(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{v})+\bar{\underset{\sim}{\phi}}^{\mathrm{T}}\underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{v}) \tag{52}$$

we will apply the Berkowitz's theorem to our maximization problem under the same constraints. Then corresponding to maximization, direction of an inequality in Clebsch condition changes to the opposite direction for the minimization. But since the third term of $\bar{\Phi}_2$ is defined with negative sign contrary to that of Φ_1, we get $\bar{\underset{\sim}{\lambda}}\geqslant\underset{\sim}{0}$, consequently. Namely as the Euler equation and the Clebsch condition for the optimal maximizer the following must be held along the optimal trajectory.

$$\dot{\underset{\sim}{x}}=\frac{\partial\bar{H}^{\mathrm{T}}}{\partial\bar{\underset{\sim}{\phi}}} \tag{53}$$

$$\dot{\bar{\underset{\sim}{\phi}}}=-\frac{\partial\bar{H}^{\mathrm{T}}}{\partial\underset{\sim}{x}} \tag{54}$$

$$\frac{\partial\bar{H}}{\partial\underset{\sim}{v}}-\bar{\underset{\sim}{\lambda}}^{\mathrm{T}}\frac{\partial\underset{\sim}{g}}{\partial\underset{\sim}{v}}=\underset{\sim}{0} \tag{55}$$

$$\underset{\sim}{g}\leqslant\underset{\sim}{0}, \qquad \bar{\underset{\sim}{\lambda}}^{\mathrm{T}}\underset{\sim}{g}=\underset{\sim}{0}, \qquad \bar{\underset{\sim}{\lambda}}\geq\underset{\sim}{0} \tag{56}$$

It is easily proved, however, that we can set $\bar{\underset{\sim}{\phi}}\equiv\underset{\sim}{\phi}$ and $\bar{\underset{\sim}{\lambda}}\equiv\underset{\sim}{\lambda}$. First, proof of $\bar{\underset{\sim}{\phi}}\equiv\underset{\sim}{\phi}$ is made with the manner similar to Berkowitz's (1964). From an evident transversality condition for the free terminal state problem, we have $\underset{\sim}{\phi}(t_1)\equiv\bar{\underset{\sim}{\phi}}(t_1)$ and furthermore since $\underset{\sim}{\phi}(t)$ and $\bar{\underset{\sim}{\phi}}(t)$ are a solution to the same linear homogeneous differential equation, we get $\bar{\underset{\sim}{\phi}}(t)\equiv\underset{\sim}{\phi}(t)$ on the interval $t_0\leqslant t\leqslant t_1$.

On the other hand, as to $\underset{\sim}{\lambda}$, it is that $\underset{\sim}{\lambda}\geqslant\underset{\sim}{0}$ and $\bar{\underset{\sim}{\lambda}}\geqslant\underset{\sim}{0}$. Furthermore since $\lambda_i g_i=0$ and $\bar{\lambda}_i g_i=0$, then $\lambda_i\equiv\bar{\lambda}_i\equiv0$ when $g_i\neq0$. Therefore letting an index set of $g_i=0$ be I_0,

$$\frac{\partial H}{\partial\underset{\sim}{u}}+\sum_{i\in I_0}\lambda_i\frac{\partial g_i}{\partial\underset{\sim}{u}}=\underset{\sim}{0}, \qquad \frac{\partial\bar{H}}{\partial\underset{\sim}{v}}-\sum_{i\in I_0}\bar{\lambda}_i\frac{\partial g_i}{\partial\underset{\sim}{v}}=\underset{\sim}{0}$$

As both λ_i and $\bar{\lambda}_i$ are of the same sign and non-negative, we can take $\lambda_i=\bar{\lambda}_i\geqslant0$, $i\in I_0$ such that the above two equations are satisfied. We will state the reason below. Let l be a number of components of I_0. Since there are $(r+q+l)$ constraint equations of the above two equations plus $\{g_i=0,\ i\in I_0\}$ against $(r+q+2l)$ free variables of $\underset{\sim}{u}, \underset{\sim}{v}, \lambda_i, \bar{\lambda}_i, i\in I_0$, there remains freedom of l. Thus we can set $\lambda_i-\bar{\lambda}_i=0$ for $i\in I_0$. Thus we can set $\bar{\underset{\sim}{\lambda}}\equiv\underset{\sim}{\lambda}$ all together and obtain Eqs. (39)$\sim$(44).

We will next explain the necessary condition (iv). For fixed $\underset{\sim}{v}=\underset{\sim}{v}^0$ we have

$$E=H(\underset{\sim}{x}^0, \underset{\sim}{u}, \underset{\sim}{v}^0, \phi_0, \underset{\sim}{\phi})-H(\underset{\sim}{x}^0, \underset{\sim}{u}^0, \underset{\sim}{v}^0, \phi_0, \underset{\sim}{\phi})\geqslant0$$

from the Weierstrass condition for minimization such that Weierstrass' E-function defined for Φ_1 should be non-negative. Next for fixed $\underset{\sim}{u}=\underset{\sim}{u}^0$ we have

$$E=H(\underset{\sim}{x}^0, \underset{\sim}{u}^0, \underset{\sim}{v}, \psi_0, \underset{\sim}{\psi})-H(\underset{\sim}{x}^0, \underset{\sim}{u}^0, \underset{\sim}{v}^0, \psi_0, \underset{\sim}{\psi})\leqslant 0$$

from Weierstrass condition for maximization such that Weierstrass' E-function defined for Φ_2 should be non-positive. Therefore the minimizer $\underset{\sim}{u}$ and the maximizer $\underset{\sim}{v}$ try to minimize and to maximize H with respect to $\underset{\sim}{u}$ and $\underset{\sim}{v}$, respectively.

The necessary conditions (iii) (v) are obtained by the manner analogous to Berkowitz's (1961).

Pursuer-Evader game is formulated as follows:

$$J=\int_{t_0}^{t_1} w_1(\underset{\sim}{x}-\underset{\sim}{y})^2+w_2\underset{\sim}{u}^2-w_3\underset{\sim}{v}^2 dt \qquad (w_i\geqslant 0: \text{ weighting function})$$

$$\dot{\underset{\sim}{x}}=f(\underset{\sim}{x}, \underset{\sim}{u}),\ \underset{\sim}{x}(t_0)=\underset{\sim}{x}_0$$

$$\dot{\underset{\sim}{y}}=f(\underset{\sim}{y}, \underset{\sim}{v}),\ \underset{\sim}{y}(t_0)=\underset{\sim}{y}_0$$

$$\underset{\sim}{g}_1(\underset{\sim}{u})\leqslant \underset{\sim}{0}$$

$$\underset{\sim}{g}_2(\underset{\sim}{v})\leqslant \underset{\sim}{0}$$

where $\underset{\sim}{x}$ is a pursuer, $\underset{\sim}{y}$ is an evader, and $\underset{\sim}{u}$, $\underset{\sim}{v}$ are respective controls. We can apply Theorem 2 directly to the problem like above. When $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are separated with respect to constraints as in above, one can formulate the problem by use of only one Lagrangian function. In order to understand meaning of Lagrange multipliers and its sign corresponding to inequality constraints, still it is convenient to consider a pair of Lagrangian functions.

Finally let us refer to the relationship between the necessary conditions of the static game and the dynamic game. Although we have studied these two types of problems individually, it can be shown that the necessary condition for the static problem (Theorem 1) i.e. the Kuhn-Tucker condition, is derived from the necessary conditions for the dynamic one (Theorem 2), i.e., the Euler, Clebsch & Weierstrass conditions. By this fact we can consider the static and dynamic optimization as a problem of the same category. (See proof in appendix.)

§ 5. Conclusion

The necessary optimality conditions for the static game and the dynamic one were derived from application of Kuhn-Tucker Theorem and Berkowitz's Theorem, respectively. Characteristics of our derivation here was to use a pair of Lagrangian functions.

APPENDIX

On the Relationship between Kuhn-Tucker Condition and Euler, Clebsch & Weierstrass Conditions.

In this section we will discuss the relationship between necessary conditions for static optimal control (Kuhn-Tucker condition) and for dynamic one (Euler, Clebsch & Weierstrass conditions). Although we consider only a minimization problem for simplicity of description below, the same relationship may be obtained for pure strategies of the game also.

The dynamic problem is defined.

$$\min_{u} \int_{t_0}^{t_1} F(x, u)dt \tag{A. 1}$$

$$\text{subj. to } \dot{x}=f_1(x, u) \qquad \text{i.e.} \qquad f(x, \dot{x}, u)=0 \tag{A. 2}$$

$$g(x, u)\leqslant 0 \qquad (g\in E^s) \tag{A. 3}$$

Define Hamiltonian and Lagrangian functions as follows, respectively.

$$H(x, u, \psi)=F(x, u)+\psi^{\mathrm{T}}f_1(x, u)$$

$$\Phi(x, \dot{x}, u, \dot{\xi}, \psi, \lambda)=F(x, u)+\psi^{\mathrm{T}}f(x, \dot{x}, u)+\lambda^{\mathrm{T}}(g(x, u)+\Xi) \tag{A. 4}$$

where Ξ is a vector which has $\xi_i^2\geqslant 0$ $(i=1, \cdots, s)$ as its components and $\xi=(\xi_1, \cdots, \xi_s)^{\mathrm{T}}$. Then necessary optimality conditions are given by the following theorem (It is assumed that appropriate assumptions like in § 2, 3 are made).

THEOREM A. 1. (BERKOWITZ 1961) There exists an non-zero vector $\{\psi(t), \lambda(t)\}\neq 0$, defined on the interval $t_0\leqslant t\leqslant t_1$, such that $\psi(t)$ and $\lambda(t)$ are continuous, except perhaps at values of t corresponding to corners of a solution curve where they possess unique right and left limits and satisfy the following conditions (for simplicity of description we consider only the case when a solution curve is normal, i.e. $\psi_0=1$):

(i) (Euler equation)

$$\frac{\partial \Phi}{\partial \psi}=0 \tag{A. 5}$$

$$\frac{\partial \Phi}{\partial x}-\frac{d}{dt}\frac{\partial \Phi}{\partial \dot{x}}=0 \tag{A. 6}$$

$$\frac{\partial \Phi}{\partial u}=0 \tag{A. 7}$$

$$\frac{\partial \Phi}{\partial \lambda}=0\rightarrow g\leqslant 0, \qquad \frac{\partial \Phi}{\partial \xi}\frac{d}{dt}\frac{\partial \Phi}{\partial \dot{\xi}}=0\rightarrow \lambda^{\mathrm{T}}g=0 \tag{A. 8}$$

(ii) (Clebsch condition)

$$\underline{\lambda} \geqslant \underline{0} \tag{A. 9}$$

(iii) (Weierstrass-Erdmann's corner condition) At the corner of the optimal trajectory, $\underline{\psi}(t)$ and $\underline{\lambda}(t)$ have well defined one sided limits that are equal.

(iv) (Weierstrass condition) From Weierstrass' E function, $E \geqslant 0$, we obtain

$$H(\underline{x}^0, \underline{u}^0, \underline{\psi}) \leqslant H(\underline{x}^0, \underline{u}, \underline{\psi}) \tag{A. 10}$$

(v) (Transversality condition)

$$\underline{\psi}(t_1) = \underline{0}$$

On the other hand the static problem is defined:

$$\min_{\underline{u}} F(\underline{x}, \underline{u}) \tag{A. 11}$$

$$\text{subj. to } \underline{f}(\underline{x}, \underline{u}) = \underline{0} \tag{A. 12}$$

$$\underline{g}(\underline{x}, \underline{u}) \leqslant \underline{0} \tag{A. 13}$$

If we consider that for every time $t \in [t_0, t_1]$ Eqs. (A. 11)$\sim$(A. 13) hold, the above static problem can be considered as

$$\min_{\underline{u}(t)} \int_{t_0}^{t_1} F(\underline{x}(t), \underline{u}(t))dt \tag{A. 14}$$

$$\text{subj. to } \underline{f}(\underline{x}(t), \underline{u}(t)) = \underline{0} \tag{A. 15}$$

$$\underline{g}(\underline{x}(t), \underline{u}(t)) \leqslant \underline{0} \tag{A. 16}$$

But since Eq. (A. 15) does not include $\dot{\underline{x}}$ as its argument, $\underline{u}^0(t)$, minimizing Eq. (A. 14) ought to be constant on the interval $t_0 \leqslant t \leqslant t_1$. Therefore $\underline{u}^0(t)$, $\underline{x}^0(t)$, the solution to the system of (A. 14)$\sim$(A. 16), become equal to $\underline{u}^0$, $\underline{x}^0$, the solution to the system of (A. 11)$\sim$(A. 13). That is, $\underline{u}^0(t) \equiv \underline{u}^0$, $\underline{x}^0(t) \equiv \underline{x}^0$ for every $t \in [t_0, t_1]$. Define the Lagrangian function (A. 4) for the system of (A. 14)$\sim$(A. 16):

$$\Phi(\underline{x}, \underline{u}, \underline{\xi}, \underline{\psi}, \underline{\lambda}) = F(\underline{x}, \underline{u}) + \underline{\psi}^{\mathrm{T}} \underline{f}(\underline{x}, \underline{u}) + \underline{\lambda}^{\mathrm{T}} (\underline{g} + \underline{\Xi}) \tag{A. 17}$$

Then from the fact that Φ does not include $\dot{\underline{x}}$, we obtain

(i) As the Euler equation

$$\frac{\partial \Phi}{\partial \underline{\psi}} = \underline{0} \tag{A. 18}$$

$$\frac{\partial \Phi}{\partial \underline{x}} = \underline{0} \tag{A. 19}$$

$$\frac{\partial \Phi}{\partial \underline{u}} = \underline{0} \tag{A. 20}$$

$$\frac{\partial \Phi}{\partial \underline{\lambda}}=\underline{0} \rightarrow \underline{g} \leqslant \underline{0}, \qquad \frac{\partial \Phi}{\partial \underline{\xi}}-\frac{d}{dt}\frac{\partial \Phi}{\partial \underline{\dot{\xi}}}=\underline{0} \rightarrow \lambda_i g_i=0 \rightarrow \underline{\lambda}^{\mathrm{T}} \underline{g}=0 \tag{A. 21}$$

(ii) The original statement of the Clebsch condition is as follows (BLISS 1946). For every non-zero vector $\{\underline{\pi}, \underline{\rho}, \underline{k}\} \neq \underline{0}$ that is a solution to the linear system

$$\frac{\partial \underline{f}_1}{\partial \underline{u}} \underline{\rho}-E \underline{\pi}=\underline{0} \qquad (E \text{ is a unit matrix}) \tag{A. 22. a}$$

$$\frac{\partial \underline{g}}{\partial \underline{u}} \underline{\rho}+2 \frac{\partial \underline{\Xi}}{\partial \underline{\xi}} \underline{k}=\underline{0} \tag{A. 22. b}$$

the following inequality holds.

$$\underline{\pi}^{\mathrm{T}} \frac{\partial^2 \Phi}{\partial \underline{\dot{x}}^2} \underline{\pi}+\underline{\rho}^{\mathrm{T}} \frac{\partial^2 \Phi}{\partial \underline{u}^2} \underline{\rho}+2 \sum_{i=1}^{s} \lambda_i k_i^2 \geqslant 0 \tag{A. 23}$$

Now apply this condition to the static problem by substituting $\underline{f}$ for $\underline{f}_1$ in Eq. (A. 22. a). Then for every vector $\{\underline{\pi}, \underline{\rho}, \underline{k}\} \neq \underline{0}$ it is necessary that the following inequality holds.

$$\underline{\rho}^{\mathrm{T}} \frac{\partial^2 \Phi}{\partial \underline{u}^2} \underline{\rho}+2 \sum_{i=1}^{s} \lambda_i k_i^2 \geqslant 0 \tag{A. 24}$$

If $g_i \leqslant 0$ at the optimal point $\{\underline{x}^0, \underline{u}^0\}$ then by Eq. (A. 21) $\lambda_i=0$. If $g_i=0$ at this point, let $\underline{\pi}=\underline{0}$, let $\underline{\rho}=\underline{0}$ and let $\underline{\kappa}$ be a vector whose i-th component is equal to one and whose others are zero. Then $\{\underline{\pi}, \underline{\rho}, \underline{\kappa}\} \neq \underline{0}$ and since $\xi_i^2=0$ by the fact that $g_i-\xi_i^2=0$, $\{\underline{\pi}, \underline{\rho}, \underline{\kappa}\}$ selected as above is a solution to the system of (A. 22. a, b). Hence from Eq. (A. 24) we obtain $\lambda_i \geqslant 0$. Consequently we always have

$$\underline{\lambda} \geqslant \underline{0} \tag{A. 25}$$

(iii) From the definition of the Weierstrass' E function

$$E=\Phi(\underline{x}^0, \underline{u}, \underline{\dot{\xi}}, \underline{\psi}, \underline{\lambda})-\Phi(\underline{x}^0, \underline{u}^0, \underline{\dot{\xi}}^0, \underline{\psi}, \underline{\lambda})$$

$$-\frac{\partial \Phi}{\partial \underline{\dot{x}}}(\underline{x}^0, \underline{u}^0, \underline{\dot{\xi}}^0, \underline{\psi}, \underline{\lambda})(\underline{\dot{x}}-\underline{\dot{x}}^0)-\frac{\partial \Phi}{\partial \underline{u}}(\underline{x}^0, \underline{u}^0, \underline{\dot{\xi}}^0, \underline{\psi}, \underline{\lambda})(\underline{u}-\underline{u}^0)$$

$$-\frac{\partial \Phi}{\partial \underline{\dot{\xi}}}(\underline{x}^0, \underline{u}^0, \underline{\dot{\xi}}^0, \underline{\psi}, \underline{\lambda})(\underline{\dot{\xi}}-\underline{\dot{\xi}}^0) \geqslant 0$$

Since $\partial \Phi / \partial \underline{\dot{x}}=\underline{0}$ and along the optimal trajectory $\partial \Phi^0 / \partial \underline{u}=\underline{0}$, $\partial \Phi^0 / \partial \underline{\dot{\xi}}=\underline{0}$ and $\underline{g}+\underline{\Xi}=\underline{0}$, we get

$$E=F(\underline{x}^0, \underline{u})+\underline{\psi}^{\mathrm{T}} \underline{f}(\underline{x}^0, \underline{u})-F(\underline{x}^0, \underline{u}^0)-\underline{\psi}^{\mathrm{T}} \underline{f}(\underline{x}^0, \underline{u}^0) \geqslant 0 \tag{A. 26}$$

Hence, $F(\underline{x}^0, \underline{u})+\underline{\psi}^{\mathrm{T}} \underline{f}(\underline{x}^0, \underline{u}) \geqslant F(\underline{x}^0, \underline{u}^0)+\underline{\psi}^{\mathrm{T}} \underline{f}(\underline{x}^0, \underline{u}^0)$.

Obviously the conditions (iii), (iv) are not necessary for the static problem.

Finally it is noted that the statement of the Euler Equation and the Clebsch condition, Eqs. (A. 18)~(A. 21), (A. 25) is surely the statement of the Kuhn-Tucker condition which was given in §2 previously.

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