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BY

MAKOTO MAEJIMA

FACULTY OF ENGINEERING
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SOME LIMIT THEOREMS FOR RENEWAL PROCESSES WITH NON-IDENTICALLY DISTRIBUTED RANDOM VARIABLES

MAKOTO MAEJIMA

Dept. of Administration Engineering, Keio University, Yokohama 223

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ABSTRACT

This paper deals with renewal processes with non-identically distributed random variables. Some limit theorems, which arise in connection with the distribution problems of the age, the lifetime and the residual lifetime of the items in a renewal process, are given. A central limit theorem for the renewal number is also shown.

1. Let $\{X_i, i=1, 2, \cdots\}$ be a sequence of independent and nonnegative random variables with $0<EX_i=\mu_i<\infty$, and let $F_i(x)$ be the distribution function of X_i. Renewal processes with independent and non-identically distributed random variables have been investigated by several authors (KAWATA 1956, 1961; HATORI 1959, 1960; CHOW and ROBBINS 1963; SMITH 1964).

When

$$\mu=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu_i \tag{1. 1}$$

exists, KAWATA (1956) proved, without assuming the nonnegativeness of X_i, that

$$\lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{T}\sum_{n=1}^{\infty}Pr(x<S_n\leq x+h)dx=\frac{h}{\mu}, \quad (h>0), \tag{1. 2}$$

where

$$S_n=\sum_{i=1}^{n}X_i,$$

under some conditions.

In this paper we shall prove a theorem which includes KAWATA's result. Furthermore, using this result, we shall give some limit theorems for the distributions of the age, the lifetime and the residual lifetime of the items in a renewal process. In the last section we shall show a central limit theorem for the renewal number in a renewal process with non-identically distributed variables.

2. Throughout the paper we use the following notations.

$N(t)=\sup(n\geq 0, S_n\leq t)$, that is, $N(t)$ is the renewal number in $(0, t]$. $(S_0\equiv 0)$.

$U(t)=t-S_{N(t)}$, that is, $U(t)$ is the age at the epoch t.

$V(t)=S_{N(t)+1}-t$, that is, $V(t)$ is the residual lifetime at the epoch t.

$X(t)=S_{N(t)+1}-S_{N(t)}$, that is, $X(t)$ is the lifetime containing the epoch t.

Furthermore let us define

$$F(t)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}F_i(t), \tag{2.1}$$

when the right hand side exists as the weak limit. $F(t)$ is not necessarily a distribution function. And set

$$\sigma_n(t)=Pr(S_n\leq t).$$

Now we state some lemmas.

LEMMA 1. Suppose $f(t)\geq 0$,

$$\int_0^\infty e^{-st}f(t)dt\sim\frac{C}{s^\gamma}, \qquad \text{as} \quad s\downarrow 0,$$

for some nonnegative number γ, then

$$\int_0^t f(u)du\sim\frac{Ct^\gamma}{\Gamma(\gamma+1)}, \qquad \text{as} \quad t\to\infty.$$

This is a well known Tauberian theorem (see e.g. WIDDER [1946, p. 192]).

LEMMA 2. Let $\{X_i, i=1, 2, \cdots\}$ be a sequence of independent and nonnegative random variables such that $0<EX_i=\mu_i<\infty$. Suppose that the distribution function $F_i(x)$ of X_i satisfies that

$$\lim_{\xi\to\infty}\int_\xi^\infty xdF_i(x)=0 \tag{2.2}$$

uniformly with respect to i. If (1. 1) exists, then

$$\lim_{s\downarrow 0}s^{\alpha+1}\sum_{n=1}^\infty n^\alpha\varphi_n(s)=\frac{\alpha!}{\mu^{\alpha+1}} \qquad \text{for} \quad \alpha=0, 1, 2, \cdots,$$

where

$$\varphi_n(s)=\int_0^\infty e^{-st}d\sigma_n(t).$$

This lemma was first proved by KAWATA (1956) in the case $\alpha=0$, and HATORI (1960) proved it for any nonnegative integer α.

NOTE. KAWATA proved (1. 2) under the same conditions as in Lemma 2.

3. In this section we shall show a theorem which includes KAWATA's result. Let us consider functions $G_n(t)$ $(n=1, 2, \cdots)$ which are defined for $t\geq 0$ with $G_n(0)=0$, and are uniformly bounded, nonnegative, nondecreasing over $[0, R]$, and zero elsewhere, R being some positive number. Furthermore we define

$$G(t)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}G_i(t)$$

when the right hand side exists weakly.

Form

$$H(t)=\sum_{n=1}^{\infty}\int_0^t G_{n+1}(t-y)d\sigma_n(y). \tag{3. 1}$$

Letting $G_n(t)\leq M$, we have

$$H(t)\leq M\sum_{n=1}^{\infty}\int_0^t d\sigma_n(y)=M\sum_{n=1}^{\infty}\sigma_n(t),$$

for which KAWATA (1956, p. 125) proved that

$$\sum_{n=1}^{\infty}\sigma_n(t)<\infty, \tag{3. 2}$$

if (2. 2) holds uniformly with respect to i, and hence the convergence of (3. 1) follows from (3. 2). Then we obtain a following theorem.

THEOREM 1. *Let $\{X_i, i=1, 2, \cdots\}$ be a sequence of independent and nonnegative random variables such that $0<EX_i=\mu_i<\infty$. Suppose that the distribution function $F_i(x)$ of X_i satisfies that (2. 2) holds uniformly with respect to i. Let $G_n(t)$ $(n=1, 2, \cdots)$ be functions which are defined for $t\geq 0$ with $G_n(0)=0$, and are uniformly bounded, nonnegative, nondecreasing over $[0, R]$, and zero elsewhere, R being some positive number. If*

$$\mu=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu_i \tag{1. 1}$$

exists and

$$G(t)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}G_i(t) \tag{3. 3}$$

holds weakly, then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T H(t)dt=\frac{1}{\mu}\int_0^\infty G(t)dt,$$

where

$$H(t)=\sum_{n=1}^{\infty}\int_0^t G_{n+1}(t-y)d\sigma_n(y),$$

(the convergence of which was assured above).

Proof. Let us denote the Laplace-Stieltjes transforms of $H(t)$, $G_n(t)$, and $\sigma_n(t)$ by

$$h(s)=\int_0^\infty e^{-st}dH(t), \qquad s\geq 0,$$

$$g_n(s)=\int_0^\infty e^{-st}dG_n(t), \qquad s\geq 0$$

and

$$\varphi_n(s)=\int_0^\infty e^{-st}d\sigma_n(t), \qquad s\geq 0,$$

respectively. Then we have

$$h(s)=\sum_{n=1}^{\infty} g_{n+1}(s)\varphi_n(s).$$

Integrating by parts, we get

$$g_n(s)=\int_0^\infty e^{-st}dG_n(t)$$

$$=s\int_0^\infty e^{-st}G_n(t)dt.$$

Here we put

$$d_n(s)=\int_0^\infty e^{-st}G_n(t)dt.$$

Then

$$h(s)=\sum_{n=1}^{\infty} sd_{n+1}(s)\varphi_n(s).$$

By the dominated convergence theorem, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}d_i(s)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}\int_0^\infty e^{-st}G_i(t)dt$$

$$=\int_0^\infty e^{-st}\left[\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}G_i(t)\right]dt$$

$$=\int_0^\infty e^{-st}G(t)dt\equiv d(s).$$

We can describe

$$\frac{1}{n}\sum_{i=2}^{n+1}d_i(s)=d(s)+\varepsilon_n(s), \tag{3.4}$$

where $\varepsilon_n(s)\to 0$ as $n\to\infty$ for any s. Furthermore, since

$$\left|d(s)-\frac{1}{n}\sum_{i=2}^{n+1}d_i(s)\right|$$

$$=\left|\int_0^\infty e^{-st}G(t)dt-\frac{1}{n}\sum_{i=2}^{n+1}\int_0^\infty e^{-st}G_i(t)dt\right|$$

$$\leq\int_0^\infty |e^{-st}|\left|G(t)-\frac{1}{n}\sum_{i=2}^{n+1}G_i(t)\right|dt$$

$$\leq\int_0^R\left|G(t)-\frac{1}{n}\sum_{i=2}^{n+1}G_i(t)\right|dt,$$

it follows from (3. 3) that

$$\frac{1}{n}\sum_{i=2}^{n+1}d_i(s)$$

converges to $d(s)$ uniformly with respect to s as $n\to\infty$. From (3. 4), we have

$$d_{n+1}(s)=d(s)+n\varepsilon_n(s)-(n-1)\varepsilon_{n-1}(s),$$

and consequently

$$\sum_{n=1}^\infty d_{n+1}(s)\varphi_n(s)=d(s)\sum_{n=1}^\infty\varphi_n(s)$$

$$+\sum_{n=1}^\infty\varphi_n(s)(n\varepsilon_n(s)-(n-1)\varepsilon_{n-1}(s)).$$

We also have

$$\lim_{s\downarrow 0}h(s)=\lim_{s\downarrow 0}\sum_{n=1}^\infty sd_{n+1}(s)\varphi_n(s) \tag{3. 5}$$

$$=\lim_{s\downarrow 0}sd(s)\sum_{n=1}^\infty\varphi_n(s)$$

$$+\lim_{s\downarrow 0}\sum_{n=1}^\infty s\varphi_n(s)(n\varepsilon_n(s)-(n-1)\varepsilon_{n-1}(s)).$$

First, using Lemma 2 ($\alpha=0$) and the dominated convergence theorem, we see that

$$\lim_{s\downarrow 0}sd(s)\sum_{n=1}^\infty\varphi_n(s)=\frac{1}{\mu}\lim_{s\downarrow 0}d(s) \tag{3. 6}$$

$$=\frac{1}{\mu}\lim_{s\downarrow 0}\int_0^\infty e^{-st}G(t)dt$$

$$=\frac{1}{\mu}\int_0^\infty\left[\lim_{s\downarrow 0}e^{-st}G(t)\right]dt$$

$$=\frac{1}{\mu}\int_0^\infty G(t)dt.$$

Second, noticing that

$$\sum_{n=1}^{\infty} |\varphi_n(s) n\varepsilon_n(s)| \leq \sup_{n=1,2,\cdots} |\varepsilon_n(s)| \sum_{n=1}^{\infty} n\varphi_n(s) < \infty,$$

we get

$$\sum_{n=1}^{\infty} \varphi_n(s)(n\varepsilon_n(s)-(n-1)\varepsilon_{n-1}(s))$$

$$=\sum_{n=1}^{\infty} n\varepsilon_n(s)(\varphi_n(s)-\varphi_{n+1}(s))$$

$$=\sum_{n=1}^{\infty} n\varepsilon_n(s)\varphi_n(s)(1-f_{n+1}(s)),$$

where

$$f_n(s)=\int_0^{\infty} e^{-st}dF_n(t).$$

Now, for any given positive number ε, we can choose an integer N independently of s such that

$$|\varepsilon_n(s)|<\varepsilon \qquad \text{for} \quad n>N,$$

in view of the fact proved above that

$$\frac{1}{n}\sum_{i=2}^{n+1} d_i(s)$$

converges to $d(s)$ uniformly with respect to s. On the other hand, KAWATA (1956) proved a following statement: there are positive constants C_1 and s_1 such that $m_n<C_1$ for $n=1, 2, \cdots$ and

$$f_n(s)=1-sm_n+s\eta_n,$$

where

$$|\eta_n|<\varepsilon(C_1+2)\equiv C_2 \qquad \text{for} \quad 0\leq s\leq s_1$$

uniformly with respect to n. Using this result, we have

$$\left| s\sum_{n=1}^{\infty} n\varepsilon_n(s)\varphi_n(s)(1-f_{n+1}(s))\right|$$

$$< s^2C(C_1+C_2)\sum_{n=1}^{N} n|\varepsilon_n(s)|+\varepsilon(C_1+C_2)s^2\sum_{n=1}^{\infty} n\varphi_n(s)$$

for $0\leq s\leq s_1$, where

$$C=\sup_{\substack{0\leq s\leq s_1\\ n=1,2,\cdots,N}} \varphi_n(s)<\infty.$$

By Lemma 2 ($\alpha=1$), we know

$$\lim_{s\downarrow 0} s^2 \sum_{n=1}^{\infty} n\varphi_n(s) = \frac{1}{\mu^2}.$$

Hence

$$\overline{\lim_{s\downarrow 0}} \left| s \sum_{n=1}^{\infty} n\varepsilon_n(s)\varphi_n(s)(1-f_{n+1}(s)) \right| \leq \varepsilon(C_1+C_2)\frac{1}{\mu^2}$$

and since ε is arbitrary, we have

$$\overline{\lim_{s\downarrow 0}}\, s \sum_{n=1}^{\infty} \varphi_n(s)(n\varepsilon_n(s)-(n-1)\varepsilon_{n-1}(s))=0 \tag{3. 7}$$

From (3. 5)—(3. 7), we obtain

$$\lim_{s\downarrow 0} h(s) = \frac{1}{\mu}\int_0^{\infty} G(t)dt. \tag{3. 8}$$

On the other hand, integrating by parts, we have

$$h(s) = \int_0^{\infty} e^{-st} dH(t)$$

$$= s\int_0^{\infty} e^{-st} H(t)dt \qquad \text{for} \quad 0 \leq s \leq s_2,$$

(for details, see KAWATA [1956]). Therefore, from (3. 8) we finally obtain

$$\lim_{s\downarrow 0} s\int_0^{\infty} e^{-st}H(t)dt = \frac{1}{\mu}\int_0^{\infty} G(t)dt,$$

which, because of Lemma 1 ($\gamma=1$), gives us the required result.

REMARK. Putting

$$G_n(t) = \frac{1}{h} \qquad \text{for} \quad 0 < t \leq h \ (h > 0),$$

$$= 0 \qquad \text{elsewhere}$$

in Theorem 1, we get (1. 2), which is no more than KAWATA's result.

4. Using the result given in the previous section, we shall prove some theorems concerning the distributions of the age, the lifetime and the residual lifetime of the items in a renewal process.

THEOREM 2. *Under the condition that* (1. 1) *and* (2. 1) *exist and* (2. 2) *holds uniformly with respect to* i, *we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{T} Pr(U(t) \leq x)dt = \frac{1}{\mu}\int_0^{x} [1-F(t)]dt. \tag{4. 1}$$

Proof. Suppose that $x<t$. The event $\{U(t)\le x\}$ occurs if and only if $S_n=y(\le t)$ and $X_{n+1}\ge t-y$ for some combination n, y. Since $U(t)=t-y\le x$, $t-x\le y\le t$ should hold. Summing over all possible n and y, we have

$$Pr(U(t)\le x)=\sum_{n=1}^{\infty}\int_{t-x}^{t}[1-F_{n+1}(t-y)]d\sigma_n(y) \qquad \text{for} \quad x<t.$$

Setting

$$Q_n(y)=F_n(y) \qquad \text{for} \quad 0\le y\le x,$$
$$=0 \qquad \text{elsewhere,}$$

we have

$$Pr(U(t)\le x)=\sum_{n=1}^{\infty}\int_{t-x}^{t}d\sigma_n(y)-\sum_{n=1}^{\infty}\int_{0}^{t}Q_{n+1}(t-y)d\sigma_n(y)$$
$$\equiv A(t, x)-B(t, x),$$

say, for $x<t$. Hence, noticing that

$$Pr(U(t)\le x)=1 \qquad \text{for} \quad x\ge t,$$

we have

$$\text{(4. 2)} \qquad \frac{1}{T}\int_0^T Pr(U(t)\le x)dt=\frac{1}{T}\int_x^T(A(t, x)-B(t, x))dt+\frac{1}{T}\int_0^x dt.$$

Defining the functions $A(t, x)$ and $B(t, x)$ even over $0\le t\le x$ by the right equalities where they were defined above, this equation turns out to be

$$\frac{1}{T}\int_0^T Pr(U(t)\le x)dt=\frac{1}{T}\int_0^T(A(t, x)-B(t, x))dt$$
$$+\frac{1}{T}\int_0^x(1-A(t, x)+B(t, x))dt.$$

Obviously, since $A(t, x)$ and $B(t, x)$ are bounded on $0\le t\le x$, the second term on the right hand side tends to 0 as $T\to\infty$, and so we obtain

$$\text{(4. 3)} \qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T Pr(U(t)\le x)dt=\lim_{T\to\infty}\frac{1}{T}\int_0^T(A(t, x)-B(t, x))dt.$$

From (1. 2), we have

$$\text{(4. 4)} \qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T A(t, x)dt=\lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}Pr(t-x<S_n\le t)dt=\frac{x}{\mu}.$$

On the other hand, Theorem 1 yields, since $Q_n(t)$ satisfies the conditions of $G_n(t)$ there, that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T B(t, x)dt=\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\sum_{n=1}^{\infty}\int_0^t Q_{n+1}(t-y)d\sigma_n(y)$$
$$=\frac{1}{\mu}\int_0^{\infty}Q(t)dt,$$

where

$$Q(t)=F(t) \qquad 0\leq t\leq x$$
$$=0 \qquad \text{elsewhere.}$$

Consequently,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T B(t, x)dt=\frac{1}{\mu}\int_0^x F(t)dt \tag{4. 5}$$

and so (4. 1) follows from (4. 3)—(4. 5).

REMARK. In Theorem 2, $F(t)$ is defined by (2. 1) and we do not assume that $F(t)$ is a distribution function. Also, μ is defined by (1. 1) and it is not necessary that μ is the expectation of a certain distribution function. However, under the condition of the theorem that (2. 2) holds uniformly with respect to i, it follows that $F(t)$ is a distribution function (statement A below), and that μ is the expectation of the distribution function $F(t)$ (statement B below). Thus (4. 1) is a distribution function, that is, writing

$$D(x)=\frac{1}{\mu}\int_0^x [1-F(t)]dt,$$

$D(0)=0$ and $D(\infty)=1$. We shall show these statements in the following way ($D(0)=0$ is trivial).

(i) Statement A. As it is obvious that $F(0)=0$, and that $F(t)$ is nondecreasing and right continuous, it suffices to prove that $F(\infty)=1$. Since

$$\int_t^\infty xdF_i(x)\geq t(1-F_i(t)),$$

we have

$$1-F_i(t)\geq\frac{1}{t}\int_t^\infty xdF_i(x).$$

The condition, that (2. 2) holds uniformly with respect to i, yields that for any $\varepsilon>0$ there exists t^* such that

$$\int_t^\infty xdF_i(x)<\varepsilon \qquad \text{for} \quad t>t^*$$

uniformly with respect to i. Thus

$$F_i(t)\geq 1-\frac{\varepsilon}{t}$$

and so

$$\frac{1}{n}\sum_{i=2}^{n+1}F_i(t)\geq 1-\frac{\varepsilon}{t}.$$

Letting $n\to\infty$, we have

$$F(t)\geq 1-\frac{\varepsilon}{t} \qquad \text{for} \quad t>t^*,$$

and consequently $F(\infty)\geq 1$. On the other hand, from the definition of $F(t)$, we have $F(t)\leq 1$ and so $F(\infty)\leq 1$, hence $F(\infty)=1$.

(ii) Statement B. For arbitrarily fixed ξ, we have

$$\begin{aligned}\mu_i&=\int_0^\infty x dF_i(x)\\&=\int_0^\xi x dF_i(x)+\int_\xi^\infty x dF_i(x)\\&=\xi F_i(\xi)-\int_0^\xi F_i(x)dx+\int_\xi^\infty x dF_i(x).\end{aligned}$$

Therefore we get

$$\begin{aligned}\mu&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}\mu_i\\&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}\left[\xi F_i(\xi)-\int_0^\xi F_i(x)dx+\int_\xi^\infty x dF_i(x)\right].\end{aligned} \tag{4.6}$$

Now, because of the dominated convergence theorem, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}\int_0^\xi F_i(x)dx=\int_0^\xi F(x)dx,$$

and we have from the definition of $F(x)$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}\xi F_i(\xi)=\xi F(\xi).$$

Hence we see that (4. 6) turns out to the equation,

$$\begin{aligned}\mu&=\xi F(\xi)-\int_0^\xi F(x)dx+\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}\int_\xi^\infty x dF_i(x)\\&=\int_0^\xi x dF(x)+\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}\int_\xi^\infty x dF_i(x).\end{aligned}$$

Letting $\xi\to\infty$, we get

$$\mu=\int_0^\infty x dF(x)+\lim_{\xi\to\infty}\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}\int_\xi^\infty x dF_i(x).$$

From the condition of the theorem that (2. 2) holds uniformly with respect to i, the second term on the right hand side is 0, and then we have

$$\mu=\int_0^\infty x dF(x),$$

which gives us

$$D(\infty)=\frac{1}{\mu}\int_0^\infty [1-F(x)]dx=\frac{1}{\mu}\int_0^\infty xdF(x)=1$$

THEOREM 3. Under the condition that (1. 1) and (2. 1) exist and (2. 2) holds uniformly with respect to i, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T Pr(V(t)\le x)dt=\frac{1}{\mu}\int_0^x [1-F(t)]dt.$$

Proof. The event $\{V(t)\le x\}$ occurs if and only if $t<S_n=y\le t+x$ and $X_{n+1}\ge t+x-y$ for some combination n, y, and hence summing over all possible n and y, we have

$$Pr(V(t)\le x)=\sum_{n=1}^\infty \int_t^{t+x} [1-F_{n+1}(t+x-y)]d\sigma_n(y).$$

Therefore we have

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T Pr(V(t)\le x)dt\\
&=\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\sum_{n=1}^\infty\int_t^{t+x}[1-F_{n+1}(t+x-y)]d\sigma_n(y)\\
&=\lim_{T\to\infty}\frac{1}{T}\int_x^{T+x} dt\sum_{n=1}^\infty\int_{t-x}^{t}[1-F_{n+1}(t-y)]d\sigma_n(y)\\
&=\lim_{T\to\infty}\frac{1}{T}\int_x^{T+x} Pr(U(t)\le x)dt\\
&=\lim_{T\to\infty}\frac{1}{T}\int_0^{T} Pr(U(t)\le x)dt\\
&\quad+\lim_{T\to\infty}\frac{1}{T}\left[\int_T^{T+x} Pr(U(t)\le x)dt-\int_0^x Pr(U(t)\le x)dt\right]\\
&=\lim_{T\to\infty}\frac{1}{T}\int_0^{T} Pr(U(t)\le x)dt,
\end{aligned}$$

and then the conclusion follows from Theorem 2.

THEOREM 4. Under the condition that (1. 1) and (2. 1) exist and (2. 2) holds uniformly with respect to i, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T Pr(X(t)\le x)dt=\frac{1}{\mu}\int_0^x t dF(t).$$

Proof. Suppose that $x<t$. The event $\{X(t)\le x\}$ occurs if and only if $S_n=y(\le t)$ and $t-y<X_{n+1}\le x$ for some combination n, y, and $t-x\le y\le t$ should hold. Summing over all possible n and y, we have

$$Pr(X(t)\leq x)=\sum_{n=1}^{\infty}\int_{t-x}^{t}[F_{n+1}(x)-F_{n+1}(t-y)]d\sigma_n(y)$$

$$=\sum_{n=1}^{\infty}\int_{t-x}^{t}F_{n+1}(x)d\sigma_n(y)$$

$$-\sum_{n=1}^{\infty}\int_{t-x}^{t}F_{n+1}(t-y)d\sigma_n(y)$$

$$\equiv A(t, x)-B(t, x),$$

say. Let us first consider $A(t, x)$. Since $F(x)$ is a distribution function which has been shown in Remark of this section above, we easily see that

$$F(x)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n+1}F_i(x)$$

holds uniformly for x, from which we can write

$$\frac{1}{n}\sum_{i=2}^{n+1}F_i(x)=F(x)+\varepsilon_n(x),$$

$\varepsilon_n(x)$ going to zero uniformly for x as $n\to\infty$. From the above equation, we get

$$F_{n+1}(x)=F(x)+n\varepsilon_n(x)-(n-1)\varepsilon_{n-1}(x).$$

Hence we have

$$A(t, x)=\sum_{n=1}^{\infty}\int_{t-x}^{t}F(x)d\sigma_n(y)$$

$$+\sum_{n=1}^{\infty}\int_{t-x}^{t}(n\varepsilon_n(x)-(n-1)\varepsilon_{n-1}(x))d\sigma_n(y)$$

$$\equiv A_1(t, x)+A_2(t, x),$$

say. Using (1. 2), we get

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T A_1(t, x)dt=\frac{x}{\mu}F(x). \tag{4. 7}$$

Noticing that

$$\sum_{n=1}^{\infty}|\sigma_n(x)n\varepsilon_n(x)|\leq\sup_{n=1,2,\cdots}|\varepsilon_n(x)|\sum_{n=1}^{\infty}n\sigma_n(x)<\infty,$$

we have

$$\sum_{n=1}^{\infty}(n\varepsilon_n(x)-(n-1)\varepsilon_{n-1}(x))d\sigma_n(y)$$

$$=\sum_{n=1}^{\infty}n\varepsilon_n(x)(d\sigma_n(y)-d\sigma_{n+1}(y)).$$

Let ε be a given positive number. Then we can choose an integer N independently of x such that

$$|\varepsilon_n(x)|<\varepsilon \qquad \text{for} \quad n>N.$$

Now we split $A_2(t, x)$ in the following way:

$$A_2(t, x)=\sum_{n=1}^{N} n\varepsilon_n(x)\left[\int_{t-x}^{t} d\sigma_n(y)-\int_{t-x}^{t} d\sigma_{n+1}(y)\right]$$

$$+\sum_{n=N+1}^{\infty} n\varepsilon_n(x)\left[\int_{t-x}^{t} d\sigma_n(y)-\int_{t-x}^{t} d\sigma_{n+1}(y)\right]$$

$$\equiv a_1(t, x)+a_2(t, x),$$

say. We then have

$$\frac{1}{T}\int_0^T a_1(t, x)dt=\frac{1}{T}\sum_{n=1}^{N} n\varepsilon_n(x)\int_0^T \{[Pr(S_n\leq t)-Pr(S_n\leq t-x)]$$

$$+[Pr(S_{n+1}\leq t)-Pr(S_{n+1}\leq t-x)]\}dt.$$

We note here that

$$\left|\int_0^T [Pr(S_n\leq t)-Pr(S_n\leq t-x)]dt\right|$$

$$=\left|\int_0^T Pr(S_n\leq t)dt-\int_{-x}^{T-x} Pr(S_n\leq t)dt\right|$$

$$=\left|\int_{T-x}^{T} Pr(S_n\leq t)dt\right|\leq x.$$

Then we have

$$\left|\frac{1}{T}\int_0^T a_1(t, x)dt\right|\leq\frac{2}{T}\sum_{n=1}^{N} nCx,$$

where

$$C=\max_{n=1,2,\cdots,N}|\varepsilon_n(x)|<\infty,$$

and hence we get

$$\overline{\lim_{T\to\infty}}\left|\frac{1}{T}\int_0^T a_1(t, x)dt\right|=0. \tag{4. 8}$$

For $a_2(t, x)$ we consider

$$\left|\frac{1}{T}\int_0^T a_2(t, x)dt\right|\leq\frac{1}{T}\left|\int_0^T dt\sum_{n=N+1}^{\infty} n\varepsilon_n(x)\{[Pr(S_n\leq t)-Pr(S_n\leq t-x)]\right.$$

$$\left.+[Pr(S_{n+1}\leq t)-Pr(S_{n+1}\leq t-x)]\}\right|$$

$$\leq\frac{1}{T}\varepsilon\int_0^T dt\sum_{n=1}^{\infty} n\{[Pr(S_n\leq t)-Pr(S_{n+1}\leq t)]$$

$$+[Pr(S_n\leq t-x)-Pr(S_{n+1}\leq t-x)]\}.$$

Note that

$$\sum_{n=1}^{\infty} n[Pr(S_n \leq t) - Pr(S_{n+1} \leq t)] = \sum_{n=1}^{\infty} Pr(S_n \leq t).$$

Then we obtain from (1. 2),

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \left| \int_0^T a_2(t, x)dt \right| \leq \lim_{T\to\infty} \varepsilon \frac{1}{T} \int_0^T dt \sum_{n=1}^{\infty} [Pr(S_n \leq t) - Pr(S_n \leq t-x)]$$

$$= \varepsilon \frac{x}{\mu}.$$

Since ε is arbitrary, we get

$$\overline{\lim_{T\to\infty}} \left| \frac{1}{T} \int_0^T a_2(t, x)dt \right| = 0. \tag{4. 9}$$

(4. 8) and (4. 9) give us

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T A_2(t, x)dt = 0. \tag{4. 10}$$

From (4. 7) and (4. 10), we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T A(t, x)dt = \frac{x}{\mu} F(x). \tag{4. 11}$$

Next, considering

$$B(t, x) = \sum_{n=1}^{\infty} \int_{t-x}^{t} F_{n+1}(t-y)d\sigma_n(y),$$

we have from the proof of Theorem 1

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T B(t, x)dt = \frac{1}{\mu} \int_0^x F(t)dt. \tag{4. 12}$$

From (4. 11) and (4. 12), we finally have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T Pr(X(t) \leq x)dt = \frac{x}{\mu} F(x) - \frac{1}{\mu} \int_0^x F(t)dt$$

$$= \frac{1}{\mu} \int_0^x t dF(t).$$

5. Finally, we shall show a central limit theorem for the renewal number in a renewal process with non-identically distributed variables. The proof is analogous to the one in the case where random variables are identically distributed. We state a well known Lindeberg central limit theorem for sums of independent random variables as a lemma:

LEMMA 3. Let $\{X_i, i=1, 2, \cdots\}$ be a sequence of independent random variables, and let $F_i(x)$ be the distribution function of X_i. Suppose that $\mu_i = EX_i$ and $\sigma_i^2 = VarX_i$ exist, and that for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{b_n^2} \sum_{i=1}^{n} \int_{|x-\mu_i|>\varepsilon b_n} (x-\mu_i)^2 dF_i(x) = 0 \tag{5.1}$$

holds, where

$$b_n^2 = \sum_{i=1}^{n} \sigma_i^2.$$

Then for every fixed x

$$Pr\left(\sum_{i=1}^{n} (X_i - \mu_i) \Big/ b_n < x\right) \xrightarrow[n\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$

THEOREM 5. Let $\{X_i, i=1, 2, \cdots\}$ be a sequence of independent and nonnegative random variables, and let $F_i(x)$ be the distribution function of X_i. Suppose that $\mu_i = EX_i$ and $\sigma_i^2 = VarX_i$ exist, that

$$\mu = \frac{1}{n} \sum_{i=1}^{n} \mu_i + o\left(\frac{b_n}{n}\right) \tag{5.2}$$

and that

$$b^2 = \lim_{n\to\infty} \frac{1}{n} b_n^2, \tag{5.3}$$

where

$$b_n^2 = \sum_{i=1}^{n} \sigma_i^2.$$

Then for every fixed x

$$Pr\left(\left(N(t) - \frac{t}{\mu}\right) \Big/ \sqrt{\frac{tb^2}{\mu^3}} < x\right) \xrightarrow[t\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du,$$

if (5.1) holds for any $\varepsilon > 0$.

Proof. We note the relation

$$Pr(N(t) < n) = Pr(S_n > t).$$

Since $X_1, X_2, \cdots$ are independent random variables, Lemma 3 gives us that

$$Pr(S_n > t) \longrightarrow Pr\left(\left(S_n - \sum_{i=1}^{n} \mu_i\right) \Big/ b_n > -x\right)$$

$$\longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} e^{-\frac{u^2}{2}} du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du,$$

when $n\to\infty$ and $t\to\infty$ in such way that

(5. 4) $$\left(t-\sum_{i=1}^{n}\mu_i\right)\Big/b_n \longrightarrow -x \qquad (x\text{: fixed}).$$

Now,

(5. 5) $$Pr(N(t)<n)=Pr\left(\left(N(t)-\frac{t}{\mu}\right)\Big/\sqrt{\frac{tb^2}{\mu^3}}<\left(n-\frac{t}{\mu}\right)\Big/\sqrt{\frac{tb^2}{\mu^3}}\right).$$

Here, we write

$$\left(n-\frac{t}{\mu}\right)\Big/\sqrt{\frac{tb^2}{\mu^3}}=\frac{n\mu-t}{b\sqrt{n}}\sqrt{\frac{n\mu}{t}}.$$

Since (5. 3) is no more than

(5. 6) $$b\sqrt{n}-b_n=o(b_n),$$

we get from (5. 2), (5. 6) and (5. 4) that

(5. 7) $$\frac{n\mu-t}{b\sqrt{n}}=\frac{\sum_{i=1}^{n}\mu_i-t+o(b_n)}{b_n+o(b_n)}\longrightarrow x$$

as $n\to\infty$, $t\to\infty$ according to (5. 4). From (5. 7), we have

$$\frac{\sqrt{n}\,\mu}{b}-\frac{t}{b\sqrt{n}}\longrightarrow x.$$

Since

$$\frac{\sqrt{n}\,\mu}{b}\longrightarrow\infty \qquad (n\to\infty),$$

we have

(5. 8) $$\frac{t}{b\sqrt{n}}\longrightarrow\infty$$

as $n\to\infty$, $t\to\infty$ according to (5. 4). Again we may write

$$\frac{n\mu-t}{b\sqrt{n}}=\frac{t}{b\sqrt{n}}\left(\frac{n\mu}{t}-1\right)\longrightarrow x.$$

From (5. 8),

(5. 9) $$\frac{n\mu}{t}\longrightarrow 1$$

as $n\to\infty$, $t\to\infty$ according to (5. 4). Thus it follows from (5. 7) and (5. 9) that

(5. 10) $$\left(n-\frac{t}{\mu}\right)\Big/\sqrt{\frac{tb^2}{\mu^3}}\longrightarrow x.$$

From (5. 5) and (5. 10), we may easily obtain the conclusion.

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