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VECTOR MAXIMUM PROBLEMS

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VECTOR MAXIMUM PROBLEMS

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ABSTRACT

A vector maximum problem means a mathematical programming problem which has several objective functions $f_1(x), f_2(x), \cdots, f_p(x)$ ($\rightarrow$maximize) where $f_i(x) (i=1,2\cdots, p)$ are numerical functions defined on the n-dimensional Euclidean space. The problems of this type were studied by DA CUNHA and POLAK (1966), HURWICZ (1964), KARLIN (1959), and KUHN and TUCKER (1951). In this paper, we first define three kinds of maximum solutions of the above problem and then investigate necessary conditions and sufficient conditions for these maximalities.

1. A Vector Maximum Problem

Let R^p, where p is a positive integer, be the p-dimensional Euclidean space with the usual norm topology. The following convention for inequalities will be used throughout this paper. If $y=(y_1, y_2, \cdots, y_p)$, $\bar{y}=(\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_p)$ are two vectors in R^p, we write

$$y \geqq \bar{y} \text{ to mean } y_i \geqq \bar{y}_i \text{ for } i=1,2,\cdots, p,$$
$$y \geq \bar{y} \text{ to mean } y \geqq \bar{y} \text{ and } y \neq \bar{y},$$
$$y > \bar{y} \text{ to mean } y_i > \bar{y}_i \text{ for } i=1, 2, \cdots, p.$$

If $y \geqq 0$, y is said to be nonnegative, if $y \geq 0$ then y is said to be semipositive and if $y>0$ then y is said to be positive. A vector maximum problem which we shall deal with is as follows.

Basic Problem: Let X be a subset of R^n and $f_i(x)$ $(i=1, 2, \cdots, p)$ be numerical functions defined on R^n, and let $f(x)=(f_1(x), f_2(x), \cdots, f_p(x))$. Find an $\hat{x} \in X$ such that $f(x) \geq f(\hat{x})$ for no $x \in X$.

In Basic Problem $f_i(x)$ $(i=1, 2, \cdots, p)$ are said to be objective functions and X

is said to be a constraint set. Now we define three kinds of maximum solutions of Basic Problem; $\hat{x}$ is a weak Pareto maximum solution (w-solution) if $\hat{x}\in X$ and $f(x)>f(\hat{x})$ for no $x\in X$, $\hat{x}$ is a Pareto maximum solution (P-solution) if $\hat{x}\in X$ and $f(x)\geq f(\hat{x})$ for no $x\in X$, and x is a strong maximum solution (s-solution) if $\hat{x}\in X$ and $f(\hat{x})\geqq f(x)$ for all $x\in X$. Between these maximum solutions, we have the relations

$$\begin{array}{l}\hat{x}:\ s\text{-solution}\\ \quad\Downarrow\\ \hat{x}:\ P\text{-solution}\\ \quad\Downarrow\\ \hat{x}:\ w\text{-solution.}\end{array}$$

2. Minimum Component Maximum Problems

In this section, a necessary and sufficient condition for $\hat{x}$ to be a w-solution of Basic Problem is shown by using some minimum component maximum problem. For this purpose, we first observe that we may assume without loss of generality $f(x)>0$ for all $x\in X$ in Basic Problem. Let $F_i(x)=\exp f_i(x)$ $(i=1, 2, \cdots, p)$, $F(x)=(F_1(x), F_2(x), \cdots, F_p(x))$, and consider the following vector maximum problem:

Problem 2-1: Find an $\hat{x}\in X$ such that $F(x)\geq F(\hat{x})$ for no $x\in X$.

Then Basic Problem is equivalent to Problem 2-1; $\hat{x}$ is a w-solution (P-solution, s-solution) of Basic Problem if and only if $\hat{x}$ is a w-solution (P-solution, s-solution) of Problem 2-1. And at the same time we have $F(x)>0$ for all $x\in X$. Therefore we may assume without loss of generality that $f(x)>0$ for all $x\in X$ in Basic Problem, and do so throughout this section.

Now we introduce the following minimum component maximum problem:

Problem 2-2: Find an $\hat{x}\in X$ that maximize $\min_{i\in I}\{u_i f_i(x)\}$ constrained by $x\in X$ where $u=(u_1, u_2, \cdots, u_p)$ is a given vector in the $(p-1)$-dimensional standard simplex $S_p=\{u\in R^p;\ u\geqq 0,\ \sum_{i=1}^{p} u_i=1\}$ and $I=\{i;\ u_i>0\}$.

Between Basic Problem and Problem 2-2 the following relation is shown:

Theorem 2-1: $\hat{x}$ is a w-solution of Basic Problem if and only if $\hat{x}$ is a solution of Problem 2-2 for some $u\in S_p$.

Proof. If $\hat{x}\in X$ is not a w-solution of Basic Problem, then there exists an $\tilde{x}\in X$ such that $f(\tilde{x})>f(\hat{x})$. Hence $\hat{x}$ is not a solution of Problem 2-2 for any $u\in S_p$. Thus we have proved the "if" part of the theorem. Conversely, let $\hat{x}$ is a w-solution of a Basic Problem. Define $\hat{u}\in R^p$ and $\hat{t}\in R$ such that $\hat{u}_i f_i(\hat{x})=\hat{t}$ $(i=1, 2, \cdots, p)$ and $\sum_{i=1}^{p}\hat{u}_i=1$. Then $\hat{u}\in S_p$, $\hat{u}>0$ and $\hat{x}$ is a solution of Problem 2-2 for $u=\hat{u}$. In fact, assume that $\hat{x}$ is not a solution of Problem 2-2 for $u=\hat{u}$, then then there exists an $\tilde{x}\in X$ such that

$$\hat{u}_i f_i(\tilde{x})>\min_j\{\hat{u}_j f_j(\hat{x})\}=\hat{t}=\hat{u}_i f_i(\hat{x}) \quad \text{for} \quad i=1, 2, \cdots, p,$$

which is a contradiction. Thus we have proved the "only if" part of the theorem.

By Theorem 2-1, the set $\hat{X}$ of all w-solutions of Basic Problem is obtained by solving Problem 2-2 for all $u \in S_p$. It should be noted that Problem 2-2 is equivalent to the following usual mathematical programming problem:

Problem 2-3: Find $\hat{t} \in R$, $\hat{x} \in X$ that maximize t constrained by $x \in X$ and $u_i f_i(x) \geqq t$ for $i \in I$.

Therefore, the set $\hat{X}$ is obtained by solving Problem 2-3 for all $u \in S_p$.

3. Fundamental Theory of Problems with Differentiable Objective Functions

In this section we assume the differentiabilities of the objective functions $f_i(x)$ $(i=1, 2, \cdots, p)$ of Basic Problem and show a necessary condition for $\hat{x}$ to be a w-solution of it. This condition will be used in deriving the Kuhn-Tucker condition in maximum problems with inequality constraints and the classical Lagrange multiplier methods in maximum problems with equality constraints. In order to state it, we define the sequential tangent cone to a set $X \subset R^n$ at $\hat{x} \in R^n$ below.

Definition 3-1: The sequential tangent cone to a set $X \subset R^n$ at $\hat{x} \in R^n$, denoted by $STC[\hat{x}; X]$, is the set of all vectors $t \in R^n$ which have the property that there exist a sequence of vectors $\{x^k\}$ in X and a sequence of positive numbers $\{\lambda^k\}$ such that

$$x^k \longrightarrow \hat{x} \quad \text{as} \quad k \longrightarrow \infty$$

and

$$\lambda^k(x^k - \hat{x}) \longrightarrow t \quad \text{as} \quad k \longrightarrow \infty.$$

Figure 3-1 depicts the sequential tangent cone to X at $\hat{x}$ in a case that $X \subset R^2$ and $\hat{x} \in R^2$.

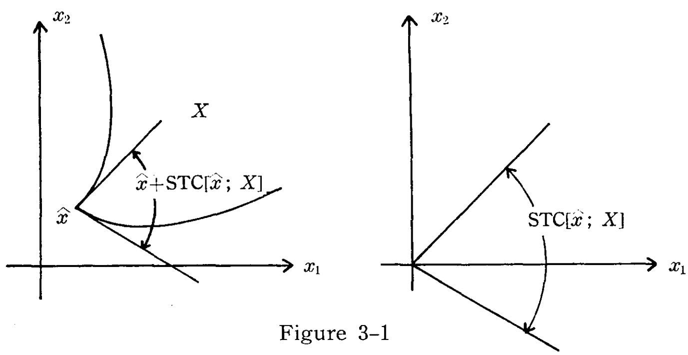

Figure 3-1

$STC[\hat{x}; X]$ is a kind of approximation set of X at $\hat{x}$. In the subsequent sections, we shall set.

$$X = \{x \in R^n;\ g_i(x) \leqq 0 \qquad (i=1, 2, \cdots, q)\}$$

or

$$X = \{x \in R^n;\ g_i(x) = 0 \qquad (i=1, 2, \cdots, q)\}$$

where $g_i(x)$ $(i=1, 2, \cdots, q)$ are differentiable numerical functions defined on R^n, and construct another kind of approximation set of X at $\hat{x}$ whose definition contains the gradients of $g_i(x)$ $(i=1, 2, \cdots, q)$ at $\hat{x}$. Note that the definition of the sequential tangent cone to X at $\hat{x}$ contains none of them even if X is given as the above form. What is called a constraint qualification implies that these two approximation sets of X at $\hat{x}$ coincide with each other.

We are now ready to show the following fundamental theorem:

Theorem 3-1: Let $f_i(x)$ $(i=1, 2, \cdots, p)$ in Basic Problem be differentiable. If $\hat{x}$ is a w-solution of Basic Problem then there exists no $t \in STC[\hat{x}; X]$ such that

$$\langle \nabla f_i(\hat{x}), t \rangle > 0 \qquad (i=1, 2, \cdots, p)$$

where $\langle \nabla f_i(\hat{x}), t \rangle$ implies the inner product of $\nabla f_i(\hat{x})$ and t.

Proof. Let $\hat{x}$ be a w-solution of Basic Problem, then $\hat{x} \in X$ and $f(x) > f(\hat{x})$ for no $x \in X$. Suppose that there exists a $t \in STC[\hat{x}; X]$ such that

$$\langle \nabla f_i(\hat{x}), t \rangle > 0 \qquad (i=1, 2, \cdots, p).$$

Then there exists a $\mu > 0$ such that

$$\langle \nabla f_i(\hat{x}), t \rangle > \mu \qquad (i=1, 2, \cdots, p).$$

It follows from $t \in STC[\hat{x}; X]$ that there exist a sequence of vectors $\{x^k\}$ in X and a sequence of positive numbers $\{\lambda^k\}$ such that

$$x^k \longrightarrow \hat{x} \quad \text{as} \quad k \longrightarrow \infty$$

and

$$\lambda^k (x^k - \hat{x}) \longrightarrow t \quad \text{as} \quad k \longrightarrow \infty.$$

For these x^k and λ^k, we have that

$$\begin{aligned} f_i(x^k) &= f_i(\hat{x} + (x^k - \hat{x})) \\ &= f_i(\hat{x}) + \langle \nabla f_i(\hat{x}), x^k - \hat{x} \rangle + 0(||x^k - \hat{x}||) \qquad (i=1, 2, \cdots, p) \end{aligned}$$

where $0(\varepsilon)/\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0+$. Hence

$$f_i(x^k) = f_i(\hat{x}) + \frac{1}{\lambda^k} \left\{ \langle \nabla f_i(\hat{x}), \lambda^k (x^k - \hat{x}) \rangle + \lambda^k ||x^k - \hat{x}|| \frac{0(||x^k - \hat{x}||)}{||x^k - \hat{x}||} \right\}$$
$$(i=1, 2, \cdots, p),$$

from which follows that there exists a positive integer $\hat{k}$ such that

$$f_i(x^k) > f_i(\hat{x}) + \frac{1}{\lambda^{\hat{k}}} \frac{1}{2} \mu > f_i(\hat{x}) \qquad (i=1, 2, \cdots, p).$$

The above inequalities contradict the fact that $\hat{x}$ is a w-solution of Basic Problem.

Q. E. D.

Corollary 3-1: If $\hat{x}$ is a w-solution of Basic Problem and in addition $STC[\hat{x}; X]$ is convex then there exists a $\hat{u} \geq 0$ such that

$$\left\langle \sum_{i=1}^{p} \hat{u}_i \nabla f_i(\hat{x}), t \right\rangle \leqq 0 \quad \text{all all} \quad t \in STC[\hat{x}; X].$$

Proof. Let $\hat{x}$ be a w-solution of Basic Problem and define the subsets Y and A of R^p such that

$$Y = \{y \in R^p ; y_i = \langle \nabla f_i(\hat{x}), t \rangle \quad (i=1, 2, \cdots, p) \quad \text{for} \quad t \in STC[\hat{x}; X]\}$$

and

$$A = \{y \in R^p ; y > 0\}$$

respectively. Since, by Theorem 3-1, $Y \cap A = \phi$, and Y and A are both convex cones, by the separation theorem for convex sets, there exists a $\hat{u} \neq 0$ such that

$$\langle \hat{u}, y \rangle \leqq 0 \quad \text{for all} \quad y \in Y$$

and

$$\langle \hat{u}, a \rangle \geqq 0 \quad \text{for all} \quad a \in A.$$

Thus $\hat{u} \geq 0$ and

$$\left\langle \sum_{i=1}^{p} \hat{u}_i \nabla f_i(\hat{x}), t \right\rangle \leqq 0 \quad \text{for all} \quad t \in STC[\hat{x}; X]. \qquad \text{Q.E.D.}$$

Corollary 3-2: If $p=1$ in Theorem 3-1 then $\langle \nabla f_1(x), t \rangle \leqq 0$ for all t contained in the closure of the convex hull of the set $STC[\hat{x}; X]$.

Corollary 3-2 is trivial from Theorem 3-1 and the continuity and the linearity of the inner product, and the proof is omitted here. Corollary 3-2 is a fundamental result in nonlinear programming problems, and it has been found in some recent papers (Canon, Cullum and Polak 1966; Guignard 1969).

4. Problems with Differentiable Objective Functions and Inequality Constraints

In this section, we deal with the following problem:

Problem 4-1: Let $f_i(x)$ $(i=1, 2, \cdots, p)$ and $g_i(x)$ $(i=1, 2, \cdots, q)$ be differentiable functions defined on R^n, and let

$$f(x) = (f_1(x), f_2(x), \cdots, f_p(x))$$
$$g(x) = (g_1(x), g_2(x), \cdots, g_q(x))$$

and

$$X = \{x \in R^n ; g(x) \leqq 0\}.$$

Find an $\hat{x} \in X$ such that $f(x) \geq f(\hat{x})$ for no $x \in X$.

First, we show a necessary condition for $\hat{x}$ to be a w-solution of Problem 4-1 under no assumption.

Theorem 4-1: If $\hat{x}$ is a w-solution of Problem 4-1 then there exist a $\hat{u} \in R^p$ and a $\hat{v} \in R^q$ satisfying the conditions

(i) $$(\hat{u}, \hat{v}) \geq 0,$$

(ii) $$\sum_{i=1}^{p} \hat{u}_i \nabla f_i(\hat{x}) - \sum_{i=1}^{q} \hat{v}_i \nabla g_i(\hat{x}) = 0,$$

(iii) $$\sum_{i=1}^{q} \hat{v}_i g_i(\hat{x}) = 0.$$

Proof. Let $\hat{x}$ be a w-solution of Problem 4-1, and define

$$I = \{i;\, g_i(\hat{x}) = 0\},$$
$$J = \{j;\, g_j(\hat{x}) < 0\}.$$

Let $g^I(x)$ be the vector function whose components are $g_i(x)$ $(i \in I)$ and $g^J(x)$ be the vector function whose components are $g_j(x)$ $(j \in J)$. Then, from the continuity of $g^J(x)$, there exists an $\varepsilon > 0$ such that

$$g^J(x) < 0 \quad \text{for all} \quad x \in B_\varepsilon(\hat{x}) = \{x \in R^n;\, ||x - \hat{x}|| < \varepsilon\}.$$

Since $\hat{x}$ is a w-solution of Problem 4-1, we have that the point $(f(\hat{x}), -g^I(\hat{x}))$ satisfies the relation $(f(x), -g^I(x)) > (f(\hat{x}), -g^I(\hat{x}))$ for no $x \in B_\varepsilon(\hat{x})$. It is obvious that $STC[\hat{x};\, B_\varepsilon(\hat{x})] = R^n$, hence, by Corollary 3-1, there exists a $(\hat{u}, v^I) \geq 0$ such that

$$\left\langle \sum_{i=1}^{p} \hat{u}_i \nabla f_i(\hat{x}) - \sum_{i \in I} v_i \nabla g_i(\hat{x}),\, t \right\rangle \leqq 0 \quad \text{for all} \quad t \in R^n.$$

Hence, by defining $\hat{v}_i = v_i^I$ $(i \in I)$ and $\hat{v}_j = 0$ $(j \in J)$, we obtain $(\hat{u}, \hat{v})$ satisfying (i), (ii) and (iii). Q. E. D.

It should be noted that there is no guarantee that $\hat{u}$ is semipositive in Theorem 4-1. In cases of $\hat{u} = 0$, it is intuitively obvious that Theorem 4-1 dose not say much more about the weak maximality of $f(x)$, because the function $f(x)$ disappears from (ii). It is possible to exclude such cases by introducing assumptions on the constraint set X. These assumptions are called constraint qualifications. In this paper, we shall introduce the Kuhn-Tucker constraint qualification (Assumption 4-1), the Arrow-Hurwicz-Uzawa constraint qualification (Assumption 4-2) and the generalized Kuhn-Tucker constraint qualification (Assumption 4-3). Here we need a new symbol. For every $\hat{x} \in X$ we define

$$K[\hat{x};\, X] = \{t \in R^n;\, \langle \nabla g_i(\hat{x}), t \rangle \leqq 0 \qquad (i \in I)\}$$

where $I = \{i;\, g_i(\hat{x}) = 0\}$. As stated in the previous section, $K[\hat{x};\, X]$ is a kind of approximation set of X at $\hat{x}$ by the use of the gradients of $g_i(x)$ $(i \in I)$ at $\hat{x} \in X$.

Assumption 4-1: For every $t \in K[\hat{x};\, X]$ there exist a positive number λ and an n-dimensional vector function $e(\tau)$ defined on the closed interval $[0, 1]$ satisfying the following conditions:

(i) $e(0) = \hat{x}$,

(ii) $e(\tau) \in X$ for all $\tau \in [0, 1]$,

(iii) $e(\tau)$ is differentiable at $\tau = 0$ and $\left.\dfrac{de(\tau)}{d\tau}\right|_{\tau=0} = \lambda t.$

Assumption 4–2: The system of inequalities

$$\langle \nabla g_i(\hat{x}), t\rangle \leqq 0 \qquad (i \in \hat{I})$$
$$\langle \nabla g_{i'}(\hat{x}), t\rangle < 0 \qquad (i' \in \tilde{I})$$

has a solution $t \in R^n$ where

$$\hat{I} = \{i;\ g_i(\hat{x}) = 0 \text{ and } g_i(x) \text{ is concave at } \hat{x}\}$$
$$\tilde{I} = \{i';\ g_{i'}(\hat{x}) = 0 \text{ and } g_{i'}(x) \text{ is not concave at } \hat{x}\}.$$

Assumption 4–3:

$$STC[\hat{x};\ X] = K[\hat{x};\ X]$$

If the set X satisfies one of these assumptions at $\hat{x} \in X$, then we can guarantee that $\hat{u}$ is semipositive in Theorem 4–1. But we shall postpone the proof of this fact until after we give the relations between these assumptions. In fact, if the set X satisfies Assumption 4–1 or Assumption 4–2 at $\hat{x} \in X$ then X satisfies Assumption 4–3 at $\hat{x}$. In order to show these relations, the following lemmata are necessary:

Lemma 4–1: If a differentiable numerical function $g(x)$ is concave at $\hat{x}$ on R^n, then

$$g(x) - g(\hat{x}) \leqq \langle \nabla g(\hat{x}), x - \hat{x}\rangle \qquad \text{for all } x \in R^n.$$

Proof. See Mangasarian (1969).

Lemma 4–2: For any given $\hat{x} \in R^n$ and $X \subset R^n$, the sequential tangent cone $STC[\hat{x};\ X]$ to X at $\hat{x}$ is a closed cone in R^n.

Proof. See Canon, Cullum and Polak (1970).

Lemma 4–3: $STC[\hat{x};\ X] \subset K[\hat{x};\ X]$ for all $\hat{x} \in X$.

Proof. See Canon, Cullum and Polak (1970).
By using the above lemmata we can prove the following theorem:

Theorem 4–2:
(i) If the set X satisfies Assumption 4–1 at $\hat{x} \in X$ then it satisfies Assumption 4–3 at $\hat{x}$.
(ii) If the set X satisfies Assumption 4–2 at $\hat{x} \in X$ then it satisfies Assumption 4–3 at $\hat{x}$.

Proof. By Lemma 4–3, we need only to prove that if the set X satisfies Assumption 4–1 or Assumption 4–2 at $\hat{x} \in X$ then $K[\hat{x};\ X] \subset STC[\hat{x};\ X]$.

(i) Let $t \in K[\hat{x};\ X]$, and let λ and $e(\tau)$ be a positive number and an n-dimensional function satisfying the conditions (i), (ii) and (iii) of Assumption 4–1. Now for a sequence of positive numbers $\{\tau^k\}$ such that $1 \geqq \tau^k > 0$ $(k=1, 2, \cdots)$ and $\tau^k \to 0$ as $k \to \infty$, define

$$\lambda^k = \frac{1}{\lambda \tau^k} \qquad (k=1, 2, \cdots)$$

and

$$x^k = e(\tau^k) \qquad (k=1, 2, \cdots),$$

then $x^k \in X$ $(k=1, 2, \cdots)$, $\lambda^k > 0$ $(k=1, 2, \cdots)$, $x^k \to x$ as $k \to \infty$ and

$$\lim_{k\to\infty} \lambda^k (x^k - x) = \frac{1}{\lambda} \lim_{k\to\infty} \frac{e(\tau^k) - e(0)}{\tau^k} = \frac{1}{\lambda} \lambda t = t.$$

Hence we have $t \in STC[\hat{x}; X]$.

(ii) Let $\tilde{t}$ be a vector satisfying the system of inequalities of Assumption 4-2. By Lemma 4-2, we need only to show that for any given $t \in K[\hat{x}; X]$

$$t + \varepsilon\tilde{t} \in STC[\hat{x}; X] \quad \text{for all} \quad \varepsilon > 0.$$

Define

$$x^k = \hat{x} + \frac{1}{k}(t + \varepsilon\tilde{t}) \qquad (k=1, 2, \cdots)$$

and

$$\lambda^k = k \qquad (k=1, 2, \cdots)$$

then for every $j \notin \hat{I} \cap \tilde{I}$ there exists a positive integer k_j such that

$$g_j(x^k) < 0 \qquad (k > k_j)$$

and for every $i \in \hat{I}$ we have

$$g_i(x^k) - g_i(\hat{x}) \leqq \left\langle \nabla g_i(\hat{x}), \frac{1}{k}(t + \varepsilon\tilde{t}) \right\rangle \leqq 0 \qquad (k=1, 2, \cdots).$$

On the other hand, for $i' \in \tilde{I}$ we have

$$\begin{aligned} g_{i'}(x^k) &= g_{i'}(\hat{x}) + \left\langle \nabla g_{i'}(\hat{x}), \frac{1}{k}(t + \varepsilon\tilde{t}) \right\rangle + 0(1/k) \\ &= \frac{1}{k} \left\{ \langle \nabla g_{i'}(\hat{x}), t + \varepsilon\tilde{t} \rangle + \frac{0(1/k)}{(1/k)} \right\} \qquad (k=1, 2, \cdots) \end{aligned}$$

where $o(\delta)/\delta \to 0$ as $\delta \to 0_+$. It follows from $\langle \nabla g_{i'}(\hat{x}), t + \varepsilon\tilde{t} \rangle < 0$ that there exists a positive integer $\tilde{k}$ such that for all $i' \in \tilde{I}$

$$g_{i'}(x^k) < 0 \qquad (k > \tilde{k}).$$

Hence, by letting $\bar{k} = \max\,\{k_j (j \notin \hat{I} \cap \tilde{I}),\ \tilde{k}\}$, we obtain that

$$\begin{aligned} &x^k \in X \qquad (k > \bar{k}), \\ &\lambda^k > 0 \qquad (k > \bar{k}), \\ &x^k \longrightarrow \hat{x} \quad \text{as} \quad k \longrightarrow \infty, \\ &\lambda^k (x^k - \hat{x}) \longrightarrow t + \varepsilon\tilde{t} \quad \text{as} \quad k \longrightarrow \infty. \end{aligned}$$

The above four relations imply $t + \varepsilon\tilde{t} \in STC[\hat{x}; X]$. Q. E. D.

Now we prove the following necessary condition for $\hat{x}$ to be a w-solution of Problem 4–1:

Theorem 4–3: Let the set X in Problem 4–1 satisfy Assumption 4–1, Assumption 4–2 or Assumption 4–3 at $\hat{x}\in X$. If $\hat{x}$ is a w-solution of Problem 4–1, then there exist $\hat{u}\in R^p$ and $\hat{v}\in R^q$ satisfying the conditions (i), (ii), (iii) of Theorem 4–1 and $\hat{u}\geq 0$.

Proof. In view of Theorem 4–2, we need only to establish the theorem under Assumption 4–3. Since $STC[\hat{x}; X]=K[\hat{x}; X]$ and $K[\hat{x}; X]$ is a convex cone, by Corollary 3–1, there exists a $\hat{u}\geq 0$ such that

$$\left\langle \sum_{i=1}^{p} \hat{u}_i \nabla f_i(\hat{x}),\, t \right\rangle \leqq 0 \quad \text{for all} \quad t\in K[\hat{x}; X].$$

Hence, by Minkowski-Farkas lemma, there exist $v_i\geqq 0$ $(i\in I)$ such that

$$\sum_{i=1}^{p} \hat{u}_i \nabla f_i(\hat{x}) = \sum_{i\in I} v_i \nabla g_i(\hat{x}).$$

Define $\hat{v}_i=v_i$ $(i\in I)$ and $\hat{v}_j=0$ $(j\notin I)$ then $(\hat{u},\hat{v})$ satisfies the conditions (i), (ii), (iii) of Theorem 4–1 and $\hat{u}\geq 0$. Q.E.D.

By Theorem 4–3, if X satisfies one of the three assumptions above and $\hat{x}$ is a w-solution of Problem 4–1 then there exists $(\hat{u},\hat{v})$ satisfying the conditions of Theorem 4–1 and $\hat{u}\geq 0$. For any given i, however, there is no guarantee that $\hat{u}_i$ is positive. We derive now the sufficient condition for $\hat{u}_i$ to be positive. This is an extention of Theorem 4–3.

Corollary 4–1: Let D be a nonempty subset of the set $\{1, 2, \cdots, p\}$ and let

$$\hat{X}=\{x\in R^n;\, g(x)\leqq 0,\ f_i(\hat{x})-f_i(x)\leqq 0\ (i\notin D)\}.$$

Let $\hat{X}$ satisfy Assumption 4–1, Assumption 4–2 or Assumption 4–3 at $\hat{x}$. If $\hat{x}$ is a P-solution of Problem 4–1 then there exists $(\hat{u},\hat{v})$ satisfying the conditions (i), (ii), (iii) of Theorem 4–1 and $\hat{u}^D\geq 0$ where $\hat{u}^D$ is the vector whose components are $\hat{u}_i$ $(i\in D)$.

The proof of Corollary 4–1 follows from Theorem 4–3 by observing that the relation $f(x)\geq f(\hat{x})$ for no $x\in X$ implies the relation $f^D(x)>f^D(\hat{x})$ for no $x\in\hat{X}$ where $f^D(x)$ is the vector function whose components are $f_i(x)$ $(i\in D)$. Note the difference between Theorem 4–3 and Corollary 4–1. In Theorem 4–3 we assume that $\hat{x}$ is a w-solution of Problem 4–1, but in Corollary 4–1 we assume that $\hat{x}$ is a P-solution of Problem 4–1. In fact, there is no guarantee that $\hat{u}^D\geq 0$ under the assumption of Theorem 4–3, because the relation $f(x)>f(\hat{x})$ for no $x\in X$ dose not necessarily imply the relation $f^D(x)>f^D(\hat{x})$ for no $x\in\hat{X}$.

5. Problems with Differentiable Objective Functions and Equality Constraints

In this section, we deal with the following problem:

Problem 5–1: Let $f_i(x)$ $(i=1, 2, \cdots, p)$ and $g_i(x)$ $(i=1, 2, \cdots, q)$ be differentiable numerical functions on R^n. Let

$$f(x)=(f_1(x), f_2(x), \cdots, f_p(x)),$$
$$g(x)=(g_1(x), g_2(x), \cdots, g_q(x))$$

and

$$X=\{x\in R^n;\ g(x)=0\}.$$

Find an $\hat{x}\in X$ such that $f(x)\geq f(\hat{x})$ for no $x\in X$.

We derive the necessary condition, what is called "Lagrange multiplier method", for $\hat{x}$ to be a w-solution of Problem 5–1. Let $p=1$ and $\hat{x}$ be a w-solution of Problem 5–1. In the Lagrange multiplier method, we require the assumption that the gradient vectors

$$g_1(\hat{x}), g_2(\hat{x}), \cdots, g_q(\hat{x}) \tag{1}$$

are linearly independent. Under this assumption, there exists a $\hat{v}\in R^q$ such that

$$\nabla f_1(\hat{x})-\sum_{i=1}^{q} v_i\nabla g_i(\hat{x})=0.$$

The linear independece of the gradient vectors (1) is a necessary assumption under which we may apply the implicit function theorem. Here we show the same necessary condition under another assumption. In order to prove it we do not use the implicit function theorem but the result of section 3. The assumption which we make here is the following one:

Assumption 5–1:

$$STC[\hat{x};\ X]=H[\hat{x};\ X]$$

where

$$H[\hat{x};\ X]=\{t\in R^n;\ \langle\nabla g_i(\hat{x}), t\rangle=0 \qquad (i=1, 2, \cdots, q)\}.$$

Assumption 5–1 is more general the assumption of the linear independece of the gradient vectors (1); that is, the following lemma is obtained:

Lemma 5–1: If the gradient vectors (1) are linearly independent then the set X satisfies Assumption 5–1 at $\hat{x}$.

Lemma 5–1 is verified by using the implicit function theorem, but proof is omitted here.

Now we state the necessary condition for $\hat{x}$ to be a w-solution of Problem 5–1.

Theorem 5–1: Let the set X satisfy Assumption 5–1 at $\hat{x}$ or the gradient vectors (1) be linearly independent. If $\hat{x}$ is a w-solution of Problem 5–1 then there exists a $(\hat{u}, \hat{v})$ satisfying the conditions

(i) $$\hat{u}\geq 0,$$

(ii) $$\sum_{i=1}^{p}\hat{u}_i\nabla f_i(\hat{x})-\sum_{i=1}^{q}\hat{v}_i\nabla g_i(\hat{x})=0.$$

Proof. In view of Lemma 5–1, we need only to establish the theorem under

Assumption 5-1. Let $\hat{x}$ be a w-solution of Problem 5-1 and the set X satisfy Assumption 5-1 at $\hat{x}$. Since $STC[\hat{x}; X]=H[\hat{x}; X]$ is a convex cone, by Corollary 3-1, there exists a $\hat{u}\geq 0$ such that

$$\left\langle \sum_{i=1}^{p} \hat{u}_i \nabla f_i(\hat{x}), t\right\rangle \leqq 0 \quad \text{for all} \quad t\in H[\hat{x}; X].$$

Hence, by the construction of the set $H[\hat{x}; X]$, there exists a $\hat{v}\in R^q$ such that

$$\sum_{i=1}^{p} \hat{u}_i \nabla f_i(\hat{x})=\sum_{i=1}^{q} \hat{v}_i \nabla g_i(\hat{x}),$$

which implies (ii). Q.E.D.

6. Saddle-Point Problems

In this section, we deal with a vector maximum problem with inequality constraints; that is:

Problem 6-1: Let M be a nonempty subset of R^n, $f_i(x)$ $(i=1, 2, \cdots, p)$ and $g_i(x)$ $(i=1, 2, \cdots, q)$ be numerical functions defined on R^n. Let

$$f(x)=(f_1(x), f_2(x), \cdots, f_p(x)),$$
$$g(x)=(g_1(x), g_2(x), \cdots, g_q(x))$$

and

$$X=\{x\in M; g(x)\leqq 0\}.$$

Find an $\hat{x}\in X$ such that $f(x)\geq f(\hat{x})$ for no $x\in X$.

Now we construct the saddle-point problem corresponding to the above vector maximum problem.

Problem 6-2: Find $\hat{u}\in R^p$, $\hat{v}\in R^q$ and $\hat{x}\in M$ satisfying the conditions

(i) $(\hat{u}, \hat{v})\geq 0$,

(ii) $L(\hat{u}, \hat{v}, x)\leqq L(\hat{u}, \hat{v}, \hat{x})\leqq L(\hat{u}, v, \hat{x})$ and all $x\in M$ and all $v\geqq 0$,

where $L(u, v, x)=\langle u, f(x)\rangle-\langle v, g(x)\rangle$.

Between Problem 6-1 and Problem 6-2 we have the following two relations.

Theorem 6-1: If $(\hat{u}, \hat{v}, \hat{x})$ is a solution of Problem 6-2 and $\hat{u}\geq 0$ then $\hat{x}$ is a w-solution of Problem 6-1.

Proof. Let $(\hat{u}, \hat{v}, \hat{x})$ be a solution of Problem 6-2 and $\hat{u}\geq 0$. From the second inequality of (ii), we have

$$\langle \hat{v}, g(\hat{x})\rangle \geqq \langle v, g(\hat{x})\rangle \qquad \text{for all } v\geqq 0.$$

Hence we see

$$\langle \hat{v}, g(\hat{x})\rangle=0$$

and

$$g(\hat{x}) \leqq 0,$$

which implies $\hat{x} \in X$. From the first inequality of (ii) and (1), we have

$$\langle \hat{u}, f(x) \rangle - \langle \hat{v}, g(x) \rangle \leqq \langle \hat{u}, f(\hat{x}) \rangle \qquad \text{for all } x \in M.$$

Since $\hat{v}$ is nonnegative, we obtain

$$\langle \hat{u}, f(x) \rangle \leqq \langle \hat{u}, f(\hat{x}) \rangle \qquad \text{for all } x \in X.$$

But $\hat{u}$ is semipositive and hence $\hat{x}$ is a w-solution of Problem 6–1. Q. E. D.

Similarly we can prove the following corollary.

Corollary 6–1: If $(\hat{u}, \hat{v}, \hat{x})$ is a solution of Problem 6–2 and $\hat{u} > 0$ then $\hat{x}$ is a P-solution of Problem 6–1.

Theorem 6–1 (or Corollary 6–1) shows a sufficient condition for $\hat{x}$ to be a w-solution (or a P-solution) of Problem 6–1 in the form of the saddle-point problem. In order to state necessary condition for $\hat{x}$ to be a w-solution (or a P-solution) of Problem 6–1 in the similar form, we must make some assumption of convexity in Problem 6–1; that is.

Assumption 6–1:
(i) M is convex subset of R^n,
(ii) $f_i(x)$ $(i=1, 2, \cdots, p)$ are all cocave on M,
(iii) $g_i(x)$ $(i=1, 2, \cdots, q)$ are all convex on M.
Under the above assumption we obtain the following theorem:

Theorem 6–2: Let M, $f(x)$ and $g(\mathrm{x})$ satisfy Assumption 6–1. If $\hat{x}$ is a w-solution of Problem 6–1 then there exists a $(\hat{u}, \hat{v})$ such that $(\hat{u}, \hat{v}, \hat{x})$ is a solution of Problem 6–2.

Proof. Let $\hat{x}$ be a w-solution of Problem 6–1. Define the index set I such that $I = \{i; g_i(\hat{x}) = 0\}$, then there exists an $\varepsilon > 0$ such that

$$g_j(x) < 0 \quad (j \notin I) \quad \text{for} \quad x \in B_\varepsilon(\hat{x})$$

where $B_\varepsilon(\hat{x}) = \{x \in M; \|x - \hat{x}\| < \varepsilon\}$. Define the subsets S and A of R^{p+q} such that

$$S = \{(y, z); y \in R^p, z \in R^q, y_i \leqq f_i(x) - f_i(\hat{x}) \ (i=1, 2, \cdots, p), \\ z_i \leqq -g_i(x) \ (i \in I) \qquad \text{for } x \in B_\varepsilon(\hat{x})\}.$$

$$A = \{(y, z); y \in R^p, z \in R^q, y > 0, z > 0\}.$$

Then A is a convex convex cone in R^{p+q} and, by Assumption 6–1, S is s convex subset of R^{p+q}. Since $\hat{x}$ is a w-solution of Problem 6–1, we see $A \cap S = \phi$. Hence, by the separation theorem for convex sets, there exists a $(\hat{u}, \hat{v}) \in R^{p+q}$ such that $(\hat{u}, \hat{v}) \geq 0$ and $\langle (\hat{u}, \hat{v}), (y, z) \rangle \leqq 0$ for all $(y, z) \in S$. Since $(y, z) \in S$ for $y=0$, $z_i = 0$ $(i \in I)$, any x_j $(j \notin I)$, we have

$$\langle \hat{v}, g(\hat{x}) \rangle = 0$$

and

$$\langle \hat{u}, f(x)\rangle - \langle \hat{v}, g(x)\rangle \leqq \langle \hat{u}, f(\hat{x})\rangle - \langle \hat{v}, g(\hat{x})\rangle \qquad \text{for all } x \in B_\varepsilon(\hat{x}).$$

But $\langle \hat{u}, f(x)\rangle - \langle \hat{v}, g(x)\rangle$ is concave on the convex set M, hence we obtain

$$\langle \hat{u}, f(x)\rangle - \langle \hat{v}, g(x)\rangle \leqq \langle \hat{u}, f(\hat{x})\rangle - \langle \hat{v}, g(\hat{x})\rangle \qquad \text{for all } x \in M.$$

Thus $(\hat{u}, \hat{v}, \hat{x})$ is a solution of Problem 6–2. Q. E. D.

Again, just as in the case of Theorem 4–1, there is no guarantee that $\hat{u}$ is semipositive in Theorem 6–1. As was done in Section 4, it is possible to exclude such cases by introducing constraint qualifications. In this paper we introduce Slater's constraint qualification, that is,

Assumption 6–2: There exists an $\tilde{x} \in M$ such that $g(\tilde{x}) < 0$.

Under Assumption 6–1 and Assumption 6–2, if $g_i(x)$ $(i=1, 2, \cdots, q)$ are differentiable then the set X satisfies Assumption 4–2. This fact has been shown by MANGASARIAN (1969).

Theorem 6–3: Let M, $f(x)$ and $g(x)$ satisfy Assumption 6–1 and Assumption 6–2. Then $\hat{x}$ is a w-solution 6–1 if and only if there exists a $(\hat{u}, \hat{v})$ such that $(\hat{u}, \hat{v}, \hat{x})$ is a solution of Problem 6–2 and $\hat{u} \geq 0$.

Proof. By Theorem 6–1, we need only to show the "only if" part of the theorem. Let $\hat{x}$ be a w-solution of Problem 6–1. Then, by Theorem 6–2, there exists a $(\hat{u}, \hat{v})$ such that $(\hat{u}, \hat{v}, \hat{x})$ is a solution of Problem 6–2. Now we show by contradiction that $\hat{u}$ is semipositive. Suppose that $\hat{u}$ is the zero vector, then from the condition (i) of Problem 6–2 we have $\hat{v} \geq 0$. On the other hand, from the condition (ii) of Problem 6–2, we see

$$\langle \hat{v}, g(x)\rangle \geqq \langle \hat{v}, g(\hat{x})\rangle \geqq \langle v, g(\hat{x})\rangle \qquad \text{for all } x \in M, \text{ all } v \geq 0,$$

hence

$$\langle \hat{v}, g(x)\rangle \geqq 0 \qquad \text{for all } x \in M. \tag{2}$$

But, by Assumption 6–2, there exists an $\tilde{x} \in M$ such that $g(\tilde{x}) < 0$. For this $\tilde{x}$ we have $\langle \hat{v}, g(\tilde{x})\rangle < 0$, which contradicts (2). Thus $\hat{u} \geq 0$. Q. E. D.

Theorem 6–3 will be reconsidered in the next section where we shall deal with the duality between objects and constraints.

7. Duality between Objects and Constraints

Suppose that we have q different kinds of resources from which we make n different kinds of products. If we make x_i units of the i-th product $(i=1, 2, \cdots, n)$ then we need z_j units of the j-th resource $(j=1, 2, \cdots, q)$ satisfying the relations

$$z_j = g_j(x_1, x_2, \cdots, x_n) \qquad (j=1, 2, \cdots, q),$$

or equivalently

$$z = g(x)$$

where $x=(x_1, x_2, \cdots, x_n)$, $z=(z_1, z_2, \cdots, z_q)$ and $g(x)=(g_1(x), g_2(x), \cdots, g_q(x))$. And at the same time we obtain $y=f(x)$ dollars as profit. Now we consider the following mathematical programming problem:

Problem 7-1: Let $X^1=\{x\in R^n;\ x\geqq 0,\ g(x)\leqq z^*\}$. Find and $\hat{x}\in X^1$ that maximizes $f(x)$ constrained by $x\in X^1$.

Let $\hat{x}$ be a solution of Problem 7-1 and define

$$X^2=\{x\in R^n;\ x\geqq 0,\ f(x)\geqq y^*\}$$

where $y^*=f(\hat{x})$. If there exists no $x\in X^2$ such that $g(\hat{x})\geq g(x)$, then the solution $\hat{x}$ of Problem 7-1 also gives the efficient use of the resources. But if $\hat{x}$ does not satisfy the above relation then there exists an $\tilde{x}\in X^2$ such that $g(\tilde{x})\leq g(\hat{x})\leqq z^*$. Since $\hat{x}$ is a solution of Problem 7-1, we see $f(\hat{x})=f(\tilde{x})$, which implies that $\tilde{x}$ is also a solution of Problem 7-1. The difference between $\hat{x}$ and $\tilde{x}$ is that $g(\tilde{x})\leq g(\hat{x})$. Therefore it is significant to consider the following vector minimum problem:

Problem 7-2: Find an $\tilde{x}\in X^2$ such that $g(x)\leq g(\tilde{x})$ for no $x\in X^2$.

The solution of this problem gives an efficient use of resources under the condition that we obtain more than or equal to y^* dollars as profit. In this section we investigate the relation between Problem 7-1 and Problem 7-2 in more general form. We call this relation "duality between objects and constraints."

Problem 7-1 is a usual mathematical programming problem which has one objective function, but Problem 7-2 is a vector minimum problem which has several objective functions. Hence they have not symmetry. In order to deal with the relation between them symmetrically, we introduce a vector maximum problem instead of Problem 7-1 and make the vector minimum problem corresponding to it.

Problem 7-3p: Let z^* be a vector in R^q, M be a nonempty subset of R^n, and $f_i(x)$ $(i=1, 2, \cdots, p)$, $g_j(x)$ $(j=1, 2, \cdots, q)$ be continuous numerical functions. Let

$$f(x)=(f_1(x), f_2(x), \cdots, f_p(x)),$$

$$g(x)=(g_1(x), g_2(x), \cdots, g_q(x))$$

and

$$X[g\leqq z^*]=\{x\in M;\ g(x)\leqq z^*\}.$$

Find an $\hat{x}\in X[g\leqq z^*]$ such that $f(x)\geq f(\hat{x})$ for no $x\in X[g\leqq z^*]$.

Problem 7-3d: Let y^* be a vector in R^q, and M, $f(x)$, $g(x)$ be the same as those of Problem 7-3p. Let

$$X[f\geqq y^*]=\{x\in M;\ f(x)\geqq y^*\}.$$

Find an $\hat{x}\in X[f\geqq y^*]$ such that $g(\hat{x})\geq g(x)$ for no $x\in X[f\geqq y^*]$.

Three kinds of maximum solotions (w-maximum solution, P-maximum solution and s-maximum solution) of Problem 7-3p and three kinds of minimum solutions (w-minimum solution, P-minimum solution and s-minimum solution) of Problem 7-3d are defined just as in the case of Basic Problem in section 1. We make some remarks on the difference between the constructions of the above two problems. Problem 7-3p has $f(x)$ as its objective function and $g(x)$ in its constraint set $X[g\leqq z^*]$, on the other hand Problem 7-3d has $g(x)$ as its objective function and $f(x)$ in its constraint set $X[f \geqq y^*]$. Both of these problems have the constraint set M in common. It is arbitary to consider which problem is primal. In this paper we regard Problem 7-3p as a primal problem and Problem 7-3d as its dual one.

Now we investigate the relations between Problem 7-3p and Problem 7-3d. First, we assume that there exists an $\bar{x}$ such that the set $X[g \leqq z^*] \cap X[f \geqq f(\bar{x})]$ is nonempty, closed and bounded. Then there exists an $\tilde{x} \in X[g \leqq z^*] \cap X[f \geqq f(\bar{x})]$ such that $f(x) \geq f(\tilde{x})$ for no $x \in X[g \leqq z^*] \cap X[f \geqq f(\bar{x})]$. Hence we have $g(\tilde{x}) \leqq z^*$, $f(\tilde{x}) \geqq f(\bar{x})$ and $f(x) \geq f(\tilde{x})$ for no $x \in X[g \leqq z^*]$. Furthermore, from the continuity of the function $f(x)$, the $X[g \leqq z^*] \cap X[f \geqq f(\tilde{x})]$ is also nonempty, closed and bounded and hence there exists an $\hat{x} \in X[g \leqq z^*] \cap X[f \geqq f(\tilde{x})]$ such that $g(x) \leq g(\hat{x})$ for no $x \in X[g \leqq z^*] \cap X[f \geqq f(\tilde{x})]$. Thus we have $g(\hat{x}) \leqq z^*$, $f(\hat{x}) \geqq f(\tilde{x})$ and $g(x) \leq g(\hat{x})$ for no $x \in X[f \geqq f(\tilde{x})]$. But it follows from $f(x) \geq f(\tilde{x})$ for no $x \in X[g \leqq z^*]$ that $f(\hat{x}) \ngeq f(\tilde{x})$. Consequently, we obtain

$$\hat{x} \in X[g \leq z^*], \tag{1}$$

$$f(x) \geq f(\hat{x}) \qquad \text{for no } x \in X[g \leqq z^*] \tag{2}$$

and

$$g(x) \leq g(\hat{x}) \qquad \text{for no } x \in X[f \geqq f(\hat{x})]. \tag{3}$$

Thus we have proved the following theorem:

Theorem 7-1: If there exists an $\bar{x}$ such that the set $X[g \leqq z^*] \cap X[f \geqq f(\bar{x})]$ is nonempty, closed and bounded then there exists an $\hat{x}$ satisfying (1), (2) and (3).

Corollary 7-1: Let $\bar{x}$ be a w-solution of Problem 7-3p. If the set $X[g \leqq z^*] \cap X[f \geqq f(\bar{x})]$ is closed and bounded then there exists an $\hat{x}$ satisfying (1), (2), (3) and

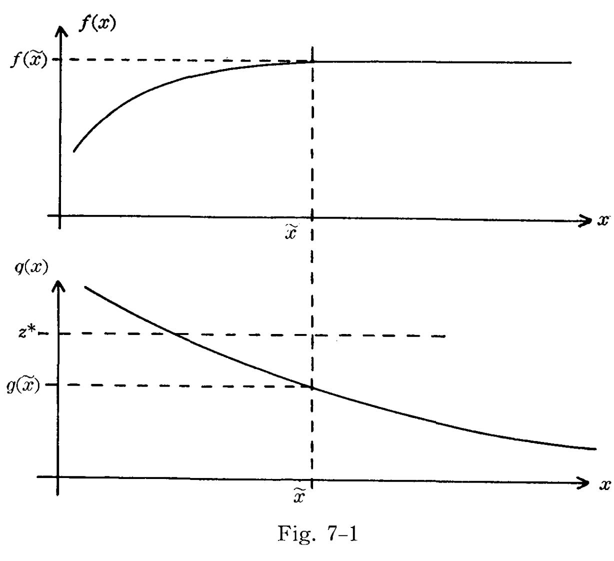

Fig. 7-1

$f(\hat{x})\geqq f(\bar{x})$. Especially, if $X[g\leqq z^*]\cap X[f\geqq f(\bar{x})]=\{\bar{x}\}$ then $\hat{x}=\bar{x}$ satisfies (1), (2) and (3).

It is should be noted that if the assumption of Corollary 7-1 does not hold then there does not necessarily exist an $\hat{x}$ satisfying (1), (2) and (3). It is apparent from the example of Fig. 7-1. In this example, we take $n=p=q=1$ and $M=\{x\in R;\ x\geqq 0\}$ in Problem 7-3p and Problem 7-3d. $\bar{x}$ is a s-solution of Problem 7-3p, but there exists no $\hat{x}$ satisfying (1), (2) and (3); because there exists no $\bar{x}$ such that the set $X[g\leqq z^*]\cap X[f\geqq f(\bar{x})]$ is nonempty, closed and bounded.

Next we derive other relations between Problem 7-3p and Problem 7-3d under more general assumptions. Let $\hat{x}$ be a w-solution of Problem 7-3p. Then there exists no $x\in X[f>f(\hat{x})]$ such that $g(x)\leqq g(\hat{x})$ where $X[f>f(\hat{x})]$ implies the set $\{x\in M;\ f(x)>f(\hat{x})\}$. It follows from the continuities of $f(x)$ and $g(x)$ that there exists no $x\in CL[X[f>f(\hat{x})]]$ such that $g(x)<g(\hat{x})$ where $CL[X[f>f(\hat{x})]]$ implies the closure of the set $X[f>f(\hat{x})]$. Hence we have $g(x)<g(\hat{x})$ for no $x\in CL[X[f>f(x)]]$. Now we introduce the following assumption.

Assumption 7-1d:

$$X[f\geqq f(\hat{x})]=CL[X[f>f(\hat{x})]].$$

If the set $X[f\geqq f(x)]$ satisfies Assumption 7-1d then we see $g(x)<g(\hat{x})$ for no $x\in X[f\geqq f(\hat{x})]$. Thus we have proved the following theorem:

Theorem 7-2: If $\hat{x}$ is a w-solution of Problem 7-3p and the set $X[f\geqq f(\hat{x})]$ satisfies Assumption 7-1d, then $\hat{x}$ is a w-solution of Problem 7-3d for $y^*=f(\hat{x})$.

Conversely, if $\hat{x}$ is a solution of Problem 7-3d for $y^*=f(\hat{x})$ and the set $X[g\leqq g(\hat{x})]$ satisfies the following assumption:

Assumption 7-1p:

$$X[g\leqq g(\hat{x})]=CL[X[g<g(\hat{x})]],$$

then $f(x)>f(\hat{x})$ for no $x\in X[g\leqq g(\hat{x})]$. Thus we obtain the following corollary:

Corollary 7-2: Let $\hat{x}\in M$, let the set $X[f\geqq f(\hat{x})]$ and the set $X[g\leqq g(\hat{x})]$ satisfy Assumption 7-1d and Assumption 7-1p respectively. Then $\hat{x}$ is a w-solution of Problem 7-3p for $z^*=g(\hat{x})$ if and only if it is a w-solution of Problem 7-3d for $y^*=f(\hat{x})$.

By Corollary 7-2 we have shown the duality between Problem 7-3p and Problem 7-3d under Assumption 7-1d and Assumption 7-1p. This is the duality between objects and constraints.

Finally, we relate Problem 7-3p and Problem 7-3d to the saddle-point problem discussed in the previous section. Let M, $f(x)$ and $g(x)$ satisfy Assumption 6-1. Let $\hat{x}$ be a w-solution of Problem 7-3p. Furthermore, we assume that the set $X[g\leqq z^*]$ satisfies Slater's constraint qualification of Problem 7-3p, that is,

Assumption 7-2p: There exists an $x\in M$ such that $g(x)<z^*$.

Then, by Theorem 6-3, there exist $\hat{u}^1\geq 0$ and $\hat{v}^1\geq 0$ such that

$$L^1(\hat{u}^1, \hat{v}^1, x)\leqq L^1(\hat{u}^1, \hat{v}^1, \hat{x})\leqq L^1(\hat{u}^1, v, \hat{x}) \qquad \text{for all } x\in M, \text{ all } v\geqq 0 \tag{4}$$

where $L^1(u, v, x)=\langle u, f(x)\rangle-\langle v, g(x)-z^*\rangle$. The first inequality of (4) is equivalent to

$$\langle \hat{u}^1, f(x)-f(\hat{x})\rangle-\langle \hat{v}^1, g(x)-g(\hat{x})\rangle \leqq 0 \qquad \text{for all } x \in M. \tag{5}$$

Now we assume that the set $X[f \geqq f(\hat{x})]$ satisfies Assumption 7–1d. It is easily verified that under Assumption 6–1 Assumption 7–1d is equivalent to Slater's constraint qualification of Problem 7–3d for $y^*=f(\hat{x})$, that is,

Assumption 7–2d: There exists an $\bar{x} \in M$ such that $f(\bar{x})>f(\hat{x})$.

Therefore, we may assume Assumption 7–2d instead of Assumption 7–1d. Under Assumption 7–2d, we have $g(\hat{x})>g(x)$ for no $x \in X[f \geqq f(\hat{x})]$. But, since the set $X[f \geqq f(\hat{x})]$ satisfies Assumption 7–2d, there exist $\hat{u}^2 \geqq 0$ and $\hat{v}^2 \geq 0$ such that

$$L^2(\hat{u}^2, \hat{v}^2, x) \leqq L^2(\hat{u}^2, \hat{v}^2, \hat{x}) \leqq L^2(u, \hat{v}^2, \hat{x}) \qquad \text{for all } x \in M, \text{ all } u \geqq 0 \tag{6}$$

where $L^2(u, v, x)=\langle u, f(x)-f(\hat{x})\rangle-\langle v, g(x)\rangle$. By observing that

$$L^2(\hat{u}^2, \hat{v}^2, \hat{x})=L(u, \hat{v}^2, \hat{x})=-\langle \hat{v}^2, g(\hat{x})\rangle \qquad \text{for all } u \geqq 0,$$

(6) is equivalent to

$$\langle \hat{u}^2, f(x)-f(\hat{x})\rangle-\langle \hat{v}^2, g(x)-g(\hat{x})\rangle \leqq 0 \qquad \text{for all } x \in M. \tag{7}$$

Note that (7) has the same form as (5). The difference between (5) and (7) is the difference between $\hat{u}^1 \geq 0$, $\hat{v}^1 \geqq 0$ and $\hat{u}^2 \geqq 0$, $\hat{v}^2 \geq 0$. By letting $\hat{u}=\hat{u}^1+\hat{u}^2$ and $\hat{v}=\hat{v}^1+\hat{v}^2$, it follows from (5) and (7) that

$$\langle \hat{u}, f(x)-f(\hat{x})\rangle-\langle \hat{v}, g(x)-g(\hat{x})\rangle \leqq 0 \qquad \text{for all } x \in M. \tag{8}$$

Conversely, it is obvious that if $\hat{x}$ satisfies (8) for some $\hat{u} \geq 0$ and $\hat{v} \geq 0$ then $\hat{x}$ is a w-solution of Problem 7–3p for $z^*=g(\hat{x})$ and a w-solution of Problem 7–3d for $y^*=f(\hat{x})$. Consequently, we obtain the following theorem:

Theorem 7–3: Let M, $f(x)$ and $g(x)$ satisfy Assumption 6–1, Assumption 7–2p for $z^*=g(\hat{x})$ and Assumption 7–2d. Then the following conditions (i), (ii) and (iii) are all equivalent:

(i) $\hat{x}$ is a w-solution of Problem 7–3p for $z^*=g(\hat{x})$,
(ii) $\hat{x}$ is a w-solution of Problem 7–3d for $y^*=f(\hat{x})$,
(iii) there exist $\hat{u} \geq 0$ and $\hat{v} \geq 0$ satisfying (8).

Furthermore, it follows from (8) that

$$\min_{u, v \geqq 0} \max_{x \in M} K(u, v, x)=K(\hat{u}, \hat{v}, \hat{x})=0 \tag{9}$$

where $K(u, v, x)=\langle u, f(x)-f(\hat{x})\rangle-\langle v, g(x)-g(\hat{x})\rangle$. If we assume that M coinsides with R^n, $f_i(x)$ $(i=1, 2, \cdots, p)$ are differentiable concave functions on R^n and $g_i(x)$ $(j=1, 2, \cdots, q)$ are differentiable convex functions on R^n, then $K(u, v, x)$ is a differentiable concave function on R^n with respect to x for every $u \geqq 0$, $v \geqq 0$. Hence the relation

$$\max_{x \in M} K(u, v, x)=K(u, v, \bar{x})$$

is equivalent to the relation

$$\sum_{i=1}^{p} u_i \nabla_i(\bar{x}) - \sum_{j=1}^{q} v_j \nabla g_j(\bar{x}) = 0 .$$

It follows from (9) that $(\hat{u}, \hat{v}, \hat{x})$ is a solution of the following problem:

Problem 7-4: Define the subset T of R^{p+q+n} such that

$$T = \left\{ (u, v, x);\ u \in R^p,\ u \geqq 0,\ v \in R^q,\ v \geqq 0,\ x \in R^n, \right.$$
$$\left. \sum_{i=1}^{p} u_i \nabla f_i(x) - \sum_{j=1}^{q} v_j \nabla g_j(x) = 0 \right\}.$$

Find a $(\hat{u}, \hat{v}, \hat{x}) \in T$ that minimizes $K(u, v, x)$ constrained by $(u, v, x) \in T$.
Thus we have proved the following theorem:

Theorem 7-4: In Theorem 7-3, if in addition $M = R^n$ and $f_i(x)$ $(i=1, 2, \cdots, p)$, $g_j(x)$ $(j=1, 2, \cdots, q)$ are differentiable then (i), (ii), (iii) of Theorem 7-3 and (iv) there exist $\hat{u} \geq 0$ and $\hat{v} \geq 0$ such that $(\hat{u}, \hat{v}, \hat{x})$ is a solution of Problem 7-4 are all equivalent.

Theorem 7-4 shows the relations between Problem 7-3p, Problem 7-3d and Problem 7-4. The relation between Problem 7-3p and Problem 7-3d means the duality between objects and constraints, while the relation between Problem 7-3p (or Problem 7-3d) and Problem 7-4 means the duality of concave (or convex) programming. Hence we have connected these two duality by Theorem 7-4.

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